MAT1300 – Notes — By Eric Hua

Contents

Appendix A. Precalculus Review

1. Real numbers and intervals

Interval Notation	**Set Notation**
$[a,b]$	$\{x \in \mathbb{R} : a \le x \le b\}$
(a,b)	$\{x \in \mathbb{R} : a < x < b\}$
$[a,b)$	$\{x \in \mathbb{R} : a \le x < b\}$
$(a,b]$	$\{x \in \mathbb{R} : a < x \le b\}$
$(a,+\infty)$	$\{x \in \mathbb{R} : x > a\}$
$[a,+\infty)$	$\{x \in \mathbb{R} : x \ge a\}$
$(-\infty,b)$	$\{x \in \mathbb{R} : x < b\}$
$(-\infty,b]$	$\{x \in \mathbb{R} : x \le b\}$
$(-\infty,+\infty)$	$\mathbb{R}$

2. Solving inequalities

Example 1 *Solve the inequality*

$$-2x - 3 \le -13.$$

Solution: We have

$$-2x - 3 \le -13 \Rightarrow -2x \le -13 + 3 \Rightarrow -2x \le -10.$$

The next step would be to divide both sides by -2. Since $-2 < 0$, the sense of the inequality is inverted, and so

$$-2x \le -10 \Rightarrow x \ge \frac{-10}{-2} \Rightarrow x \ge -5.$$

Example 2 *Solve the inequality*

$$x^2 + 2x - 35 < 0.$$

Solution: Observe that $x^2 + 2x - 35 = (x-5)(x+7)$, which vanishes when $x = -7$ or when $x = 5$. Now we construct the table:

$x \in$	$(-\infty,-7)$	$(-7,5)$	$(5,+\infty)$
$x+7$	$-$	$+$	$+$
$x-5$	$-$	$-$	$+$
$(x+7)(x-5)$	$+$	$-$	$+$

On the last row, the sign of the product $(x+7)(x-5)$ is determined by the sign of each of the factors $x+7$ and $x-5$.

From the sign diagram above we see that

$$\{x \in \mathbb{R} : x^2 + 2x - 35 < 0\} = (-7, 5).$$

Notice that we exclude both $x = -7$ and $x = 5$ in the set, as $(x+7)(x-5)$ vanishes there.

3. Absolute Values

Definition 1 *Let $x \in \mathbb{R}$. The* absolute value of x*—denoted by $|x|$—is defined by*

$$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

Example 3 *Let $x > 10$. Then $|3 - |5 - x|| = |3 - (x - 5)| = |8 - x|$.*

- $|x| \leq t \iff -t \leq x \leq t.$
- $|x| \geq t \iff x \geq t \quad \text{or} \quad x \leq -t.$
- Triangle Inequality: Let a, b be real numbers. Then $|a + b| \leq |a| + |b|$.

Example 4 *Solve the inequality $|2x - 1| \leq 1$.*

Solution:

$$|2x-1| \leq 1 \iff -1 \leq 2x - 1 \leq 1 \iff 0 \leq 2x \leq 2 \iff 0 \leq x \leq 1 \iff x \in [0,1].$$

The solution set is the interval $[0, 1]$.

4. Exponents and radicals

Properties of exponents:

- $x^0 = 1, \quad x \neq 0.$
- $x^{-n} = \frac{1}{x^n}, \quad x \neq 0.$
- $x^{1/n} = \sqrt[n]{x}$, $x^{m/n} = \sqrt[n]{x^m}$.
- $x^m x^n = x^{m+n}$, $x^m / x^n = x^{m-n}$.
- $(x^m)^n = x^{mn}$.
- $x^n y^n = (xy)^n$.

For Example,

$$\frac{x^{3/2}+5x^2}{x^{1/2}}=x(1+5x^{1/2}).$$

5. Factoring Polynomials

- $a^2-b^2=(a-b)(a+b)$.
- $a^3-b^3=(a-b)(a^2+ab+b^2)$ and $a^3+b^3=(a+b)(a^2-ab+b^2)$.
- $x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+x^2y^{n-3}+xy^{n-2}+y^{n-1})$.
- $(a\pm b)^2=a^2\pm 2ab+b^2$.
- $(a+b)^3=a^3+3a^2b+3ab^2+b^3$ and $(a-b)^3=a^3-3a^2b+3ab^2-b^3$.

Example 5

$$\begin{aligned} x^4+x^2+1 &= x^4+2x^2+1-x^2 \\ &= (x^2+1)^2-x^2 \\ &= (x^2+1-x)(x^2+1+x). \end{aligned}$$

Example 6 $x^2-8x-9=(x-9)(x+1)$.

6. Rationalizing denominator or numerator

- If the denominator is $\sqrt{a}$, then multiply both top and bottom by $\sqrt{a}$.
- If the denominator is $\sqrt{a}\pm\sqrt{b}$, then multiply both top and bottom by $\sqrt{a}\mp\sqrt{b}$.

Example 7

$$\frac{x}{\sqrt{x+4}-2}=\frac{x(\sqrt{x+4}+2)}{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}=\frac{x(\sqrt{x+4}+2)}{x}=\sqrt{x+4}+2.$$

1.1 The Cartesian Plane and Distance Formula

1. Distance formula: The distance between two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

2. Midpoint formula: The midpoint between two points (x_1, y_1) and (x_2, y_2) is

$$\text{Midpoint} = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}).$$

3. Translating points: A point (x_0, y_0) is translated r units to the right, u units up, it gives a new point $(x_0 + r, y_0 + u)$.

1.2 Graphs of Equation

1. Sketch the graph of a line and parabola.

2. Finding intercepts:

   - x-intercepts: Let $y = 0$ and solve the equation for x;
   - y-intercepts: Let $x = 0$ and solve the equation for y.

   Example 8 *Find x-intercepts of $y = 2x^2 + x - 1$.*

   Solution: Let $y = 0$, we have $2x^2 + x - 1 = 0, \Rightarrow x = 0.5, -1$.

3. Circle: The circle with center (h, k) and radius r is: $(x - h)^2 + (y - k)^2 = r^2$.

4. Points of intersection: Common points on both graphs.

   Example 9 *Find the points of intersection of $y = 2x^2 + x - 1$ and $y = x^2 + 1$.*

   Solution: Let $2x^2 + x - 1 = x^2 + 1, \Rightarrow x^2 + x - 2 = 0, \Rightarrow (x+2)(x-1) = 0, \Rightarrow x = 1, -2 \Rightarrow$ two points of intersection $(1, 2)$, $(-2, 5)$.

5. Application in finance:

   - Break-even point: The total cost = the total revenue;
   - Supply equation: Relationship between the unit price p and quantity supplied x;
   - Demand equation: Relationship between the unit price p and quantity demanded x;
   - Equilibrium point: Intersection between supply equation and demand equation.

Example 10 *A store has been selling Häagen-Dazs bar at the price of \$4.00 per bar and, at this price, the store can sell 60 bars per day. If the store raises its price, it will sell 2 fewer bars per day for each \$0.4 increase in price. Assume that each bar costs the store \$2.*

a. Find the demand equation (p as a function of quantity x)
b. Find $R(x)$, revenue as a function of quantity x.
c. What is the total cost $C(x)$?
d. What is the total profit $P(x)$?
e. What is the break-even point?
f. If the supply equation is $S(x) = 2x + 3$, find the equilibrium points.
g. Find the maximum profit.

Solution: a. $x = 60 - 2\left(\frac{p-4.00}{0.4}\right), \Rightarrow x = 60 - 5p + 20, \Rightarrow p = 16 - 0.2x$.
b. $R(x) = xp = 16x - 0.2x^2$.
c. $C(x) = 2x$.
d. $P(x) = R(x) - C(x) = 14x - 0.2x^2$.
e. Let $P(x) = 0$, we have $14x - 0.2x^2 = 0 \Rightarrow x = 0, 70$.
f. Let $S(x) = p, \Rightarrow 2x + 3 = 14 - 0.2x \Rightarrow x = 5$.
g.

$$\begin{aligned} P(x) &= 14x - 0.2x^2 = -0.2(x^2 - 70x) \\ &= -0.2(x-35)^2 + 245. \end{aligned}$$

max $P(x) = 245$.

1.3 Lines in the plane and slope

1. Slope or Rate of Change (of y with respect to x) through two points (x_1, y_1) and (x_2, y_2):
$$m = \frac{\text{the change in } y}{\text{the change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$
2. Line of slope-intercept form: $y = mx + b$.
3. Line of slope-point form: $y - y_0 = m(x - x_0)$, where (x_0, y_0) is a point on the line.
4. General form of a line: $Ax + By + C = 0$.
5. Vertical line: $x = a$.
6. Horizontal line: $y = b$.
7. Parallel lines: $L_1 : y = m_1x + b_1$, $L_2 : y = m_2x + b_2$. Then $L_1 // L_2 \Leftrightarrow m_1 = m_2$.
8. Perpendicular lines: $L_1 : y = m_1x + b_1$, $L_2 : y = m_2x + b_2$. Then $L_1 \perp L_2 \Leftrightarrow m_1 = -\frac{1}{m_2}$.

Example 11 *Find a line whose graph passes through points* $(1, 2)$ *and* $(3, 6)$.

Solution: $m = \frac{6-2}{3-1} = 2$, $y - 2 = 2(x - 1) \Rightarrow y = 2x$.

Example 12 *A line is given by* $y = -3x + 1$. *If* x *increases by 5, how does* y *change?*

Solution: $\Delta x = 5$, $\Delta y = m\Delta x = (-3)(5) = -15$. So y decreases by 15.

1.4 Functions

Function: *A function is a relationship between two variables (one is called independent variable, another is called dependent variable) such that to each value of the independent variable there corresponds exactly one value of the dependent variable.*

- Domain of the function $y = f(x)$: $D =$ The set of all values of the independent variable x for which the function is defined.
- Range of the function: $R =$ The set of all values taking on by the dependent variable y.

Four ways to represent functions:

- Verbally: by a description in words;
- numerically: by a table of values;
- visually: by a graph $\{(x, f(x))|x \in D\}$;
- algebraically: by an explicit formula.

