COMM 225: "REVISED" SOLUTION TO REVIEW QUESTIONS (FINAL EXAM MATERIALS) W2012

TOPIC: STATISTICAL PROCESS CONTROL

Q 3.1: *A certain molded part is examined for conformity in diameter, and for the presence of excessive flashing. Equal-sized samples are taken at 15-min intervals throughout the eight-hour working day. For each sample, the mean and the range of the diameters are recorded, and a notion is made if any of the sampled parts have excessive flashing.*

Sample	*Mean*	*Range*	*Sample*	*Mean*	*Range*	*Sample*	*Mean*	*Range*
1	1.975	0.2	6	2	0.3	11	2.05	0.1
2	2.025	0.3	7	1.925	0.2	12	1.975	0.1
3	1.9	0.3	8	1.975	0.2	13	1.975	0.3
4	1.95	0.1	9	2	0.2	14	2.025	0.2
5	2.1	0.2	10	2.025	0.2			

- *The sum of the mean and range of the 14 samples shown above are 27.9 and 2.9 respectively.*
- *The fifteenth sample is: {2.00, 2.00, 1.90, 2.10, 2.40, 2.10, 1.95, 2.00}*

(a) *Calculate the control for the appropriate process control charts limits using all the information you have. Explain if the process appears to be in control?*

Solution:

- The appropriate process control charts is $\bar{x}$-chart and R-chart as they deal with the mean and the range of the samples.
- Note that, we have the information about 15 samples and hence we have to use all the 15 samples for calculating the control limits. Also sample size = n = 8.

The mean, $\bar{x}$ for the 15th sample is 2.05625 and the range R is 2.40-1.90 = 0.50.

Therefore, $\bar{\bar{x}} = \frac{27.90 + 2.05625}{15} = 1.9971$ and $R = \frac{2.90 + 0.05}{15} = 0.2267$

- Control Limits for the $\bar{x}$-chart can be computed using $UCL = \bar{\bar{x}} + A_2\bar{R}$ and $LCL = \bar{\bar{x}} - A_2\bar{R}$ as follows: (A_2 = 0.37 for n=8)

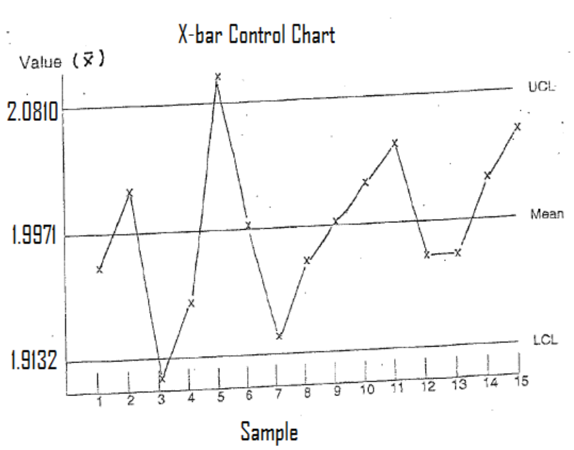
Upper Control Limit, $UCL = \bar{\bar{x}} + A_2\bar{R} = 1.9971 + 0.37(0.2267) = 2.0810$

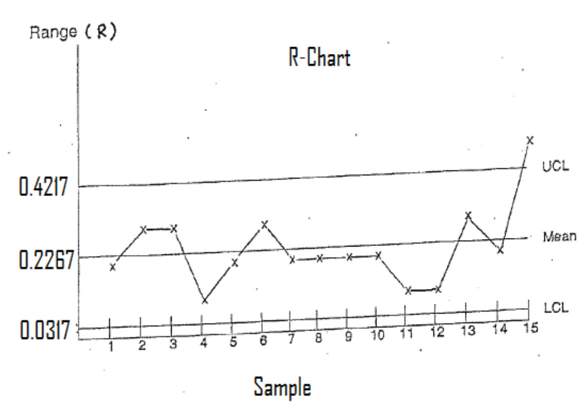
Lower Control Limit, $LCL = \bar{\bar{x}} - A_2\bar{R} = 1.9971 - 0.37(0.2267) = 1.9132$

- Control Limits for R-charts can be computed using $UCL = D_4\bar{R}$, and $LCL = D_3\bar{R}$ as follows:

Upper Control Limit, $UCL = D_4\bar{R} = 1.86(0.2267) = 0.4217$ (D_4 = 1.86 for n=8)

Lower Control Limit, $LCL = D_3\bar{R} = 0.14(0.2267) = 0.0317$ (D_3 = 0.14 for n=8)





- From the above two charts, we see that the process is OUT of control. Sample 3 and 5 for the $\bar{x}$ chart and sample 15 for R-chart are out of control for the control limits. The causes must be investigated.

