MATH 3705* B Test 1 January 2007

SOLUTIONS

Questions 1-5 are multiple choice. Circle the correct answer. Only the answer will be marked. [12 marks]

1. **[2]** $\mathcal{L}\{e^{-2t}\cos(3t)\} =$

$(a)\ \dfrac{s+2}{(s+2)^2+9}$ $(b)\ \dfrac{s}{s^2+9}$ $(c)\ \dfrac{s-2}{(s-2)^2+9}$ (d) None of the above

2. **[2]** $\mathcal{L}\{t\sin(3t)\} =$

$(a)\ \dfrac{6s}{(s^2+9)^2}$ $(b)\ \dfrac{s^2-9}{(s^2+9)^2}$ $(c)\ \dfrac{9-s^2}{(s^2+9)^2}$ (d) None of the above

3. **[2]** $\mathcal{L}\{u(t-1)\,e^{2t}\} =$

$(a)\ \dfrac{e^{-s}}{s-2}$ $(b)\ \dfrac{e^{1-s}}{s+2}$ $(c)\ \dfrac{e^{2-s}}{s-2}$ (d) None of the above

4. **[3]** $\mathcal{L}^{-1}\left\{\dfrac{2s}{s^2-25}\right\} =$

$(a)\ e^{5t}-e^{-5t}$ $(b)\ e^{5t}+e^{-5t}$ $(c)\ -\cos(5t)$ (d) None of the above

5. **[3]** $\mathcal{L}^{-1}\left\{\dfrac{se^{-3s}}{s^2+4}\right\} =$

$(a)\ u(t-3)\,\cos(2t)$ $(b)\ u(t+3)\,\cos[2(t+3)]$

$(c)\ u(t-3)\,\cos[2(t-3)]$ (d) None of the above

Answers: 1.(a), 2.(a), 3.(c), 4.(b), 5.(c).

6. [**4 marks**] Find $\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} t, & \text{if } 0 \le t < 2; \\ e^{5t}, & \text{if } t \geq 2. \end{cases}$

Solution:

Rewrite $f(t)$ using the unit step function $u(t-a)$:

$$f(t) = [u(t) - u(t-2)]t + u(t-2)e^{5t} = u(t)t + u(t-2)(-t + e^{5t}).$$

Find the Laplace transform of each term of the expression above, using the Second Shifting theorem with $a = 0$ and $a = 2$ respectively:

$$\mathcal{L}\{u(t)t\} = e^{-0t}\mathcal{L}\{t\} = \frac{1}{s^2},$$

$$\mathcal{L}\{u(t-2)(e^{5t} - t)\} = \mathcal{L}\{u(t-2)(e^{5(t-2)} \cdot e^{10} - (t-2) - 2)\} = e^{-2s}\mathcal{L}\{e^{10}e^{5t} - t - 2\} =$$

$$= e^{-2s}\left\{\frac{e^{10}}{s-5} - \frac{1}{s^2} - \frac{2}{s}\right\}.$$

7. [**7 marks**] Employ the Laplace transform to solve the initial-value problem

$$y'' + 6y' + 10y = 0, \quad y(0) = 0, \ y'(0) = 2.$$

Solution:

Take the Laplace transform of both parts of the equation:

$$[s^2Y(s) - sy(0) - y'(0)] + 6[sY(s) - y(0)] + 10Y(s) = 0.$$

Substitute $y(0) = 0, \ y'(0) = 2$ and solve for $Y(s)$:

$$(s^2 + 6s + 10)Y(s) = 2, \ \Rightarrow \ Y(s) = \frac{2}{s^2 + 6s + 10} = \frac{2}{(s+3)^2 + 1}.$$

Using the First Shifting Theorem with $a = -3$, find

$$\mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\{\frac{1}{(s+3)^2 + 1}\} = 2e^{-3t}\sin(t) = y(t).$$

8. [**7 marks**] Find $f(t)$, if

$$f(t) - 3\int_0^t f(x)\,dx = 6t.$$

Solution:

Let $\mathcal{L}\{f(t)\} = F(s)$. Take the Laplace transform of both parts of the equation:

$$F(s) - 3\frac{1}{s}F(s) = 6\frac{1}{s^2} \quad \Rightarrow F(s)(1 - \frac{3}{s}) = \frac{6}{s^2} \quad \Rightarrow F(s) = \frac{6}{s(s-3).}$$

To invert $F(s)$, one can use **(a)** partial fractions or **(b)** $\mathcal{L}\left\{\int_0^t g(x)\,dx\right\} = \frac{G(s)}{s}.$

(a) $F(s) = \dfrac{6}{s(s-3)} = -\dfrac{2}{s} + \dfrac{2}{s-3} \quad \Rightarrow \quad y(t) = -2 + 2e^{3t}.$

OR

(b) $\dfrac{6}{s(s-3)} = \dfrac{1}{s}\cdot\dfrac{6}{s-3} = \dfrac{1}{s}\cdot G(s).$ Then $\mathcal{L}^{-1}\left\{G(s)\right\} = \mathcal{L}^{-1}\left\{\dfrac{6}{s-3}\right\} = 6e^{3t} = g(t),$
and

$$y(t) = \int_0^t g(x)\,dx = 6\int_0^t e^{3x}\,dx = 6\cdot\frac{1}{3}e^{3x}\Big|_0^t = 2e^{3t} - 2.$$