Example 13 *$f(x) = \frac{x^2}{x^2-3x+2}$ is a function, $D = \{x : x \neq 1, 2\}$.*

Example 14 *$f(x) = \sqrt{4 - x^2}$ is a function, $D = \{x : -2 \leq x \leq 2\}$.*

Example 15 *$f(x) = \pm x^2$ is not a function.*

Example 16 *Piecewise defined functions: $f(x) = \begin{cases} 2x, & x \leq 0; \\ 3x, & x > 0. \end{cases}$*

Example 17 *Evaluate a function: Let $f(x) = x^2 - 2x + 1$. Then $f(5) = 5^2 - 2(5) + 1 = 16$. $f(x + 3) = (x + 3)^2 - 2(x + 3) + 1 = x^2 + 4x + 4$.*

Basic functions such as:

- Power function: $f(x) = x^c$, where c is a real number.
- Polynomials: $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, where n is a positive integer (which is called the degree of $P(x)$).
- Rational functions: $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

Some important functions:

- One-to-one function: $y = f(x)$ is 1-1 $\Leftrightarrow$ for each $y \in R$, there is only one $x \in D$. Horizontal line test can be used to check this.

  Example 18 *$f(x) = x^2$ is not 1-1; $g(x) = x^2$, $x > 0$ is 1-1.*

- Composite function $f(g(x))$ or $f \circ g(x)$. The domain of $f \circ g(x)$ is the set of all $x \in D(g)$ such that $g(x) \in D(f)$.

  Example 19 *Let $f(x) = \sqrt{2x - 4}$, $g(x) = \sqrt{3 - 2x}$.*

Then $f(g(x)) = \sqrt{2\sqrt{3-2x}-4}$ with $D(f \circ g) = \{x \leq -0.5\}$ (the solution of $3-2x \geq 0$ and $2\sqrt{3-2x}-4 \geq 0$). $g(f(x)) = \sqrt{3-2\sqrt{2x-4}}$, $D(g \circ f) = \{2 \leq x \leq 3.125\}$ (the solution of $2x-4 \geq 0$ and $3-2\sqrt{2x-4} \geq 0$).

- Inverse function: $y = f(x) \to x = f^{-1}(y)$. We write it as $y = f^{-1}(x)$.
  - The graph of f^{-1} and the graph of f are symmetric about the line $y = x$.
  - Cancelation: $f(f^{-1}(y)) = y$, $f^{-1}(f(x)) = x$.

To find inverse, we have two ways:

- Method 1: Using the coordinate geometry;
- Method 2: Using variable interchange.

Example 20 *Let* $f(x) = \frac{3x+2}{5x-4}$, *find the inverse* $f^{-1}(x)$.

Strategy to use variable interchange:
1) Write $y = \frac{3x+2}{5x-4}$;
2) Switch x and y: $x = \frac{3y+2}{5y-4}$;
3) Isolate y: $y = \frac{4x+2}{5x-3}$;
4) Answer: $y = f^{-1}(x) = \frac{4x+2}{5x-3}$.

Example. let $f(x) = 2x^3 + 1$, find the inverse $f^{-1}(x)$.

1.5 Limits

Definition 2 *We write*

$$\lim_{x \to a} f(x) = L$$

and say "as x approaches a, the limit of $f(x)$ is L." If L is a finite number, we say that the limit exists, otherwise, the limit does not exist.

Properties:

- $$\lim_{x \to a} P(x) = P(a), \quad P(x) \text{ is a polynomial.}$$

- $$\lim_{x\to a}(cf(x) \pm dg(x)) = cf(a) \pm dg(a), \quad \text{c, d are constants.}$$

- $$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad if \ \lim_{x\to a} g(x) \neq 0.$$

- $$\lim_{x\to a}[f(x)]^n = [f(a)]^n.$$

- $$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{f(a)}.$$

Example 21

$$\lim_{x\to 1}(x^2-3) = 1^2 - 3 = -2, \quad \lim_{x\to 1}\frac{3x^4+8x-2}{x-2} = \frac{3(1)^4+8(1)-2}{1-2} = -9.$$

Special case:

$$\lim_{x\to a}\frac{f(x)}{g(x)} \quad where \ g(a) = 0.$$

- If $f(a) \neq 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ does not exist.
- If $f(a) = 0$, simplify $\frac{f(x)}{g(x)}$ first, then study the limit.

Example 22

$$\lim_{x\to 2}\frac{3x^4+8x-2}{x-2} \nexists, \quad \lim_{x\to 2}\frac{x-2}{x-2} = 1.$$

Example 23

$$\begin{aligned}
\lim_{x\to 2}\frac{x^2-4}{x-2} &= \lim_{x\to 2}(x+2) = 4, \\
\lim_{h\to 0}\frac{(h+1)^2-1}{h} &= \lim_{h\to 0}\frac{h(h+2)}{h} = \lim_{h\to 0}(h+2) = 2, \\
\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x} &= \lim_{x\to 0}\frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)} = \lim_{x\to 0}\frac{x}{x(\sqrt{x+4}+2)} \\
&= \lim_{x\to 0}\frac{1}{\sqrt{x+4}+2} = \frac{1}{4}.
\end{aligned}$$

One-sided limit

Definition 3 *We write*

$$\lim_{x\to a^-} f(x) = L$$

and say that the limit of $f(x)$ is L as x approaches a from the left. Similarly, We write

$$\lim_{x\to a^+} f(x) = L$$

and say that the limit of $f(x)$ is L as x approaches a from the right.

Theorem 1

$$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.$$

Example 24 *Consider the Heaviside function*

$$H(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0. \end{cases}$$

$$\lim_{t\to 2} H(t) = 1,$$

$$\lim_{t\to 0^+} H(t) = 1, \lim_{t\to 0^-} H(t) = 0, \Rightarrow \lim_{t\to 0} H(t) \nexists.$$

Example 25 $\lim_{x\to 0} \frac{|x|}{x} \nexists.$

$\because \lim_{x\to 0^-} \frac{|x|}{x} = -1, \lim_{x\to 0^+} \frac{|x|}{x} = 1.$

Example 26 *Let*

$$f(x) = \begin{cases} x - 5, & x < 0; \\ x^2 + 3x, & 0 \leq x \leq 1; \\ x^4 - x^3 + 4, & x > 1. \end{cases}$$

Then $\lim_{x\to 0} f(x) \nexists$ *and* $\lim_{x\to 1} f(x) = 4.$

Unbounded Behavior

$\lim_{x\to 1^-} \frac{1}{1-x} = \infty, \lim_{x\to 1^+} \frac{1}{1-x} = -\infty.$

1.6 Continuity

Definition 4 *If* $\lim_{x\to a} f(x) = f(a)$*, then* $f(x)$ *is continuous at* $x = a$*, otherwise,* $f(x)$ *is discontinuous at* $x = a$*. If* $f(x)$ *is continuous at any point on an interval, then* $f(x)$ *is continuous on the interval.*

Example 27 *Explore discontinuity from graph.*

Example 28 *Consider* $f(x) = \frac{x^2-2x+1}{x-1}$ *at* $x = 1$*. Sol:* $f(x)$ *is undefined at* $x = 1$*. But* $\lim_{x\to 1} f(x) = 0$*. So the discontinuous point* $x = 1$ *is* **removable** *if we define* $f(1) = 0$*.*

Example 29 *Determine the continuity of* $f(x) = \frac{|x|}{x}$*.*

Sol: $x = 0$ is not removable.

Definition 5 *If* $\lim_{x\to a^-} f(x) = f(a)$*, then* $f(x)$ *is continuous from the left at* $x = a$*; if* $\lim_{x\to a^+} f(x) = f(a)$*, then* $f(x)$ *is continuous from the right at* $x = a$*. If* $f(x)$ *is continuous in* (a, b)*, and left-continuous at* b*, right-continuous at* a*, then we say that* $f(x)$ *is continuous on* $[a, b]$*.*

Example 30 *Determine the left and right continuity at* $x = 0$*:*

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0; \\ 1, & x = 0. \end{cases}.$$

Sol: continuous from right at $x = 0$, discontinuous from left at $x = 0$.

Theorem 2 *If* $f(x)$ *and* $g(x)$ *are continuous at* a*, then*

$$f \pm g, fg, cf (c \textit{ is a constant}), \frac{f}{g} (\textit{if } g(a) \neq 0)$$

are continuous.

Theorem 3 *Polynomials, rational functions, root functions are continuous in their domain.*

Example 31 *The greatest integer function:* $f(x) = [x] =$ *greatest integer less than or equal to* x *is discontinuous at any integer.*

[3.5]=3, [-3.5]=-4, [5]=5.

4.1 Exponential functions

An exponential function is defined as:

$$y = f(x) = a^x, \quad a > 0.$$

- Domain: x can be any real number.
- Exponential growth: $a > 1$; $\lim_{x\to-\infty} a^x = 0$, $\lim_{x\to\infty} a^x = \infty$.
- Exponential decay: $0 < a < 1$; $\lim_{x\to-\infty} a^x = \infty$, $\lim_{x\to\infty} a^x = 0$.

Laws of exponents:

$$a^{x+y} = a^x a^y, \quad a^{x-y} = a^x/a^y, \quad (a^x)^y = a^{xy}, \quad a^x b^x = (ab)^x.$$

Example 32 *Sketch the graph of* $y = 2^x + 5$.

Solution: HA: $\lim_{x\to-\infty} y = 5$.

Example 33 *Sketch the graph of* $y = 2^{-x} + 5$.

Solution: HA: $\lim_{x\to\infty} y = 5$.

4.2 Natural Exponential function

Natural exponential function is defined as:

$$y = f(x) = e^x, \quad a > 0,$$

where $e = \lim_{x\to 0}(1 + x)^{1/x} = \doteq 2.71828....$
Remark. Logistic growth function is defined as: $f(t) = \frac{a}{1+be^{-kt}}$.
Application in compounded interest. Let A represent the amount of money after a certain amount of time, P represent the principle or the amount of money you start with

(Present value), r represent the interest rate, t represent the amount of time in years, n represent the number of times per year. Then

- Compounded n times per year:
$$A = P\left(1+\frac{r}{n}\right)^{nt} = P(1+r_{eff})^t,$$
where
$$r_{eff} = \left(1+\frac{r}{n}\right)^{n} - 1$$
is called **effective rate.**

- Compounded continuously:
$$A = Pe^{rt}.$$

Example 34 *Suppose $12000 is put into an account that pays 11% annually. How much will be in the account after 25 years?*

a) compounded continuously.

b) compounded quarterly. What is the effective rate?

Solution: a)
$$A = Pe^{rt} = 12000e^{0.11(25)} = 187711.58.$$

b)
$$A = P\left(1+\frac{r}{n}\right)^{nt} = 12000\left(1+\frac{0.11}{4}\right)^{4(25)} 180869.07$$
$$r_{eff} = \left(1+\frac{r}{n}\right)^{n} - 1 = \left(1+\frac{0.11}{4}\right)^{4} - 1 = 0.1146.$$

4.4 Logarithmic functions

$$y = a^x \quad \xrightarrow{inversefunction} \quad y = \log_a x,$$
$$y = e^x \quad \xrightarrow{inversefunction} \quad y = \log_e x = \ln x,$$
$$y = 10^x \quad \xrightarrow{inversefunction} \quad y = \log_{10} x = \log x.$$

Definition: $y = \log_a x$ is called logarithmic function with the base a. Domain $= \{x > 0\}$. $y = \ln x$ is called natural logarithmic function.