(b) *Every-day 32 samples were taken. Last week the number of samples for which there was a notion of excessive flashing was 2, 4, 2, 2, 3, 3, and 8. The 8 looks higher than the rest. Use an appropriate control char, based on these seven data, to determine whether the presence of excessive flashing is out of control. Sketch the control chart, showing the upper and lower control limits.*

Although we have the information about the number of samples and sample size, but we do not have the results of all the samples. Therefore, we must use c-chart.

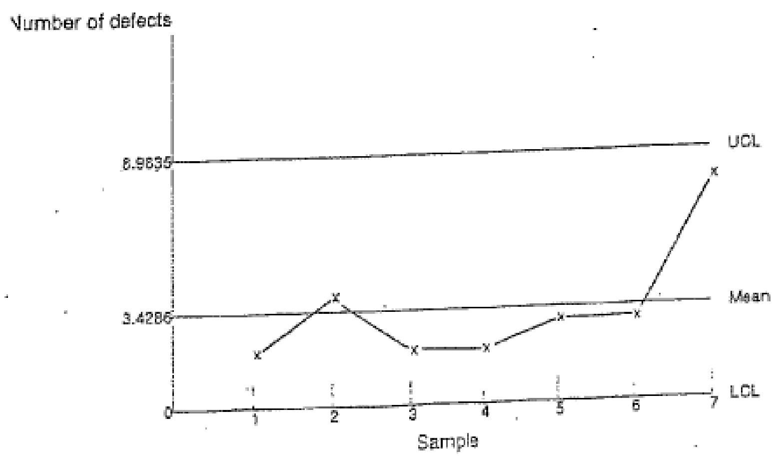
$$\bar{c} = \frac{2+4+2+2+3+3+8}{7} = 3.4286$$

The control limits of the c-chart can be computed (using sigma=3) as follows:

$$UCL = 3.4286 + 3\sqrt{3.4268} = 8.9835$$

$$LCL = 3.4286 - 3\sqrt{3.4268} = -2.1263 \cong 0$$

Although the process is in control, sample #7 is very close to UCL, the process must be investigated.



Q 3.2: *The following table lists the number of defective 40-watt light bulbs found in samples of 100 light bulbs selected over 25 days from a manufacturing process. Construct an appropriate chart to monitor the process and plot the data using the 3-sigma limit. Comment on the process.*

Day	*Defective*	*Day*	*Defective*	*Day*	*Defective*	*Day*	*Defective*	*Day*	*Defective*
1	3	6	4	11	2	16	2	21	2
2	2	7	4	12	4	17	3	22	2
3	5	8	5	13	4	18	1	23	3
4	8	9	6	14	4	19	4	24	5
5	3	10	1	15	0	20	0	25	3

Solution: k = 25 (number of samples) and sample size, n = 100

Proportion defective in the first sample = 3/100
Proportion defective in the second sample = 2/100
… …
Proportion defective in the 25th sample = 3/100

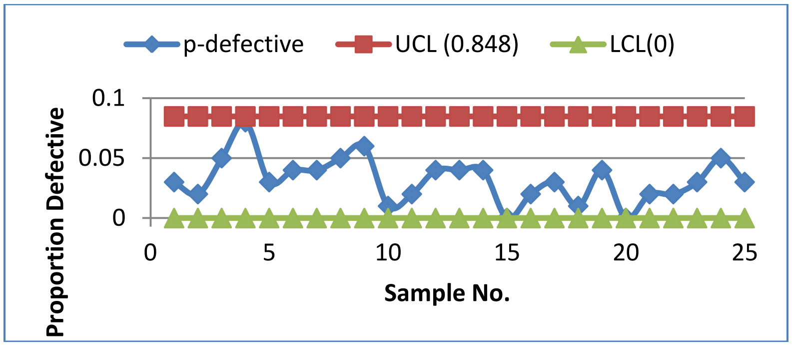
- Average proportion defective for the group of 13 samples = $\bar{p} = \frac{80}{25*100} = 0.032$
- The sample standard deviation can be computed as $\sigma_p = \sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = \sqrt{\frac{0.032(1-0.032)}{100}} = 0.0176$

- The control limits can be computed as follows:

$$UCL = \bar{p} + z\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0.032 + 3\sqrt{\frac{0.032(1-0.032)}{100}} = 0.0848$$

$$LCL = \bar{p} - z\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0.032 - 3\sqrt{\frac{\overline{0.032}(1-\overline{0.032})}{100}} = 0\ (cannot\ be - ve)$$

- An UCL of 0.0848 indicates that the fraction defective is 8.48 in a sample size of 100. Therefore, the process may not be in control.



***Q 3.3**: An advertising agency tracks the complaints, by week received, about the billboards in its city:*

Week	*1*	*2*	*3*	*4*	*5*	*6*
No. of Complaints	*4*	*5*	*4*	*11*	*3*	*9*

(a) What type of control chart would you use to monitor this process and why?
(b) What are the 3-sigma control limits for this process? Assume that the historical complaint rate is unknown. Is the process mean in control, according to the control limits?
(c) Assume that the historical complaint rate has been 4 calls a week. Is the process mean in control, according to the control limits?

