MATH 3705*
Test 1 - Solutions

January 2006

Questions 1-6 are multiple choice. Circle the correct answer. Only the answer will be marked. [12 marks]

1. $\mathcal{L}\{e^{4t}\cos(5t)\} =$

   $(a)\ \dfrac{s+4}{(s+4)^2+25} \quad (b)\ \dfrac{s+4}{s^2+25} \quad (c)\ \dfrac{s-4}{(s-4)^2+25} \quad (d)$ None of the above

2. $\mathcal{L}\{t\sin(2t)\} =$

   $(a)\ \dfrac{4s}{(s^2+4)^2} \quad (b)\ \dfrac{s^2-4}{(s^2+4)^2} \quad (c)\ \dfrac{4-s^2}{(s^2+4)^2} \quad (d)$ None of the above

3. $\mathcal{L}\left\{\int_0^t e^x \sin(t-x)\,dx\right\} =$

   $(a)\ \dfrac{e^{-s}}{s^2+1} \quad (b)\ \dfrac{1}{(s-1)(s^2+1)} \quad (c)\ \dfrac{1}{(s-1)^2+1} \quad (d)$ None of the above

4. $\mathcal{L}^{-1}\left\{\dfrac{6}{s^2-9}\right\} =$

   $(a)\ e^{3t}-e^{-3t} \quad (b)\ e^{3t}+e^{-3t} \quad (c)\ -\sin(3t) \quad (d)$ None of the above

5. $\mathcal{L}^{-1}\left\{\dfrac{3s+6}{s^2-4s+13}\right\} =$

   $(a)\ 3e^{2t}\cos(3t)+4e^{2t}\sin(3t) \quad (b)\ 3e^{-2t}\cos(3t)+\dfrac{9}{4}e^{-2t}\sin(3t)$

   $(c)\ 3e^{2t}\cos(3t)+\dfrac{9}{4}e^{2t}\sin(3t) \quad (d)$ None of the above

6. $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2-6s+25}\right\} =$

   $(a)\ u(t-2)e^{3(t-2)}\sin[4(t-2)] \quad (b)\ \dfrac{1}{4}u(t-2)e^{3t-6}\sin(4t-8)$

   $(c)\ \dfrac{1}{4}u(t-2)e^{3t}\sin[4(t-2)] \quad (d)$ None of the above

Answers: 1.(c), 2.(a), 3.(b), 4.(a), 5.(a), 6.(b),

7. [9 marks] Employ the Laplace transform to solve the initial-value problem

$$y'' + 2y' + 10y = 0, \quad y(0) = 1, \ y'(0) = 2.$$

Solution:

Take the Laplace transform of both parts of the equation:

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 10Y(s) = 0.$$

Substitute $y(0) = 1, \ y'(0) = 2$ and solve for $Y(s)$:

$$(s^2 + 2s + 10)Y(s) = s + 4, \ \Rightarrow \ Y(s) = \frac{s+4}{s^2 + 2s + 10} = \frac{(s+1)+3}{(s+1)^2 + 9}.$$

Using the First Shifting Theorem with $a = -1$, find

$$\mathcal{L}^{-1}\{Y(s)\} = e^{-t}\cos(3t) + e^{-t}\sin(3t) = y(t).$$

8. [9 marks] Find $f(t)$, if

$$f(t) - 2\int_0^t f(x)\,dx = 2t.$$

Solution:

Let $\mathcal{L}\{f(t)\} = F(s)$. Take the Laplace transform of both parts of the equation:

$$F(s) - 2\frac{1}{s}F(s) = 2\frac{1}{s^2} \quad \Rightarrow F(s)(1 - \frac{2}{s}) = 2\frac{1}{s^2} \quad \Rightarrow F(s) = \frac{2}{s(s-2).}$$

To invert $F(s)$, one can use (a) partial fractions or (b) $\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{F(s)}{s}$.

(a) $F(s) = \dfrac{2}{s(s-2)} = -\dfrac{1}{s} + \dfrac{1}{s-2} \quad \Rightarrow \quad y(t) = -1 + e^{2t}.$

OR

(b) $\dfrac{2}{s(s-2)} = \dfrac{1}{s} \cdot \dfrac{2}{s-2} = \dfrac{1}{s} \cdot F(s).$ Then $\mathcal{L}^{-1}\left\{F(s)\right\} = \mathcal{L}^{-1}\left\{\dfrac{2}{s-2}\right\} = 2e^2t = f(t),$

and

$$y(t) = \int_0^t f(x)\,dx = 2\int_0^t e^{2x}\,dx = 2 \cdot \frac{1}{2}e^{2x}\Big|_0^t = e^{2t} - 1.$$