Properties: Let $B, C > 0$. Then

1. $\ln(BC) = \ln B + \ln C$,

2. $\ln(\frac{B}{C}) = \ln B - \ln C$,

3. $\ln(B^n) = n \ln B$,

4. $\ln(e^x) = x$, $\ln e = 1$,

5. $e^{\ln B} = B$,

6. $\log_e 1 = 0$.

Example 35 *Convert a^x to base e.*

$$a^x = e^{x \ln a}.$$

Example 36 *Solve for x:*

$$3^{2x-1} = 4, \quad \ln x + \ln(x-8) = \ln 9.$$

2.1 The derivative and the slope of a graph

Definition 6 *Let $P = (a, f(a))$ be a point on the curve $y = f(x)$. The tangent of $f(x)$ at P is the line through P with slope*

$$m = \lim_{\Delta x \to 0} m_{secant} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 37 *Find the slope and the equation of the tangent line to the curve*

$$y = f(x) = 3x^2 - 6x + 1$$

at the point $(2, 1)$. Sketch the curve.

Sol. $a = 2$, $f(a) = 1$.

$$m = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{3x^2 - 6x + 1 - 1}{x - 2} = \lim_{x \to 2} (3x) = 6.$$

The tangent line is

$$y - 1 = 6(x - 2), \Longrightarrow, y = 6x - 11.$$

To sketch the curve, $y = 3(x-1)^2 - 2$.

Example 38 *Find the tangent line to the hyperbola $xy = 4$ at the point $(1, 4)$.*

Sol. $a = 1$, $f(x) = y = 1/x$.

$$m = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h\to 0} \frac{-1}{1+h} = -1.$$

The tangent line is

$$y = -x + b, \Longrightarrow, 4 = -1 + b, b = 5, \quad y = -x + 5.$$

Example 39 *Find the slope of the tangent line to the curve $y = \frac{1}{\sqrt{x+1}}$ at the point $(0, 1)$.*

Sol. $a = 0$,

$$m = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{0+h+1}} - 1}{h} = \lim_{h\to 0} \frac{1 - \sqrt{h+1}}{h\sqrt{h+1}}$$

$$= \lim_{h\to 0} \frac{-h}{h\sqrt{h+1}(1+\sqrt{h+1})} = -\frac{1}{2}.$$

Definition 7 *(Definition of the Derivative): The derivative of the function $y = f(x)$ at a is*

$$y'(a) = \frac{dy}{dx} = f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Meaning: $f'(a) =$

- instantaneous rate of change of $f(x)$ at a, or
- rate of change of $f(x)$ at a, or
- the slope of the tangent line to the curve at a.

Example 40 *Let $f(x) = x^2$. Calculate $f'(5)$.*

Sol:

$$f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0} \frac{(5+h)^2 - 5^2}{h} = 10.$$

Example 41 *The volume of a sphere of radius r is given by*

$$V = \frac{4}{3}\pi r^3.$$

Calculate $\frac{dV}{dr}$ by definition. What's the meaning of this derivative?

Sol:

$$\frac{dV}{dr} = \lim_{h\to 0}\frac{V(r+h)-V(r)}{h} = \lim_{h\to 0}\frac{\frac{4}{3}\pi(r+h)^3 - \frac{4}{3}\pi r^3}{h}$$

$$= \frac{4}{3}\pi\lim_{h\to 0}\frac{(r+h)^3 - r^3}{h} = \frac{4}{3}\pi\lim_{h\to 0}\frac{3r^2h + 23rh^2 + h^3}{h} = 4\pi r^2.$$

The derivative is the surface area.

Definition 8 *The derivative of the function $y = f(x)$ is the function $f'(x)$:*

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Example 42 *Let $f(x) = \sqrt{x-3}$. Find $f'(x)$ and state the domains of f and f'.*

Sol:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h-3}-\sqrt{x-3}}{h}$$

$$= \lim_{h\to 0}\frac{(\sqrt{x+h-3}-\sqrt{x-3})(\sqrt{x+h-3}+\sqrt{x-3})}{h(\sqrt{x+h-3}+\sqrt{x-3})}$$

$$= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h-3}+\sqrt{x-3})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h-3}+\sqrt{x-3}} = \frac{1}{2\sqrt{x-3}}.$$

The domain of f: $x-3 \geq 0$, $x \geq 3$.

The domain of f': $x-3 \geq 0$ and $2\sqrt{x-3} \neq 0$, $x > 3$.

Example 43 *Let $f(x) = \frac{x}{x-3}$. Find $f'(x)$ and state the domains of f and f'.*

Example 44 *Find $f'(x)$ from the graph of f.*

Definition 9 *The function f is differentiable at a if $f'(a)$ exists. It is differentiable on an interval if $f'(a)$ exists for any a on the interval.*

Example 45 *$f(x) = |x|$ is not differentiable at $x = 0$.*

Theorem 4 *If a function is differentiable at $x = c$, then the function is continuous at $x = c$.*

2.2 Some Rules for differentiation

- Constant rule: If $f(x) = c$, then $f'(x) = 0$ or $\frac{d}{dx}(c) = 0$.
- Power Rule: If $f(x) = x^n$, n is any real number. Then $f'(x) = nx^{n-1}$.
- Constant multiple rule: $[cf(x)]' = cf'(x)$.

Example 46 *Let $f(x) = 5$. Then $f'(x) = 0$.*
Let $f(x) = x^5$. Then $f'(x) = 5x^4$.
Let $f(x) = \frac{1}{x^5}$. Then $f'(x) = (x^{-5})' = -5x^{-6}$.
Let $f(x) = \frac{1}{\sqrt{x}}$. Then $f'(x) = (x^{-0.5})' = -0.5x^{-1.5}$.
Let $f(x) = \frac{2}{\sqrt{x}}$. Then $f'(x) = 2\left(\frac{1}{\sqrt{x}}\right)' = -x^{-1.5}$.

- Sum rule and difference rule: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$.

Example 47 *Let $f(x) = 4x^3 + 6x^2 - 23x + 7$.*
1) Find interval(s) such that $f'(x) \leq 1$.
2) Find the equation of the tangent line at $(1, -6)$.

Sol: 1) $f'(x) = 12x^2 + 12x - 23$. Then from $f'(x) \leq 1$ we imply that

$$12x^2 + 12x - 23 \leq 1, \Rightarrow 12x^2 + 12x - 24 \leq 0, \Rightarrow x^2 + x - 2 \leq 0, \Rightarrow (x-1)(x+2) \leq 0.$$

Therefore, $-2 \leq x \leq 1$.

2) Let $y = mx + b$ be the tangent line. Then

$$m = f'(1) = 1, \Rightarrow y = x + b.$$

Sub $(1, -6)$: $-6 = 1 + b, \Rightarrow b = -7, \Rightarrow y = x - 7$.

Example 48 *Find the equation of the tangent line to $f(x) = 2\sqrt{x} - 3$ at $(4, 1)$.*

Sol: $f'(x) = x^{-1/2} \Rightarrow f'(4) = 1/2 \Rightarrow y = 1/2x - 1$.

Example 49 *Find the equation of the line(s) that pass through the point $P(2, 9)$ and are tangent to $f(x) = -x^2 + 2x$. Sketch the graph.*

Sol: $f'(x) = -2x + 2$. Let $(a, f(a))$ be a point on the curve whose tangent line goes through $P(2, 9)$. Then $m = -2a + 2 \Rightarrow$

$$\frac{f(a) - 9}{a - 2} = -2a + 2, \Rightarrow a = -1, 5.$$

When $a = -1$, $m = 4 \Rightarrow y = 4x + 1$;
When $a = 5$, $m = -8 \Rightarrow y = -8x + 25$.

2.4 The product and Quotient rules

- Product rule:
$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).$$
- Quotient rule:
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Example 50 *Let* $f(x) = (\sqrt{x} + x^2)(x^3 + x)$. *Calculate* $f'(4)$.

Sol: By the product rule,

$$f'(x) = (\frac{1}{2\sqrt{x}} + 2x)(x^3 + x) + (\sqrt{x} + x^2)(3x^2 + 1) \Rightarrow f'(4) = (\frac{1}{4} + 8)(68) + 18(49).$$

Example 51 *Let* $f(x) = (x^3 + 4x^2)(x^5 + x + 1)$. *Calculate* $f'(1)$ *and the tangent at* $(1, 15)$.

Sol: By the product rule,

$$f'(x) = (3x^2 + 8x)(x^5 + x + 1) + (x^3 + 4x^2)(5x^4 + 1) \Rightarrow f'(1) = 63 \Rightarrow y = 63x - 48.$$

Example 52 *Let* $f(x) = \frac{\sqrt{x}+x^2}{x^3+x}$. *Calculate* $f'(4)$.

Sol: Use the quotient rule.

Example 53 . *Let* $f(x) = \frac{x^3+4x^2}{x^5+x+1}$. *Calculate* $f'(1)$.

Sol: Use the quotient rule.

Example 54 *At what point(s) on the curve* $y = \frac{x^2-4}{x+1}$ *is the tangent line*

a) parallel to $y = 3x$*?*

b) perpendicular to $y = -0.5x$*?*

Solution: By quotient rule,

$$y' = \frac{(x^2-4)'(x+1) - (x^2-4)(x+1)'}{(x+1)^2} = \frac{2x(x+1) - (x^2-4)1}{(x+1)^2} = \frac{x^2+2x+4}{(x+1)^2}.$$

a) Let $y' = 3 \Rightarrow \frac{x^2+2x+4}{(x+1)^2} = 3 \Rightarrow 2x^2 + 4x - 1 = 0 \Rightarrow x = -1 \pm \frac{\sqrt{6}}{2}$.

b) $y' = -\frac{1}{-0.5} = 2 \Rightarrow \frac{x^2+2x+4}{(x+1)^2} = 2 \Rightarrow x^2 + 2x - 2 = 0 \Rightarrow x = -1 \pm \frac{\sqrt{3}}{2}$.

2.5 The chain rule

- Chain Rule:

$$[f(g(x))]' = f'(g(x))g'(x), \quad \frac{df(g(x))}{dx} = \frac{df(v)}{dv} \cdot \frac{dg(x)}{dx}, v = g(x), \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- General Power Rule:

$$[u(x)^n]' = nu^{n-1}u'(x).$$

Example 55 *Let $f(x) = (x^2 - x - 1)^{100}$. Calculate $f'(x)$.*

Sol: $f'(x) = 100(x^2 - x - 1)^{99}(x^2 - x - 1)' = 100(x^2 - x - 1)^{99}(2x - 1)$.