Solution:

(a) The appropriate control chart here is c-chart. The goal here is to control the number of complaints per day. It is counting the number of occurrences per week.

(b) **If the process mean is unknown**, then using the information in the table, we get
Mean = $\bar{c}$ = 36/6 = 6 complaints per week.
Sample Standard deviation = $\sqrt{\bar{c}} = \sqrt{6} = 2.45$
The control limits can be computed as follows:

$$UCL = 6 + 3(2.45) = 13.35$$

$$LCL = 6 - 3(2.45) \Rightarrow 0$$

The process is within control.

(c) **If the process mean is known to be 4** complaints per week, then
Standard deviation = $\sqrt{\bar{c}} = \sqrt{4} = 2$
The control limits can be computed as follows:

$$UCL = 4 + 3(2.00) = 10.00$$

$$LCL = 4 - 3(2.00) \Rightarrow 0$$

The process may not be in control.

Q 3.4: *As a part of an insurance company's training program, participants learn how to conduct a fast but an effective analysis of client's insurability. The goal is to have participants achieve a time less than 45 minutes. There is no minimum time, but the quality of assessment should be acceptable. Test results for three participants were, Jerry, a mean of 37 minutes and a standard deviation of 2.5 minutes; Armand, a mean of 39 minutes and a standard deviation of 3 minutes; and Lau, a mean of 37.5 and standard deviation of 2.5 minutes. Which of the participants would you judge to be capable and why?*

Solution:

Given: Upper Specification = 45 minutes, Lower Specification = 0.

We evaluate the process capability index of the three participants as follows:

- $Jerry\text{: } C_{pk} = min\left[\frac{\bar{\bar{x}}-Lower\ Spec.}{3\sigma}, \frac{Upper\ Spec.-\bar{\bar{x}}}{3\sigma}\right] = min\left[\frac{37-0}{3*2.5}, \frac{45-37}{3*2.5}\right] = 1.066$
- $Armand\text{: } C_{pk} = min\left[\frac{\bar{\bar{x}}-Lower\ Spec.}{3\sigma}, \frac{Upper\ Spec.-\bar{\bar{x}}}{3\sigma}\right] = min\left[\frac{39-0}{3*3}, \frac{45-39}{3*3}\right] = 0.066$
- $Lau\text{: } C_{pk} = min\left[\frac{\bar{\bar{x}}-Lower\ Spec.}{3\sigma}, \frac{Upper\ Spec.-\bar{\bar{x}}}{3\sigma}\right] = min\left[\frac{37.5-0}{3*2.5}, \frac{45-37.5}{3*2.5}\right] = 1$

Lau and Jerry have a process capability of 1 and 1.006, and hence both of them are capable of meeting design specifications.

TOPIC: SERVICE DESIGN

Q 5.1: *Fore and Aft Marina is a new marina planned for a location on the Ohio River near Madison, Indiana. Fore and Aft expects a mean arrival rate of 7 boats per hour. Fore and Aft is evaluating the following two options:*

- *Option A: One dock with a mean service rate of 12 boats per hour.*
- *Option B: Two docks with a single queue. Each dock is capable of serving on average 10 boats per hour.*

Calculate the following for both the options (Option A and Option B)

a) *What is the average time a boat will wait for service?*
b) *What is the probability that more than 2 boats are waiting for services?*
c) *What is the probability that a boat has to wait before getting the services?*

Solution:

Option A:

- Arrival rate (λ) = 7 boats/hour
- Service rate (μ) = 12 boats/hour
- No. of Servers = 1. Hence, the problem can be solved using the basic single server model.

➢ Average time a boat will wait for service, $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{7}{12(12-7)} = 0.1166\ hrs = 7\ min$

➢ Probability that more than 2 boats are <u>waiting for services</u> implies that there are 3 <u>boats in the system</u>. Hence, this can be calculated as follows:

Probability of more than 3 boats in the system

$= P_4 + P_5 + \cdots = 1 - (P_0 + P_1 + P_2 + P_3)$)=1-0.882 =0.118

- Probability of no boats in the system $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{7}{12} = 0.4166$
- Probability of 1 boat waiting for service $P_1 = \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) = \frac{7}{12}\left(1 - \frac{7}{12}\right) = 0.243$
- Probability of 2 boats in the system $P_2 = \left(\frac{\lambda}{\mu}\right)^2\left(1 - \frac{\lambda}{\mu}\right) = 0.141$
- Probability of 3 boats in the system $P_3 = \left(\frac{\lambda}{\mu}\right)^3\left(1 - \frac{\lambda}{\mu}\right) = 0.08225$

- Probability that a boat has to wait before getting the services is same as the probability that the server is busy. This is also the probability that there is 1 boat and more than 1 boat in the system.
  - Probability of the server is busy = 1-$P_0 = \frac{\lambda}{\mu} = \frac{7}{12} = 0.5833$

- Probability of 1 boat or more in the system $= P_1 + P_2 + P_3 + \cdots = 1 - P_0 = \frac{\lambda}{\mu} = 0.5833$

Option B:

- Arrival rate (λ) = 7 boats/hour
- Service rate (μ) = 10 boats/hour
- No. of Servers = 2. Hence, the problem can be solved using the multiple-server model.

- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=s-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]+\frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s\frac{s\mu}{s\mu-\lambda}} = \frac{1}{\left[1+\frac{7}{10}\right]+\frac{1}{2!}\left(\frac{7}{10}\right)^2\frac{2*10}{2*10-7}} = 0.481$

The expected queue length and the average time before service can be computed as:

- $L_q = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!(s\mu-\lambda)^2}P_o = \frac{7*10*(0.7)^2}{(20-7)^2} \times 0.481 = 0.10$ boats
- $W_q = \frac{L_q}{\lambda} = \frac{0.10}{7}$=0.01428 hours = 0.857 minutes.

Probability that more than 2 boats are waiting for services implies that there are 4 boats in the system (2 in service and 2 waiting in line). Hence, this can be calculated as follows:

- $P_n = \begin{cases} \frac{1}{s!s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0 \text{ for } n > s \\ \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 \text{ for } n \leq s \end{cases}$

- $P_1 = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 = 0.7 * 0.481 = 0.337$
- $P_2 = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 = 0.5*(0.7)^2 * 0.481 = 0.118$
- $P_3 = \frac{1}{s!s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0 = \frac{1}{2*2}(0.7)^3 0.481 = 0.0412$
- $P_4 = \frac{1}{s!s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0 = \frac{1}{2*2*2}(0.7)^4 0.481 = 0.0144$
- Probability that more than 2 boats are waiting for services =
- Probability of more than 4 boats in the system
  $= P_5 + P_6 + \cdots = 1 - (P_0 + P_1 + P_2 + P_3 + P_4)$ =1-(0.481+0.337+0.118+0.0412+0.0144) =0.0084

➢ Probability that a boat has to wait before getting the services =

$$P_w = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu-\lambda}P_0 = \frac{1}{2!}\left(\frac{7}{10}\right)^2 \frac{2*10}{2*10-7} \times 0.481 = 0.181$$

***Q 5.2**: A warehouse has three loading docks. Trucks wait in a single line until signaled to enter in the next available dock. Currently every dock requires a team of two persons who can load a truck on average in 10 minutes. Trucks arrive at the rate of 12 per hour.*

a) Determine the expected queue length as well as the average time before a truck is assigned to a dock.

b) Determine the probability that a truck get the dock without waiting in the queue.

c) It is estimated that each truck waiting in the system costs $30/hour. The warehouse manager is evaluating the following two options listed below. Which option would you recommend i.e. current system, option 1 or option 2? Justify your answer by actual calculations. (Assume warehouse operates 10 hours per day and 300 days per year)

- *Option 1: One Automated Loading System (fixed service time), Loading Rate: 20 trucks per hour, Total Cost = $35,000/year*
- *Option 2: One Semi-automated loading System, Loading Rate: 15 trucks per hour, Total Cost = $18,000/year*

Solution:

- Arrival rate (λ) = 12 trucks/hour
- Service rate (μ) = 6 trucks/hour
- No. of Servers = s = 3. Because all the trucks wait in the single line and there are three docks, the problem can be solved using the multiple-server model.

- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=s-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right]+\frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s\frac{s\mu}{s\mu-\lambda}} = \frac{1}{\left[1+2+\frac{1}{2}(2)^2\right]+\frac{1}{3!}(2)^3\frac{3*6}{3*6-12}} = \frac{1}{4+5} = \frac{1}{9} = 0.1111$

(a) The expected queue length and the average time before a truck is assigned to a dock can be computed as:

- $L_q = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!(s\mu-\lambda)^2}P_o = \frac{12*6(2)^3}{2(18-12)^2} \times 0.1111 = 0.89 \text{ trucks}$
- $W_q = \frac{L_q}{\lambda} = \frac{0.89}{12} =$ 0.0741 hours = 4.45 minutes

(b) The probability that a truck arriving at the dock must wait for service is

- $P_w = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^S \frac{s\mu}{s\mu-\lambda}P_0 = \frac{1}{3!}(2)^3\frac{3*6}{3*6-12} \times 0.1111 = 0.4444$

Therefore, the probability that a truck get the dock without waiting in the queue = $1 - P_w = 0.5556$

(c) Note that we are given with the truck's waiting cost in the system. Hence, we will compute the <u>average waiting time in the system</u> for the three options to compare them.