Example 56 *Let $h(x) = g(f(x))$, where $f'(2) = 3$, $f(2) = 4$, $g'(3) = -5$, $g(4) = 8$, $g'(4) = 7$. Find $h'(2)$.*

Solution: $h'(x) = g'(f(x))f'(x) \Rightarrow h'(2) = g'(f(2))f'(2) = g'(4)(3) = 7(3) = 21$.

Example 57 *Let $y = \sqrt{x + \sqrt{x^2 + x}}$. Calculate y'.*

Solution:

$$y' = \frac{1}{2}\frac{1}{\sqrt{x + \sqrt{x^2 + x}}}(x + \sqrt{x^2 + x})'$$

$$= \frac{1}{2\sqrt{x + \sqrt{x^2 + x}}}\left(1 + \frac{1}{2}\frac{1}{\sqrt{x^2 + x}}(x^2 + x)'\right) = \frac{1}{2\sqrt{x + \sqrt{x^2 + x}}}\left(1 + \frac{2x + 1}{2\sqrt{x^2 + x}}\right)$$

2.7 Implicit differentiation

Implicit Differentiation: Assume $f(x, y) = C$. To find y',

- consider x as an independent variable, y as a dependent variable;
- differentiate both sides with respect to x;
- isolate y'.

Example 58 *Find y' from $y^2 + x^2 = 1$.*

Sol:

$$\frac{d}{dx}(y^2 + x^2) = \frac{d1}{dx}, \Rightarrow 2yy' + 2x = 0, \Rightarrow y' = -\frac{x}{y}.$$

Example 59 *Let*

$$y^2 + x^2 = xy + 3.$$

1) Find the equation of the tangent line to the curve at $(0, \sqrt{3})$.
2) Find all the points on the curve where the tangent line is either horizontal or vertical.

Sol: 1)

$$\frac{d}{dx}(y^2 + x^2) = \frac{d}{dx}(xy + 3), \Rightarrow 2yy' + 2x = y + xy', \Rightarrow y' = \frac{y - 2x}{2y - x}.$$

$$m = y'|_{(0,\sqrt{3})} = 0.5, \Rightarrow y = 0.5x + \sqrt{3}.$$

2) Horizontal tangent line: $y' = 0 \Rightarrow y - 2x = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1, y = 2orx = -1, y = -2$.

Vertical tangent line: $y' = \infty \Rightarrow 2y - x = 0 \Rightarrow y^2 = 1 \Rightarrow y = 1, x = 2ory = -1, x = -2$.

4.3 Derivatives of Exponential Functions

Derivative of exponential function:

$$(e^x)' = e^x, \quad (e^{u(x)})' = e^{u(x)}u'(x).$$

Example 60

$$(e^{4x})' = 4e^{4x}.$$

Example 61 *Show that $e^x \geq 1 + x$ for $x \geq 0$.*

Proof. Let $f(x) = e^x - (1 + x)$. Then $f'(x) = e^x - 1 \geq 0$ when $x \geq 0$. Thus $f(x)$ is increasing when $x \geq 0$. Note that $f(0) = 0$, so $f(x) \geq 0$ for $x \geq 0$.

Example 62 *At what point(s) on the curve $y = x^2e^x - 3e^x$ is the tangent line horizontal?*

Solution: $y' = (2x + x^2 - 3)e^x$. Let $y' = 0$, we imply that $x = 1, -3$.

Example 63 *The catenary is the theoretical shape of a hanging flexible chain or cable when supported at its ends and acted upon by a uniform gravitational force (its own weight) and in equilibrium. Let*

$$y = 30(e^{x/60} + e^{-x/60}), \quad -30 \leq x \leq 30.$$

Find the lowest point.

Solution. $y' = \frac{1}{2}(e^{x/60} - e^{-x/60})$. From $y' = 0$, $x = 0$.

x	$x < 0$	$x > 0$
y'	-	$+$

By the first-derivative test, the minimum point is $(0, 60)$.

4.5 Derivatives of Logarithmic Functions

Derivative of inverse functions

$$\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))}.$$

Some special results:

- Derivatives of log functions:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad (\ln f(x))' = \frac{f'(x)}{f(x)},$$

$$(\log_a |x|)' = \frac{1}{x \ln a}, \quad (\log_a f(x))' = \frac{f'(x)}{f(x) \ln a},$$

$$(a^x)' = a^x(\ln a), \quad [a^{u(x)}]' = a^{u(x)}(\ln a)u'(x).$$

Change of base:

$$\log_a b = \frac{\ln b}{\ln a}.$$

Example 64 *Differentiate* $\ln(x^2 + 1)$.

Example 65 *Differentiate* $y = \ln \frac{x^2+x+5}{(x+1)^2}$.

4.6 Exponential Growth and Decay

Exponential growth/decay is growth in which the rate of growth is proportional to the current size:

$$\frac{dy}{dt} = ky, \Rightarrow$$
$$y = Ce^{kt},$$

where C is the initial value, k is the constant of proportionality.

- Exponential growth: $k > 0$;
- Exponential decay: $k < 0$.
- k is the continuous growing or decaying rate.

Special cases:

- Half-life (exponential decay): The time required for the quantity to be reduced to half. Let H be the half-life, then

$$P(t+H) = \frac{1}{2}P(t) \Rightarrow P(t) = P_0(\frac{1}{2})^{t/H}.$$

- Doubling-time (exponential growth): The time required for the quantity to be doubled. Let D be the doubling time, then

$$P(t+D) = 2P(t) \Rightarrow P(t) = P_0(2)^{t/D}.$$

Example 66 *An air-freshener with 30 grams and evaporates at a continuous rate of 2% a day. Find the quantity $Q(t)$ after t days. What is the half-life?*

Answer: $Q(t) = 30e^{-0.02t}$. $H = 50\ln 2$.
Population: we say that $P(t)$ is an exponential function of t with base a,

$$P(t) = P_0 a^t = P_0 e^{kt}, \quad a > 0, a \neq 1,$$

where P_0 is the initial quantity.

- Population growth: $a > 1$;
- Population decay: $0 < a < 1$.
- $a = P(t+1)/P(t)$.
- k is the exponential growth rate.

Example 67 *The world population since World War II has been growing exponentially at the rate of 1.9% per year. The world population in 1975 was (approximately) 4 billion.*
(a) What will the population of the world be in the year 2000?
(b) When will the world population be 7 billion?

Solution: (a) Let $P(t)$ be the world population (in billions) at time t, in years since 1975. Then $P(0) = 4$. $k = 0.019$. So the exponential model for this problem is

$$P(t) = 4e^{0.019t}.$$

The year 2000 corresponds to $t = 25$, so

$$P(25) = 4e^{0.019(25)} \doteq 6.43.$$

(b)

$$P(t) = 7 = 4e^{0.019t}, \Rightarrow \ln 1.75 = 0.019t, \Rightarrow t = (\ln 1.75)/0.019 \doteq 29.4$$

According to the model, the world population should be 7 billion around the years 1975+29.4.

2.8 Related Rates

Let $y = f(x)$. Then $f'(x)$ measures how fast $f(x)$ is increasing or decreasing.

Solving a Related Rates Problem

Step 1: Identify the changing quantities.

Step 2: Construct an equation that relates the changing quantities.

Step 3: Differentiate both sides of the equation with respect to the third variable.

Step 4: Isolate the required quantity, or substitute the given values in the derived equation you obtained above, and solve for the required quantity.

Example. The area of a circle is growing at a rate of $12cm^2/s$. How fast is the radius growing when the radius equals 10 cm?

Sol:

$$\frac{dA(r)}{dt} = 2\pi r \frac{dr}{dt}.$$

Example. A 13-foot ladder leaning against the wall starts to slide down the wall at a rate of 4 ft/min. How fast is the base of the ladder moving when it is 5 feet from the wall?

3.1 Increasing and Decreasing Functions

Definition 10 *$y = f(x)$ is increasing on an interval I if $f(x_1) \le f(x_2)$ for any $x_1 < x_2, x_1, x_2 \in I$; $y = f(x)$ is decreasing on an interval I if $f(x_1) \ge f(x_2)$ for any $x_1 < x_2, x_1, x_2 \in I$.*

INCREASING/DECREASING TEST (I/D TEST):

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.
- If $f'(x) = 0$ on an interval, then f is a constant on that interval.

Example 68 *Let $f(x) = x^3 - 3x^2$. State all the intervals of increase and decrease.*

Solution:
(a) $f'(x) = 3x^2 - 6x = 3x(x-2)$. Let $f'(x) = 0$. We have $3x(x-2) = 0$, which gives $x = 0, 2$.
(b) Look at the following table

x	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
$f'(x)$	+	-	+
$f(x)$	increase	decrease	increase

Therefore,
The intervals of increase: $-\infty < x < 0$, $2 < x < \infty$.
The intervals of decrease: $0 < x < 2$.

Definition 11 *Critical number: A point p in the domain such that $f'(p) = 0$ or $f'(p)$ undefined is called a critical number, $(p, f(p))$ is a critical point, $f(p)$ is a critical value.*

Example 69 *Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.*

Sol: $f'(x) = \frac{12-8x}{5x^{2/5}}$. $f'(x) = 0 \Rightarrow x = 1.5$; $f'(x)$ undefined at $x = 0$. The critical numbers are 1.5 and 0.

Example 70 *Find the critical numbers of $f(x) = |x|$.*

Sol:
$$f'(x) = \begin{cases} 1, & x > 0; \\ -1, & x < 0. \end{cases}$$
$f'(x)$ does not exit at $x = 0$.

3.2 Extrema and the First-Derivative Test

- Relative extrema: $f(x)$ has a relative minimum at p if $f(p) \leq f(x)$ for points x near p; $f(x)$ has a relative maximum at p if $f(p) \geq f(x)$ for points x near p.
- Absolute (Global) Maximum and Minimum: $f(x)$ has a Global (Absolute) Maximum at p if $f(p) \geq f(x)$ for all x in the domain; $f(x)$ has Absolute (global) Minimum at p if $f(p) \leq f(x)$ for all x in the domain.

First Derivative Test: Let p be a critical number. If f' changes from - to + at p, then f has a local minimum at p; If f' changes from + to - at p, then f has a local maximum at p.