- $W_{CURRENT\ SYSTEM} = W_q + \frac{1}{\mu} = 0.0741 + 0.1666 = \mathbf{0.24\ hrs} = 14.45\ min$

Option 1: This option of one automated loading system with fixed service time can be modeled appropriately using single-server model with constant service time.

- Arrival rate (λ) = 12 trucks/hour
- Service rate (μ) = 20 trucks/hour
- $W_q = \frac{L_q}{\lambda} = \frac{\lambda}{2\mu(\mu-\lambda)} = \frac{12}{2*20(20-12)} = 0.0375\ hrs = 2.25\ min$
- $W_{OPTION\ 1} = W_q + \frac{1}{\mu} = 0.0375\ hr + 0.05\ hr = \mathbf{0.0875\ hrs} = 5.25\ min$

Option 2: This option of one semi-automated loading system can be modeled appropriately using basic single-server model.

- Arrival rate (λ) = 12 trucks/hour
- Service rate (μ) = 15 trucks/hour
- $W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{12}{15(15-12)} = 0.2666\ hrs = 16\ min$
- $W_{OPTION\ 2} = W_q + \frac{1}{\mu} = 0.2666 + 0.0666 = \mathbf{0.3332\ hrs} = 19.996\ min$

	Current System	Option 1	Option 2
Arrival Rate	12 trucks/hour	12 trucks/hour	12 trucks/hour
Expected No. of Truck arrivals/year	12*10*300 = 36,000	12*10*300 = 36,000	12*10*300 = 36,000
Average Waiting Time in System	0. 240 hours	0.0875 hours	0.3332 hours
Waiting Cost	$30/hour	$30/hour	$30/hour
Total Waiting Cost/year	36,000*0.240*30 = $ 259,200	36,000*0.0875*30 = $ 94,500	36,000*0.3332*30 = $ 359, 856
Annual Fixed Cost	-	$ 35,000	$ 18,000
TOTAL ANNUAL COST	**$ 259,200**	**$ 129,500 (LOWEST)**	**$ 377,856**

Option 1 is recommend as it is most cost effective.

Q 5.3*: The Chattanooga Furniture store gets an average of 50 customers per shift. Marilyn Helms, the manager, wants to calculate whether she should hire 1, 2, 3 or 4 salespeople. She has determined that average waiting times will be 7 minutes with one salesperson, 4 minutes with two salespeople, 3 minutes with three salespeople, and 2 minutes with four salespeople. She has estimated the cost per minute that customer wait at $1. The cost per salesperson per shift (including fringe benefits) is $70. How many sales people should be hired?*

Solution:

- Arrival rate (λ) = 50 customer/shift
- Cost of Waiting = $1/min/customer
- Cost per salesperson per shift = $70
- The costs associated with four options are as follows:

	Option 1: 1 Salesperson	**Option 2: 2 Salespersons**	**Option 3: 3 Salespersons**	**Option 4: 4 Salespersons**
Average Waiting Time	7 min	4 min	3 min	2 min
Total Cost of Waiting	50*1*7 = $ 350	50*1*4 = $ 200	50*1*3 = $ 150	50*1*2 = $ 100
Labor Cost	70*1 = $ 70	70*2 = $ 140	70*3 = $ 210	70*4 = $ 280
TOTAL COST	$420	$340 (LOWEST)	$360	$380

- The optimal policy is to hire 2 salespersons as it has the lowest total cost.

Q 5.4*: A hockey arena experiences the highest volume of ticket sales on Fridays from 10:00 a.m. to 6:00 p.m. Currently, there are three ticket windows serving their own line in front of each window. A consultant hired recently carried out a time study and found that the verbal communication between the customer and the salesperson takes three minutes on the average. The same study also revealed that there were no customers in the system 20% of the time. Based on Poisson arrival rates and negative exponential service times, answer the following questions. Show all work.*

a) What is the arrival rate per hour at each line?
b) How long on the average must a customer wait before the start of the service by the salesperson?
c) What is the average length of each queue?
d) What is the probability of finding at most two customers waiting in each queue?

After studying the sales operations, the same consultant proposed having a single queue to be served by three sales persons.

e) What is the arrival rate for this configuration?
f) What is the probability of finding no customers in the system?
g) How long on the average must a customer wait before the start of service by the salesperson?
h) Calculate the total amount of customer waiting time saved on Fridays if the consultant's proposal is accepted.

Solution:

- Service rate (μ) = 20 customers/hour
- Probability that there were no customers in the system = $P_0 = 0.20$
- Note that there are three ticket windows serving their own line in front of each window. This implies that this is a single-server model. Furthermore, all the three systems are identical; hence it suffices to analyze one of the three systems.

For a single server model, $P_0 = 1 - \frac{\lambda}{\mu} \rightarrow 0.20 = 1 - \frac{\lambda}{20}\ or\ \lambda = 16\ customers/hour$

- Hence, arrival rate at each line is 16 customers per hour.
- $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{16}{20(20-16)} = 0.20\ hrs = 12\ min$
- $L_q = \lambda W_q = 16 * 0.20 = 3.2\ customers$
- Probability of finding at most 2 customers waiting in queue is same as the probability of finding at most 3 customers in the system. Hence, this can be calculated as follows:

$$= P_{\leq 3} = P_0 + P_1 + P_2 + P_3 = (1-0.8) + 0.8*(1-0.8) + 0.8^2*(1-0.8) + 0.8^3*(1-0.8) = 0.5904$$

If the consultant proposes having a single queue to be served by three sales persons, then this becomes a multiple-server model with

- Arrival rate (λ) = 16+16+16 = 48 customers/hour
- Service rate (μ) = 20 trucks/hour
- No. of Servers (S) = 3

Probability of finding no customers in the system is

- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=s-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s\frac{s\mu}{s\mu-\lambda}} = \frac{1}{\left[1+2.4+\frac{1}{2}(2.4)^2\right]+\frac{1}{3!}(2.4)^3\frac{3*20}{3*20-48}} = \frac{1}{6.28+11.51} = 0.056$

The average time a customer must wait before the start of service by the salesperson can be computed as:

- $L_q = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!(s\mu-\lambda)^2}P_o = \frac{48*20(2.4)^3}{2(60-48)^2} \times 0.056 = 2.59$ customers
- $W_q = \frac{L_q}{\lambda} = \frac{2.59}{48} = 0.0536$ hours = 3.24 minutes

Note that a customer waits on an average for 12 minutes in queue in the current system, whereas the average waiting time reduced to 3.24 minutes in the proposed system. Hence, there is a reduction of 8.86 minutes or 0.1477 hours of waiting time per customer. This amounts to 48 customers/hour*0.1477 hours/customer* 8 hours/day = 56.72 hours/day.

***Q 5.5**: Customers arrive at the lobby of the exclusive and expensive Ritz Hotel at the rate of 40 per hour (Poisson distributed) to check in. The hotel normally has three clerks available at the desk to check guests in. The average time for a clerk to check in a guest is four minutes (exponentially distributed). Clerks are paid $12 per hour and the hotel assigns a good will cost of $2 per minute for the time guest must wait in the line.*

a) Determine the current waiting time for guests before getting the services.

b) Determine if the present check-in system is cost effective; if it is not, recommend what hotel management should do.

Solution:

- Arrival rate (λ) = 40 customers/hour
- Service rate (μ) = 60/4 min per customer= 15 customers/hour
- No. of Servers = s = 3
- The problem can be solved using the multiple-server model.

Probability of finding no customers in the system is

- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu-\lambda}} = \frac{1}{\left[1+2.67+\frac{1}{2}(2.67)^2\right]+\frac{1}{3!}(2.67)^3 \frac{3*15}{3*15-40}} = \frac{1}{7.23+28.55} = 0.028$

The average time a customer must wait before the start of service by the clerk can be computed as:

- $L_q = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!(s\mu-\lambda)^2} P_o = \frac{40*15(2.667)^3}{2(40-15)^2} \times 0.028 = 6.38$ customers
- $W_q = \frac{L_q}{\lambda} = \frac{6.38}{40} = 0.1595$ hours = 9.57 minutes

Clerks are paid $12 per hour and there are 3 clerks, hence the total labor cost is $36 per hour.

The arrival rate is 40 customers per hour and on an average every customer waits 9.57 minutes. The hotel assigns a good will cost of $2 per minute for the time guest must wait in the line. Hence, the total waiting cost/hour = 40*9.57*2 = $765.6. This is very high compared to the labor cost of $36 per hour. Hence, the management should increase the number of clerks in order to reduce the total waiting time and goodwill costs (thereby increasing service level).

Q 5.6: *Customers arrive at the drive-up window of a fast food restaurant at the rate of 25 per hour. The employee working the window can serve one customer every 2 minutes. Assume Poisson arrivals and exponential service rates.*

(a) *What is the average utilization of the employee?*
(b) *What is the average number of customers in line?*
(c) *What is the average number of customers in the system?*
(d) *What is the average time spent waiting in line?*
(e) *What is the average time waiting in the system?*
(f) *What is the probability that exactly 2 cars will be waiting in line?*
(g) *What is the probability that an arriving customer will have an actual waiting time (in the system) of more than 20 minutes?*

If the manager decides to train the employees to reduce the service time variability as much as possible, what would be the change in the above performance measures? Use appropriate model to do the calculations.