Example 71 *Let $f(x) = x^3 - 3x^2$. Find the local minimum points and all the local maximum points.*

Solution:
(a) $f'(x) = 3x^2 - 6x = 3x(x-2)$. Let $f'(x) = 0$. We have $3x(x-2) = 0$, which gives $x = 0, 2$.
(b) Look at the following table

x	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
$f'(x)$	+	-	+
$f(x)$	increase	decrease	increase

(c) Note that at $x = 0$, $f'(x)$ changes from + to -; at $x = 2$, $f'(x)$ changes from - to +. By the First Derivative Test, $f(x)$ has a local maximum at $x = 0$ and a local minimum at $x = 2$.
FERMAT THEOREM: If f has a relative maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.
CLOSED INTERVAL METHOD: To find a global maximum or minimum for $f(x)$ on a closed interval $[a, b]$:

1. Find all the critical numbers, e.g., x_1, ..., x_n.

2. global minimum $= \min\{f(x_1), ..., f(x_n), f(a), f(b)\}$;

   global maximum $= \max\{f(x_1), ..., f(x_n), f(a), f(b)\}$.

Example 72 *Find the absolute maximum and minimum of the function*

$$f(x) = 2x^3 - 3x^2 - 12x + 7, \quad [-2, 0].$$

Sol: Step 1) $f'(x) = 6x^2 - 6x - 12$, $f'(x) = 0 \Rightarrow x = -1, 2$, $f'(x)$ is defined anywhere. Hence $x = -1$ is the only one critical number in $(-2, 0)$.

Step 2) global minimum $= \min\{f(-2), f(-1), f(0)\} = \min\{3, 14, 7\} = 3$;

global maximum $= \max\{f(-2), f(-1), f(0)\} = \max\{3, 14, 7\} = 14$.

3.3 Concavity and Second-Derivative Test

Definition 12 *(CONCAVITY)If the graph of f lies above all of its tangents on an interval I (f' is increasing on I), it is called concave upward on I. If the graph of f lies below all of its tangents on I (f' is decreasing on I), it is called concave downward on I.*

CONCAVITY TEST: If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition 13 *Point of inflection: If $f(x)$ changes concavity at p, then p is an inflection point, and $f''(p) = 0$ or undefined.*

Example 73 *Let $f(x) = x^3 - 3x^2$. State intervals of concavity and find all the points of inflection.*

Solution: $f''(x) = 6x - 6 = 0 \Rightarrow x = 1$.

x	$-\infty < x < 1$	$1 < x < \infty$
$f''(x)$	-	$+$
$f(x)$	concave down	concave up

Concave up: $1 < x < \infty$; Concave down: $-\infty < x < 1$.

Since $f(x)$ changes concavity at $x = 1$, $x = 1$ is a point of inflection.

Example 74 *Consider the function*

$$f(x) = \frac{x}{x^2 - 1}.$$

Study the concavity and find all the points of inflection.

Solution: The domain of the function: $x \neq \pm 1$.

$$f'(x) = \frac{-1 - x^2}{(x^2 - 1)^2}, \quad f'' = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

$$f''(x) = 0, \Rightarrow x = 0.$$

x	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
$f''(x)$	-	+	-	+
$f(x)$	concave down	concave up	concave down	concave up

Second Derivative Test: Let p be a critical number.

- If $f''(p) > 0$, then f has a local minimum at p;
- If $f''(p) < 0$, then f has a local maximum at p;
- If $f''(p) = 0$, then nothing.

Example 75 *Let* $f(x) = x^3 - 3x^2 - 80$*. Find all the local minimum points and all the local maximum points by Second Derivative Test.*

Sol: From $f'(x) = 3x^2 - 6x$ we get $f''(x) = 6x - 6$. $f'(x) = 0 \Rightarrow x = 0, 2$. Note that $f''(0) = -6 < 0$ $f''(2) = 6 > 0$ By the Second Derivative Test, $f(x)$ has a local maximum at $x = 0$ and a local minimum at $x = 2$.

3.4 Optimization Problems

GUIDELINES FOR SOLVING MAX./MIN. PROBLEMS

1. If possible, draw a sketch or diagram of the problem to be solved. Pictures are a great help in organizing and sorting out your thoughts.

2. Define variables to be used and carefully label your picture or diagram with these variables. This step is very important because it leads directly or indirectly to the creation of mathematical equations.

3. Write down all equations which are related to your problem or diagram. Clearly denote that equation which you are asked to maximize or minimize. Experience will show you that MOST optimization problems will begin with two equations. One equation is a "constraint" equation and the other is the "optimization" equation. The "constraint" equation is used to solve for one of the variables. This is then substituted into the "optimization" equation before differentiation occurs. Some problems may have NO constraint equation. Some problems may have two or more constraint equations.

4. Change your equations to a function of only one variable. Then differentiate the function.

5. Verify that your result is a maximum or minimum value using the first or second derivative test for extrema.

Example 76 *Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is a maximum.*

Solution: Let variables x and y represent two nonnegative numbers. We wish to MAXIMIZE the PRODUCT

$$P = xy^2 \tag{1}$$

subject to

$$x + y = 9. \tag{2}$$

Substitute (2) into (1),

$$P(x) = x(9-x)^2, \quad 0 \le x \le 9.$$

Now differentiate this equation using the product rule and chain rule, getting

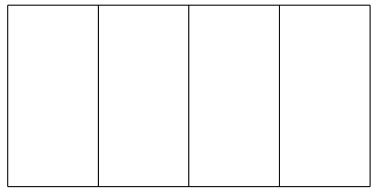
$$P'(x) = 3(9-x)((3-x). \quad P'(x) = 0 \Rightarrow x = 3, 9.$$

Note that

$$P(0) = 0, P(9) = 0, P(3) = 108.$$

When $x = 3$, $y = 6$, we have the maximum product 108.

Example 77 *Build a rectangular garden with three parallel partitions using 500 feet of fencing (See the graph). What dimensions will maximize the total area of the garden?*



Solution: Let variables x and y represent the width and the length of the garden. We wish to MAXIMIZE the Area

$$A = xy \tag{3}$$

subject to

$$5x + 2y = 500. \tag{4}$$

Substitute (4) into (3),

$$A(x) = 250x - 2.5x^2, \quad 0 \le x \le 100.$$

Now differentiate this equation, getting

$$A'(x) = 250 - 5x. \quad A'(x) = 0 \Rightarrow x = 50.$$

Note that

$$A(0) = 0, A(100) = 0, A(50) = 6250.$$

When $x = 50$, $y = 125$, we have the maximum area 6250.

Example 78 *Find the point(s)* (x, y) *on the graph of* $y = \sqrt{x}$ *nearest the point* $(4, 0)$*.*

Solution: We wish to MINIMIZE the distance

$$D = \sqrt{(x-4)^2 + (y-0)^2} \tag{5}$$

subject to

$$y = \sqrt{x}. \tag{6}$$

Substitute (6) into (5),

$$D(x) = \sqrt{(x-4)^2 + x} = \sqrt{x^2 - 7x + 16}, \quad x \ge 0.$$

Now differentiate this equation, getting

$$D'(x) = \frac{2x - 7}{2\sqrt{x^2 - 7x + 16}}. \quad D'(x) = 0 \Rightarrow x = 3.5.$$

Note that

$$D(0) = 4, D(3.5) = \sqrt{7.5}.$$

When $x = 3.5$, $y = \sqrt{3.5}$, we have the minimum distance $\sqrt{7.5}$..

3.5 Business and Economics Applications

- Average cost: let C be the total cost, and x be the number of units produced, then

$$\bar{C} = \frac{C}{x}.$$

- Price elasticity of demand
  - Price elasticity of demand (PED or Ed) is a measure used in economics to show the responsiveness, or elasticity, of the quantity demanded to a change in its price. More precisely, it gives the percentage change in quantity demanded in response to a one percent change in price.
  - Price elasticities are almost always negative.
  - In determining whether to increase or decrease prices, a firm needs to know what the net effect will be. Elasticity provides the answer: The percentage change in total revenue is equal to the percentage change in quantity demanded plus the percentage change in price. (One change will be positive, the other negative.)
  - Let x be quantity demanded, $p(x)$ be the price. Then

$$\eta := \frac{\text{rate of change in demand}}{\text{rate of change in price}} = \frac{\Delta x/x}{\Delta p(x)/p(x)} \approx \frac{\frac{p(x)}{x}}{p'(x)}.$$

  - For a given price, the demand is elastic if $|\eta| > 1$, when the price is raised, the total revenue falls, and vice versa; The demand is inelastic if $|\eta| < 1$, when the price is raised, the total revenue rises, and vice versa; the demand has unit elasticity if $|\eta| = 1$, the percentage change in quantity is equal to that in price, so a change in price will not affect total revenue.

- The quantity effect: an increase in unit price will tend to lead to fewer units sold, while a decrease in unit price will tend to lead to more units sold.

- Not to be confused with Price elasticity of supply.

Example 79 *A store has been selling Häagen-Dazs bar at the price of \$3.00 per bar and, at this price, the store can sell 60 bars per day. If the store raises its price, it will sell 2 fewer bars per day for each \$0.4 increase in price. Assume that x bars cost the store* $0.1x^2 + 0.4$.

a. Find the demand equation (p as a function of quantity x)

b. Find the minimum average cost.

c. Find the maximum revenue.

d. Find the maximum profit.

e. Find the intervals on which the demand is elastic, inelastic, and of unit elasticity.

f. Use e to describe the behavior of revenue

Solution: a. $x = 60 - 2\left(\frac{p-3.00}{0.4}\right), \Rightarrow x = 60 - 5p + 15, \Rightarrow p(x) = 15 - 0.2x. \quad 0 \le x \le 75.$

b. $C(x) = 0.1x^2 + 0.4. \Rightarrow \bar{C} = \frac{C}{x} = 0.1x + \frac{0.4}{x}$. $\bar{C}'(x) = 0.1 - \frac{0.4}{x^2}$. $\bar{C}'(x) = 0 \Rightarrow x = \pm 2$. So $x = 2$. $\bar{C}''(2) = 100 > 0 \Rightarrow$ we have minimum average cost 0.8 when $x = 2$.

c. $R(x) = xp(x) = 15x - 0.2x^2$. $R'(x) = 0 \Rightarrow x = \frac{15}{0.4} = 37.5$. Maximum revenue is $R(37.5)$.

d. $P(x) = R(x) - C(x) = 15x - 0.3x^2 - 0.4$. $P'(x) = 0 \Rightarrow x = 25$. Maximum profit is $P(25)$.

e.

$$\eta = \frac{\frac{p(x)}{x}}{p'(x)} = \frac{\frac{15}{x} - 0.2}{-0.2} = -\frac{75}{x} + 1.$$

$$|\eta| = 1 \Rightarrow x = 37.5, \quad |\eta| > 1 \Rightarrow x < 37.5, |\eta| < 1 \Rightarrow x > 37.5.$$

f. The revenue is increasing in $(0, 37.5)$ and decreasing in $(37.5, 75)$.