Solution:

- Arrival rate (λ) = 25 customer/hour
- Service rate (μ) = 1 customer every 2 min = 30 customers/hour.
- No. of Servers = 1
- Hence, the problem can be solved using the basic single server model.

a) The average utilization of the employee = $\rho = \frac{\lambda}{\mu} = \frac{25}{30} = 0.833 \text{ or } 83.33\%$

b) The average number of customers in line = $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{25*25}{30*(30-20)} = 4.17 \text{ customers}$

c) The average number of customers in the system = $L = L_q + \frac{\lambda}{\mu} == 4.17 + 0.83 = 5 \text{ customers}$

d) The average time spent waiting in line = $W_q = \frac{L_q}{\lambda} == \frac{4.17}{25} = 0.1666 \text{ hours} = 10 \text{ min}$

e) The average time spent waiting in the system = $W = \frac{L}{\lambda} == \frac{5}{25} = 0.2 \text{ hours} = 12 \text{ min}$

f) Probability that exactly 2 cars will be waiting in line = Probability that there are exactly 3 cars in the system = $P_3 = \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right) = \left(\frac{25}{30}\right)^3 \left(1-\frac{25}{30}\right) = 0.096$

If the manager decides to train the employees to reduce the service time variability as much as possible, what would be the change in the above performance measures? Use appropriate model to do the calculations.

Reducing the service time variability as much as possible implies making the service time constant
For the single server model with constant service times, the calculations are as follows:

a) The average utilization of the employee = $\rho = \frac{\lambda}{\mu} = \frac{25}{30} = 0.833 \text{ or } 83.33\%$ (Unchanged)

b) The average number of customers in line = $L_q = \frac{\lambda^2}{2\mu(\mu-\lambda)} = \frac{25*25}{2*30*(30-20)} = 2.08 \text{ customers}$

(Reduces by Half)

c) The average number of customers in the system = $L = L_q + \frac{\lambda}{\mu} == 2.08 + 0.83 = 2.8833 \text{ customers}$

d) The average time spent waiting in line = $W_q = \frac{L_q}{\lambda} == \frac{2.08}{25} = 0.0832 \text{ hours} = 5 \text{ min}$ (Reduces by Half)

e) The average time spent waiting in the system = $W = \frac{L}{\lambda} == \frac{2.8833}{25} = 0.115 \text{ hours} = 6.92 \text{ min}$

f) Probability that exactly 2 cars will be waiting in line = Probability that there are exactly 3 cars in the system = $P_3 = \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right) = \left(\frac{25}{30}\right)^3 \left(1-\frac{25}{30}\right) = 0.096$ (Unchanged)

Q 5.7: *The manager of an amusement park wants to hire a repair team for the bumper car section. On average 2 cars breakdown every hour. The loss of revenue due to breakdown is $40/hour. The average repair (service) time is 30 minutes for a team of size 1, 20 minutes for a team of size 2, and 15 minutes for a team of size 3.*

(a) *Assuming that each repair person costs $20/hour, how many people should be hired?*
(b) *Suppose that the director of the amusement park has asked the manager to go with the repair team size of 2. However, to keep the cars in working condition, more than one repair team may be needed. How many teams should be hired?*

Solution:

- Arrival rate (λ) = 2 cars/hour
- No. of Servers = s = 1
- Note that a repair team works together to get the repair job done, hence a team acts as a server here (e.g. a team of three repair person is one single server, and not 3 servers).
- The problem can be solved using the basic single-server model.

For a team of 1 repair person:

- Service Time = 30 minutes/car
- Service rate (μ) = 2 cars/hour
- We cannot proceed with the calculations as the arrival rate = service rate, L_q = infinity

For a team of 2 repair persons:

- Service Time = 20 minutes/car
- Service rate (μ) = 3 cars/hour

- We can proceed with the calculations as the arrival rate is less than service rate.
- The average length of the queue = $L = \frac{\lambda}{(\mu-\lambda)} = \frac{2}{(3-2)} = 2\ cars$
- Total cost per hour = 2 repair-person *$20/hour + 2 cars * 40/hour = **$120/hour**

For a team of 3 repair persons:
- Service Time = 15 minutes/car
- Service rate (μ) = 4 cars/hour
- We can proceed with the calculations as the arrival rate is less than service rate.
- The average length of the queue = $L = \frac{\lambda}{(\mu-\lambda)} = \frac{2}{(4-2)} = 1\ car$
- Total cost per hour = 3 repair-person *$20/hour + 1 car * 40/hour = **$100/hour**

Hire 3 repair persons as it is least among the two feasible options that we have.

(b): Suppose that the director of the amusement park has asked the manager to go with the repair team size of 2. However, to keep the cars in working condition, more than one repair team may be needed. How many teams should be hired?

This is a multiple-sever model, because we are talking of more than one team, where each team comprises of 2 repair person.

For a team of 2 repair persons:
- Arrival rate (λ) = 2 cars/hour
- Service rate (μ) = 3 cars/hour
- No. of Servers = s = 1
- $L = \frac{\lambda}{(\mu-\lambda)} = \frac{2}{(3-2)} = 2\ cars$
- Total cost per hour = 2 repair-person *$20/hour + 2 cars * 40/hour = **$120/hour**

For 2 teams of 2 repair persons:
- Arrival rate (λ) = 2 cars/hour
- Service rate (μ) = 3 cars/hour
- No. of Servers = s = 2
- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu-\lambda}} = 0.5$
- $L = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!(s\mu-\lambda)^2} P_o + \frac{\lambda}{\mu}$=0.75 cars
- Total cost per hour = 2 teams * 2 repair-person *$20/hour + 0.75 car * 40/hour = **$110/hour** (approx).