3.6 Asymptotes

Definition 14 *The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:*

$$\lim_{x\to a^-} f(x) = \pm\infty, \lim_{x\to a^+} f(x) = \pm\infty, \lim_{x\to a} f(x) = \pm\infty.$$

Example 80 $\lim_{x\to 0} \frac{1}{x^2} = \infty$, $\lim_{x\to 1} \frac{-1}{(x-1)^2} = -\infty$.

Example 81 *Verify VA from graph, sketch graph.*

Definition 15 *If $\lim_{x\to\pm\infty} f(x) = L$, then $y = L$ is a HA (Horizontal Asymptote).*

Example 82 *Find HA:*

$$f(x) = \frac{2x^2 - 1}{x^2}, \quad \frac{P(x)}{Q(x)}, \quad \frac{\sqrt{x^2+1}}{x}.$$

Example 83 *Verify HA from graph, sketch graph.*

3.7 Curve Sketching

The following checklist is intended as a guide to sketching a curve y = f(x) by hand.

Not every item is relevant to every function. For instance, a given curve might not have an asymptote or possess symmetry. However, the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

- A. DOMAIN
- B. INTERCEPTS
- C. SYMMETRY
  1. EVEN FUNCTION: f(-x) = f(x) for all x in D. the curve is symmetric about the y-axis. This means that our work is cut in half.
  2. ODD FUNCTION: f(-x) = -f(x) for all x in D. the curve is symmetric about the origin. This means that our work is cut in half.
- D. ASYMPTOTES
  - HORIZONTAL: $\lim_{x\to\pm\infty} f(x) = L$, then $y = L$ is a HA.
  - VERTICAL: $\lim_{x\to a^{\pm}} f(x) = \pm\infty$, then $x = a$ is a VA.
- E. INTERVALS OF INCREASE OR DECREASE: use I/D Test.
- F. LOCAL MAXIMUM AND MINIMUM VALUES: First Derivative Test or Second Derivative Test.
- G. CONCAVITY AND POINTS OF INFLECTION

Example 84 *Sketch the graph*

$$f(x) = \frac{x}{x^2 - 1}.$$

Solution:

A. The domain of the function: $x \neq \pm 1$.

B. x-intercept: $y = 0 \Rightarrow x = 0$; y-intercept: $x = 0 \Rightarrow y = 0$.

D. $\lim\limits_{x\to\pm\infty} f(x) = 0$, $y = 0$ is HA; VA: let $x^2 - 1 = 0$, we have $x = 1$ and $x = -1$.

E.

$$f'(x) = \frac{-1 - x^2}{(x^2 - 1)^2}, \quad f'' = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

Note that $f'(x) < 0$ for any x, so f is decreasing in the domain.

G.

$$f''(x) = 0, \Rightarrow x = 0.$$

x	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
$f''(x)$	-	+	-	+
$f(x)$	concave down	concave up	concave down	concave up

Example 85 *Consider the function*

$$f(x) = \left[x^2(6-x)\right]^{1/3}, \qquad -\infty < x < \infty.$$

a) Find f' and f''.

b) Find all the critical numbers of $f(x)$.

c) Find all the intervals of increasing and decreasing, and classify all the critical numbers as local maxima, minima, or neither.

d) Study the concavity and find the point(s) of inflection.

e) Sketch the graph.

Solution: a)

$$f'(x) = \frac{4-x}{\sqrt[3]{x(6-x)^2}}, \quad f''(x) = \frac{-8}{\sqrt[3]{x^4(6-x)^5}}.$$

b)

Sol: From $f'(x) = 0$ we get $x = 4$;

From $f'(x)$ undefined we imply that $x = 0, 6$.

So we have three critical numbers $x = 4, 0, 6$.

c) Look at the following table

x	$-\infty < x < 0$	$0 < x < 4$	$4 < x < 6$	$6 < x < \infty$
$f'(x)$	-	+	-	-
$f(x)$	decreasing	increasing	decreasing	decreasing

By the First Derivative Test, $f(0) = 0$ is a local minimum, $f(4) = \sqrt[3]{32}$ is a local maximum, $f(6)$ is neither.

d) Note that $f''(x)$ undefined at $x = 0, 6$. Look at the following table

x	$-\infty < x < 0$	$0 < x < 6$	$6 < x < \infty$
$f''(x)$	-	-	+
$f(x)$	concave down	concave down	concave up

$x = 6$ or $(6, f(6))$ is a point of inflection.

e) Sketch the graph. (Remark. $f(x)$ has no vertical and horizontal asymptotes).

5.1 Antiderivatives and Indefinite Integral

Definition 16 *A function $F(x)$ is called an antiderivative of $f(x)$ on an interval I if $F'(x) = f(x)$ for all x in I. The antiderivative of f is denoted by $\int f(x)dx$, which is called an indefinite integral.*

Some basic results:

function	antiderivative	formula
k	$kx + C$	$\int kdx = kx + C$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1} + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C;\ (n \neq -1)$
		$\int kf(x)dx = k \int f(x)dx$
		$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx.$

Example 86 *Calculate*

$$\int \frac{3x^5 - 5x^7}{x^3} dx.$$

Solution:

$$\int \frac{3x^5 - 10x^7}{x^3} dx = \int (3x^2 - 10x^4)\, dx = x^3 - 2x^5 + C.$$

Example 87 *An object moves along a coordinate line with velocity*

$$v(t) = 2 - 3t + t^2 \quad units/s.$$

Its initial position is 2 units to the right of the origin(when t=0). Find the position of the object.

Sol: Let $s(t)$ be the position. Then $s(0) = 2$. Since $s'(t) = v(t)$,

$$s(t) = \int s'(t)dt = \int v(t)dt = 2t - \frac{3}{2}t^2 + \frac{1}{3}t^4 + C.$$

$s(0) = 2 \Rightarrow C = 2$ and

$$s(t) == 2t - \frac{3}{2}t^2 + \frac{1}{3}t^4 + 2. \Rightarrow s(4) = 7\frac{1}{3}.$$

5.2 Integration by Substitution and the General Power Rule

General Power Rule: If $u(x)$ is differentiable, then

$$\int u^n u' dx = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

Substitution (or Change of variables): If $u(x)$ is differentiable, then

$$\int u^n u' dx = \int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

Guidelines of Substitution for $\int f(x)dx$::

1. Come up with a substitution $u = u(x)$.

2. Ideally you may want to find the inverse function of $u(x)$, meaning that you will find $x = x(u)$.

3. Differentiate to find $dx = x'(u)du$.

4. Back to the given integral and make the appropriate substitutions

$$\int f(x)dx = \int f(x(u))x'(u)du,$$

5. Check after algebraic simplifications that the new integral is easier than the initial one. Otherwise, come up with another substitution.

6. Do not forget that the answer is a function of x. Therefore once you have finished doing all your calculations, you should substitute back to the initial variable x.

Example 88 *Evaluate*

$$\int (2x-1)(x^2-x)^{100}\, dx.$$

Solution: Let $u = x^2 - x$. Then $du = (2x-1)dx$. Thus

$$\int (2x-1)(x^2-x)^{100}\, dx = \int u^{100} du = \frac{u^{101}}{101} + C = \frac{(x^2-x)^{101}}{101} + C.$$

Example 89

$$\int x\sqrt{x^2+1}\, dx.$$

Solution: Let $u = x^2 + 1$. Then $du = (2x)dx$. Thus

$$\int x\sqrt{x^2+1}\,dx = \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\frac{u^{3/2}}{3/2} + C = \frac{(x^2+1)^{3/2}}{3} + C.$$

Marginal Propensity to Consume: = The rate of change of consumption with respect to income. Let x be the income, $Q(x)$ be the income consumed. Then

$$\textbf{Marginal Propensity to Consume} = \frac{dQ}{dx}.$$

The marginal propensity to consume (MPC) indicates what the household sector does with extra income.

Example 90 *For a family of four in 2005, the marginal propensity to consume income x can be modeled by*

$$\frac{dQ}{dx} = \frac{0.98}{(x-19999)^{0.02}}, \qquad x \geq 20000, Q(20000) = 20000.$$

Would the family consumed more than 30000 if the family income is 33000?

Solution:

$$Q = \int \frac{dQ}{dx}dx = \int \frac{0.98}{(x-19999)^{0.02}}dx = (x-19999)^{0.98} + C.$$

$$Q(20000) = 20000 \Rightarrow C = 19999 \Rightarrow Q = (x-19999)^{0.98} + 19999 \Rightarrow Q(33000) = 30756.$$

5.3 Integral of Exponential and Logarithm

function	antiderivative	formula
e^x	$e^x + C$	$\int e^x dx = e^x + C$
$e^u u'$	$e^u + C$	$\int e^u u' dx = e^u + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
$\frac{u'}{u}$	$\ln\lvert u\rvert + C$	$\int \frac{u'}{u}dx = \int \frac{1}{u}du = \ln\lvert u\rvert + C$

Example 91

$$\int (3e^x + \sqrt{x})dx = \int 3e^x dx + \int x^{1/2}dx = 3e^x + \frac{2}{3}x^{3/2} + C.$$

Example 92 *Find*

$$\int x^2 e^{2x^3} dx.$$

Solution: Let $u = 2x^3$. Then $du = 6x^2 dx \Rightarrow \frac{1}{6} du = x^2 dx$.

$$\int x^2 e^{2x^3} dx = \int e^u \frac{1}{6} du = \frac{1}{6} e^u + C = \frac{1}{6} e^{2x^3} + C.$$

Example 93 *Find*

$$\int \frac{1}{2-3x} dx.$$

Solution: Let $u = 2 - 3x$. Then $du = -3dx \Rightarrow \frac{1}{-3} du = dx$.

$$\int \frac{1}{2-3x} dx = \int \frac{1}{u} (\frac{1}{-3}) du = -\frac{1}{3} \int \frac{1}{u} du = -\frac{1}{3} \ln |2 - 3x| + C.$$

Example 94 *Find*

$$\int \frac{2x^2+4x+2}{x^2+1} dx.$$

Solution:

$$\int \frac{2x^2+4x+2}{x^2+1} dx = \int \left(2 + \frac{4x}{x^2+1} \right) dx = 2x + \int \frac{4x}{x^2+1} dx$$

$$= 2x + \int \frac{2}{u} du = 2x + \ln |u| + C = 2x + \ln |x^2+1| + C = 2x + \ln(x^2+1) + C,$$

where $u = x^2 + 1$, then $du = 2xdx$.

Example 95 *Find*

$$\int \frac{1}{e^{-x}+1} dx.$$

Solution:

$$\int \frac{1}{e^{-x}+1} dx = \int \frac{e^x}{1+e^x} dx.$$

Let $u = 1 + e^x$, then $du = e^x dx$.

$$\int \frac{e^x}{1+e^x} dx = \int \frac{1}{u} du = \ln |u| + C = \ln |1+e^x| + C = \ln(1+e^x) + C.$$

5.4 Area and Fundamental Theorem of calculus

Definition 17 *The area of the region bounded by the curves $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$ is:*

$$Area = \int_a^b f(x)dx,$$

where $f(x)$ is nonnegative and continuous on $[a, b]$. $\int_a^b f(x)dx$ is called definite integral of f from the lower limit a to the upper limit b.

The Fundamental Theorem of Calculus: If $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b.$$

Some basic properties about definite integral:

- $\int_a^b c dx = c(b - a)$;
- $\int_a^a f(x)dx = 0$;
- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$;
- $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$;
- Constant multiple: $\int_a^b cf(x)dx = c\int_a^b f(x)dx$;
- $\int_a^a f(x)dx = -\int_b^a f(x)dx$.

Example 96 *Let $\int_1^5 f(x)\,dx = 3$, $\int_1^5 g(x)\,dx = 5$. Calculate $\int_1^5 [2f(x) - g(x) - 1]\,dx$.*

Answer:

$$\int_1^5 [2f(x) - g(x) - 1]\,dx = 2\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx - \int_1^5 1\,dx$$
$$= 2(3) - 5 - 1(5 - 1) = -3.$$

Example 97 *Find the area of the region bounded by the x-axis and the graph of*

$$f(x) = x^2 - 1, \quad 2 \leq x \leq 3.$$

Solution:

$$Area = \int_a^b f(x)dx = \int_2^3 (x^2-1)dx = (\frac{1}{3}x^3 - x)|_2^3 = (\frac{1}{3}(3^3)-3) - (\frac{1}{3}(2^3)-2) = \frac{16}{3}.$$

Example. Calculate $\int_0^3 f(x)dx$, where

$$f(x) = \begin{cases} x, & 0 \le x \le 1; \\ 1, & 1 \le x \le 2; \\ 3-x, & 2 \le x \le 3. \end{cases}$$

Solution:

$$\int_0^3 f(x)dx = \text{the area of the region bounded by the } x\text{-axis and the graph of } f(x)$$

$$= \frac{(1+3)1}{2} = 2.$$

Example 98 *Calculate* $\int_0^2 e^{(\ln 3)t}dt$.

Solution: Let $f(t) = e^{(\ln 3)t}$, then $F(t) = \frac{1}{\ln 3}e^{(\ln 3)t} + C$.

$$\int_0^2 e^{(\ln 3)t}dt = F(2) - F(0) = \frac{8}{\ln 3}.$$

Example 99 *Calculate* $\int_0^3 |2x-4|dx$.

Solution: Note that

$$|2x-4| = \begin{cases} -(2x-4), & x < 2; \\ 2x-4, & x \ge 2. \end{cases}$$

$$\int_0^3 |2x-4|dx = \int_0^2 -(2x-4)dx + \int_2^3 (2x-4)dx = (-x^2+4x)|_0^2 + (x^2-4x)|_2^3 = 5.$$

Average value of $f(x)$ from a to b =

$$\frac{1}{b-a}\int_a^b f(x)dx.$$

Example 100 *Find the average value of* x^3 *over* $[0,2]$.

Solution:

$$\text{the average} = \frac{1}{2-0}\int_0^2 x^3 dx = 2.$$

Even and odd functions: If $f(-x) = -f(x)$ for all x, then $f(x)$ is odd; if $f(-x) = f(x)$ for all x, then $f(x)$ is even.

- If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$;
- If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.

Example 101 $\int_{-2}^{2} xe^{5x^4+x^2}dx = 0$.

Annuity: An annuity is a type of plan in which the same amount is invested at a time interval and the interest rate remains fixed. If $c(t)$ is income function, t is time, r is continuous interest rate, T is the term of annuity, then

$$\text{amount of annuity} = e^{rT}\int_0^T c(t)e^{-rt}dt.$$

Example 102 *You deposit 2000 each year for 15 years in an individual retirement account paying 5% interest. How much will you have after 15 years?*

Solution:

$$\text{amount of annuity} = e^{rT}\int_0^T c(t)e^{-rt}dt = e^{(0.05)15}\int_0^{15} 2000e^{-0.05t}dt \doteq 44680.$$

5.5 Area between two Graphs

Area of a region bounded by graphs. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then the area of the region bounded by

$$y = f(x), y = g(x), x = a, x = b$$

is

$$\int_a^b (f(x) - g(x))dx.$$

Example 103 *Calculate the area of the region bounded by*

$$y = x^2 - 4x + 7, y = -x^2 + 4x + 1, x = 0, x = 2.$$

Solution: Step 1. Find intersections: Let $(-x^2+4x+1) = (x^2-4x+7), \Rightarrow x^2-4x+3 = 0, \Rightarrow x = 1, x = 3$.

Step 2. By using the intersections, the interval $(0, 2)$ is divided into $(0, 1)$ and $(1, 2)$. In $(0, 1)$: $x^2 - 4x + 7 > -x^2 + 4x + 1$; In $(1, 2)$: $-x^2 + 4x + 1 > x^2 - 4x + 7$. Therefore

$$area = \int_0^1 [(x^2 - 4x + 7) - (-x^2 + 4x + 1)]dx + \int_1^2 [(-x^2 + 4x + 1) - (x^2 - 4x + 7)]dx$$

$$= \int_0^1 (2x^2 - 8x + 6)dx + \int_1^2 (-2x^2 + 8x - 6)dx.$$

Example 104 *Find the area of the region between $y = x^{1/2}$ and $y = x^{1/3}$ for $0 \le x \le 1$.*

Example 105 *Calculate the area of the region bounded by*

$$y = 2x, y = 8x^3.$$

Solution: Step 1. Find intersections: Let $2x = 8x^3, \Rightarrow x = -0.5, 0, 0.5$.

Step 2. By using the intersections, we get two intervals $(-0.5, 0)$ and $(0, 0.5)$. In $(-0.5, 0)$: $8x^3 > 2x$; In $(0, 0.5)$: $2x > 8x^3$. Therefore

$$area = \int_{-0.5}^0 (8x^3 - 2x)dx + \int_0^{0.5} (2x - 8x^3)dx = (2x^4 - x^2)|_{-0.5}^0 + (x^2 - 2x^4)|_0^{0.5} = 0.25$$

Consumer Surplus and Producer Surplus: Consumer surplus measures the welfare that consumers derive from their consumption of goods and services, or the benefits they derive from the exchange of goods. Consumer surplus is the difference between what consumers are willing to pay for a good or service (indicated by the position of the demand curve) and what they actually pay (the market price).

Producer surplus is a measure of producer welfare. It is the difference between what producers are willing and able to supply a good for and the price they actually receive.

Let $p = D(x)$ be a demand function, and $p = S(x)$ be a supply function, $p_0 = D(x_0) = S(x_0)$ be the equilibrium point. Then

- The Consumer Surplus = The area of the region bounded by the graph of the demand function $p = D(x)$, the horizontal line $p = p_0$, and the vertical line $x = 0$:

$$\text{Consumer Surplus} = \int_0^{x_0} [D(x) - p_0]dx;$$

- The Producer Surplus = The area of the region bounded by the graph of the supply function $p = S(x)$, the horizontal line $p = p_0$, and the vertical line $x = 0$:

$$\text{Producer Surplus} = \int_0^{x_0} [p_0 - S(x)]dx.$$

Example 106 *Let Demand:* $p = -0.36x + 9$*, Supply:* $p = 0.14x + 2$*. Find the consumer and supply surpluses.*

$-0.36x + 9 = 0.14x + 2 \Rightarrow x = 14, p_0 = -0.36(14) + 9 = 3.96$. Solution:

$$\text{Consumer Surplus} = \int_0^{x_0} [D(x) - p_0]dx = \int_0^{14} [-0.36x + 9 - 3.96]dx = 35.28;$$

$$\text{Producer Surplus} = \int_0^{x_0} [p_0 - S(x)]dx = \int_0^{14} [3.96 - (0.14x + 2)]dx = 13.72.$$

6.1 Integration By Parts and Present Value

Recall the product rule:

$$(uv)' = u'v + uv'$$

or

$$uv' = (uv)' - u'v.$$

Integrating both sides, we have that
Integration by parts:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx.$$

When to Use Integration By Parts:

1. When U-substitution does not work
2. When there is a mix of two types of functions such as an exponential and polynomial, polynomial and log, etc.
3. With $\ln x$.
4. When all else fails.

Example 107 *Evaluate*

$$\int x^2 e^x \, dx.$$

Solution: Integration by parts

$$\int x^2 e^x \, dx = \int x^2 \, de^x = x^2 e^x - \int (2x) e^x \, dx$$

$$= x^2 e^x - 2 \int x \, de^x = x^2 e^x - 2xe^x + 2 \int e^x \, dx$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

Example 108 *Evaluate*

$$\int_0^1 x^2 e^x \, dx.$$

Solution: Since

$$\int x^2 e^x\, dx = x^2 e^x - 2xe^x + 2e^x + C,$$

$$\int_0^1 x^2 e^x\, dx = (x^2 e^x - 2xe^x + 2e^x)|_0^1 = e - 2.$$

Example 109 *Evaluate*

$$\int 4x^3 \ln x\, dx.$$

Solution: Integration by parts

$$\int 4x^3 \ln x\, dx = \int \ln x\, dx^4 = x^4 \ln x - \int x^4\, d(\ln x)$$

$$x^4 \ln x - \int x^4 \frac{dx}{x} = x^4 \ln x - \int x^3\, dx$$

$$= x^4 \ln x - \frac{1}{4} x^4 + C$$

Present Value Formula

If $c(t)$ is the continuous annual income over T years with an inflation rate r, then the present value can by found by

$$\int_0^T c(t) e^{-rt} dt.$$

Example 110 *Your B-Bond brings you an annual income of* $2000t$ *dollars where* t *is the number of years since the bond begins. The bond will expire in 20 years. A business has offered to purchase the bond from you. How much should you ask for it? Assume an inflation rate of* 5%.

Solution: This question is a present value problem. Since there is inflation, your later earnings will be worth less than this year's earnings. The formula to determine this is given by

$$PV = \int_0^{20} 2000te^{-0.05t} dt.$$

Let $u = -0.05t$, then $du = -0.05dt$; when $t = 0$, $u = 0$, $t = 20$, $u = -1$. By U-substitution, we get

$$PV = \int_0^{-1} \frac{2000}{(-0.05)(-0.05)} ue^u du = \int_0^{-1} 800000ue^u du$$

$$= 800000(ue^u - e^u)_0^{-1} = 211393.$$

6.4 Improper Integrals

6.4.1 Type I (Infinite Interval)

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx, \quad \int_{-\infty}^b f(x)dx = \lim_{t\to\infty}\int_t^b f(x)dx,$$
$$\int_{-\infty}^\infty f(x)dx = \int_c^\infty f(x)dx + \int_{-\infty}^c f(x)dx.$$

Definition 18 *Integral is convergent* $\Leftrightarrow$ *Integral is a finite number.*

Example 111

$$\int_1^\infty \frac{1}{x^2}dx = (-\frac{1}{x})|_1^\infty = 1.$$

Example 112

$$\int_1^\infty \frac{1}{x}dx = (\ln x)|_1^\infty = \infty.$$

Example 113

$$\int_{-\infty}^\infty e^{-|x|}dx = \int_{-\infty}^0 e^{-|x|}dx + \int_0^\infty e^{-|x|}dx = \int_{-\infty}^0 e^{-(-x)}dx + \int_0^\infty e^{-x}dx = 2.$$

6.4.2 Type 2 (Discontinuous Integrand)

If $f(x)$ is continuous on $[a,b)$, then

$$\int_a^b f(x)dx = \lim_{t\to b}\int_a^t f(x)dx;$$

If $f(x)$ is continuous on $(a,b]$, then

$$\int_a^b f(x)dx = \lim_{t\to a}\int_a^b f(x)dx;$$

If $f(x)$ is **discontinuous** at c: $a < c < b$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Example 114

$$\int_2^3 \frac{1}{\sqrt{3-x}}dx = (-2\sqrt{3-x})|_2^3 = 2.$$

Example 115

$$\int_0^2 \frac{1}{x-1}dx = \int_0^1 \frac{1}{x-1}dx + \int_1^2 \frac{1}{x-1}dx = -\infty + \infty = \mathit{divergent}.$$

7.1 The 3D Coordinate System

We will use Right-Hand System.

- Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$Distance = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

- Midpoint of the two points (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$Midpoint = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

- The sphere with center (h, k, l) and radius r is:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$

Example 116 *Find an expression for a sphere of radius 5 with center at* $(1, 2, 3)$.

Solution: $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 25$.

7.3 Functions of Several Variables

Functions of Two variables:

Definition 19 *Let D be a set of ordered pairs. If for each $(x, y) \in D$, there exists a unique $f(x, y)$, then f is called a function of the two variables x and y. D is the domain,range $R = \{f(x, y) : (x, y) \in D\}$. We write $z = f(x, y)$. The graph of the function $z = f(x, y)$ is the set*

$$\{(x, y, z) : z = f(x, y), (x, y) \in D\}.$$

Similarly we can define function of several variables.

Example 117 *Find the domain and range of $f(x, y) = \sqrt{9 - x^2 - y^2}$.*

Contour map and level curve: Level curves of $z = f(x, y)$: $f(x, y) = c$ **for different** c**.** A contour map consists of level curves with equal space.

Example 118 *Find three level curves of $f(x, y) = \sqrt{9 - x^2 - y^2}$.*

Solution: Let $c = 0, 1, 2$. We have $x^2 + y^2 = 9, 8, 5$.

7.4 Partial Derivatives

- Partial derivatives in two variables: Let $z = f(x, y)$, then

$$\frac{\partial z}{\partial x} := \frac{\partial f}{\partial x} := f_x(x, y) := D_x f := \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

  which is the derivative of f with respect to x.

$$\frac{\partial z}{\partial y} := \frac{\partial f}{\partial y} := f_y(x, y) := D_y f := \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h},$$

  which is the derivative of f with respect to y.

- Meaning: f_x means the rate of change of f with respect to x when y is fixed.

Computing partial derivatives algebraically

Strategy: To calculate $f_x(x, y)$, we consider y as constant, then $f_x(x, y)$ is the ordinary derivative of f to x. Similarly to y.

Example 119 *Find f_x and f_y: $z = f(x, y) = xe^{2xy^3} + x^2$.*

Solution: By product rule and chain rule,

$$f_x(x, y) = e^{2xy^3} + 2xy^3e^{2xy^3} + 2x, \quad f_y(x, y) = 6x^2y^2e^{2xy^3} + 0.$$

Example 120 *Find f_x, f_y : $w = f(x, y) = \sqrt{x^2 + y^2}$.*

Solution: By power rule and chain rule,

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}, \quad f_y(x, y, z) = \frac{y}{\sqrt{x^2 + y^2}}.$$

Higher-order partial derivatives:

- $f_{xx} = (f_x)_x$, $f_{xy} = (f_x)_y = f_{yx} = (f_y)_x$, $f_{yy} = (f_y)_y$.
- $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}, \cdots$

Example 121 *Find f_{xy} and f_{yy}: $z = f(x, y) = 2x^3y^2 + x^2$.*

Solution: By product rule,

$$f_x(x, y) = 6x^2y^2 + 2x, \Rightarrow f_{xy}(x, y) = 12xy^2 + 2.$$

$$f_y(x, y) = 4x^3y, \Rightarrow f_{yy}(x, y) = 4x^3.$$

Example 122 *Find* $f_{xy}(1, \ln 2)$*:* $z = f(x, y) = x^2 e^{xy}$.

Solution: By product rule,

$$f_x(x, y) = 2xe^{xy} + x^2 ye^{xy}, \Rightarrow f_{xy}(x, y) = 3x^2 e^{xy} + x^3 e^{xy}.$$

$$f_{xy}(1, \ln 2) = 6 + 2 \ln 2.$$

Example 123 *Find* f_x*:* $z = f(x, y) = x^3 e^{xy^2} + x \ln(2x + y^3)$.

Solution: By product rule and chain rule,

$$f_x(x, y) = 3x^2 e^{xy^2} + x^3 y^2 e^{xy^2} + \ln(2x + y^3) + x \frac{2}{2x + y^3}$$

$$= 3x^2 e^{xy^2} + x^3 y^2 e^{xy^2} + \ln(2x + y^3) + \frac{2x}{2x + y^3}.$$

7.5 Extrema of Functions of Two Variables

Definition 20 *We say that a function* $f(x, y)$ *has a relative (local) maximum at a point* (x_0, y_0) *if there is a circle centered at* (x_0, y_0) *such that*

$$f(x, y) \leq f(x_0, y_0)$$

for all (x, y) *in that circle;* $f(x, y)$ *has a relative (local) minimum at a point* (x_0, y_0) *if there is a circle centered at* (x_0, y_0) *such that*

$$f(x, y) \geq f(x_0, y_0)$$

for all (x, y) *in that circle.*

Definition 21 *The critical points of a function* $f(x, y)$ *are those points* (x_0, y_0) *for which* $f_x(x_0, y_0) = 0$ *and* $f_y(x_0, y_0) = 0$*, or if* $f_x(x_0, y_0)$ *or* $f_y(x_0, y_0)$ *is undefined.*

Example 124 *Find the critical point(s) of* $f(x, y) = x^3 - 3xy + y^3$

Solution: The first partial derivatives are $f_x(x,y) = 3x^2 - 3y,\ f_y(x,y) = -3x + 3y^2$. Setting $f_x = 0$ and $f_y = 0:\ 3x^2-3y = 0,\ -3x+3y^2 = 0$. We imply that $(x,y) = (0,0),(1,1)$.

First-Partials Test for Relative Extrema: If f has a relative extrema at (a,b), and the first partial derivatives exist in a circle centered at (a,b), then (a,b) is a critical point.

Second-Partials Test for Relative Extrema: Assume that f has a continuous partial derivatives on an open region containing (a,b). Let (a,b) be a critical point of f. Denote

$$d(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2 .$$

1. If $d > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a relative minimum.
2. If $d > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a relative maximum.
3. If $d < 0$, then $f(a,b)$ has a saddle point.
4. If $d = 0$, then the second derivatives test gives nothing.

Example 125 *Find the critical point(s) of $z = (x+y)(xy+xy^2)$ and classify them.*

Solution:

$$\frac{\partial z}{\partial x} = y(2x+y)(y+1), \quad \frac{\partial z}{\partial y} = x\left[3y^2 + 2y(x+1) + x\right].$$

$$\frac{\partial z}{\partial x}(x) = 0, \Rightarrow y = 0, -1, -2x.$$

Set

$$\frac{\partial z}{\partial y} = 0.$$

- If $y = 0$, then

$$0 = \frac{\partial z}{\partial y} = x\left[3y^2 + 2y(x+1) + x\right] = x^2, \Rightarrow x = 0.$$

- If $y = -1$, then

$$0 = \frac{\partial z}{\partial y} = x\left[3y^2 + 2y(x+1) + x\right] = x(1-x), \Rightarrow x = 0, 1.$$

- If $y = -2x$, then

$$0 = \frac{\partial z}{\partial y} = x\left[3(-2x)^2 + 2(-2x)(x+1) + x\right] = 4x^2(2x-1), \Rightarrow x = 0, \frac{1}{2}.$$

So critical points are

$$(a, b) \in \{(0, 0), (0, -1), (1, -1), (\frac{1}{2}, -1)\}.$$

To test all of them, we use the Second-Partials Test.

$$d(x, y) = f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2 = 2y(y+1)(2(3y+x+1))x - (3y^2 + y(4x+2) + 2x)^2.$$

- $d(0, -1) = -1$, $d(1, -1) = -1$, so $(0, -1), (1, -1)$ are saddle points.
- $d(0, 0) = 0$, $d(\frac{1}{2}, -1) = 0$, the test fails.

Example 126 *Classify the critical points of* $f(x, y) = x^3 - 3xy + y^3$

Solution:

$$f_x(x, y) = 3x^2 - 3y, \quad f_y(x, y) = -3x + 3y^2, \Rightarrow$$

$$f_{xx}(x, y) = 6x, \quad f_{yy}(x, y) = 6y, \quad f_{xy}(x, y) = -3.$$

Setting $f_x = 0$ and $f_y = 0 :\ 3x^2 - 3y = 0,\ -3x + 3y^2 = 0.$ We imply that $(x, y) = (0, 0), (1, 1)$.

$$d(x, y) = f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2 = 36xy - 9.$$

- $d(0, 0) = -9$, $d(1, -1) = -1$, so $(0, 0)$ is a saddle point.
- $d(1, 1) = 27$, $f_{xx}(1, 1) = 6 > 0$, so $f(1, 1)$ is a relative minimum.