For 3 teams of 2 repair persons:
- Arrival rate (λ) = 2 cars/hour
- Service rate (μ) = 3 cars/hour
- No. of Servers = s = 3
- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu-\lambda}} = 0.512$
- $L = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!(s\mu-\lambda)^2} P_o + \frac{\lambda}{\mu}$=0.68 cars
- Total cost per hour = 3 teams * 2 repair-person *$20/hour + 0.68 car * 40/hour = **$147/hour** (approx).

Note that the total cost has started increasing, hence having two teams of 2 repair person is optimal.

Q 5.8: *The manager of a regional warehouse must decide on the number of loading docks to request for a new facility in order to minimize the sum of dock costs and driver-truck costs. The manager has learned that each driver-truck combination represents a cost of $300 per day and that each dock plus loading crew represents a cost of $1,100 per day.*

a) *How many docks should be requested if trucks arrive at the rate of four per day, each dock can handle five trucks per day assuming both rates are Poisson?*

For 1 dock:

- Arrival rate (λ) = 4 trucks/day
- Service rate (μ) = 5 trucks/day
- No. of Servers = s = 1
- Average no. of trucks in system = $L = \frac{\lambda}{(\mu-\lambda)} = \frac{4}{(5-4)} = 4\ trucks$
- Cost of 1 dock (with loading crew) per day = $1100.
- Cost of 4 trucks waiting (along with driver) per day = 4*300 = $1200
- Total cost per day = **$2300.**

For 2 docks:

- Arrival rate (λ) = 4 trucks/day
- Service rate (μ) = 5 trucks/day
- No. of Servers = s = 2
- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu-\lambda}} = 0.429$
- Average no. of trucks in system $= L = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!(s\mu-\lambda)^2} P_o + \frac{\lambda}{\mu}$=0.95 trucks
- Cost of 2 dock (with loading crew) per day = 2*$1100 = $ 2200.
- Cost of 4 trucks waiting (along with driver) per day = 0.95*300 = $285
- Total cost per day = **$2485.**

b) *An employee has proposed adding new equipment that would speed up the loading rate to 5.71 trucks per day. The equipment would cost $100 per day for each dock. Should the manager invest in the new equipment?*

This new equipment will change the service rate (μ) to 5.71.

For 1 dock:

- Arrival rate (λ) = 4 trucks/day
- Service rate (μ) = 5.71 trucks/day
- No. of Servers = s = 1
- Average no. of trucks in system = $L = \frac{\lambda}{(\mu-\lambda)} = \frac{4}{(5.71-4)} = 2.34\ trucks$
- Cost of 1 dock (with loading crew) per day = $1100.
- Cost of 4 trucks waiting (along with driver) per day = 2.34*300 = $702.
- Cost of new equipment per day = $100.
- Total cost per day = **$1902.**

For 2 docks:

- Arrival rate (λ) = 4 trucks/day
- Service rate (μ) = 5.71 trucks/day
- No. of Servers = s = 2
- $P_0 = \frac{1}{\left[\sum_{n=0}^{n=s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu-\lambda}} = 0.481$

- Average no. of trucks in system $= L = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^s}{(s-1)!(s\mu-\lambda)^2} P_o + \frac{\lambda}{\mu}$=0.80 trucks
- Cost of 2 dock (with loading crew) per day = 2*$1100 = $ 2200.
- Cost of 4 trucks waiting (along with driver) per day = 0.80*300 = $240
- Cost of two of these new equipments per day = 2*$100 = $200.
- Total cost per day = **$2640.**

Note that the total cost has started increasing, hence having 1 dock and investing in this new equipment is optimal.

TOPIC: SALES AND OPERATIONS PLANNING

Q 6.1: *Wetski Water Ski (WWS) is the world's largest producer of water skis. As you might suspect, water skis exhibit a highly seasonal demand pattern, with peaks during the summer months and valleys during the winter months. The company likes to zero out its inventory at the end of a year so that it can start fresh each January. The company currently uses a level production strategy but would like to evaluate other options. Given the following costs and quarterly sales forecasts, create a production plan and calculate the cost of the plan for three strategies: (a) Level production, (b) Chase demand, (c) Mixed strategy: Produce 50,000 in period 1, and 134,000 in periods 2 through 4. Which plan would you recommend to WWS? (Note: No backordering is allowed.)*

Quarter	*Sales Forecast*
1	50,000
2	150,000
3	200,000
4	52,000

- *Beginning workforce: 50 employees*
- *Production per employee: 1,000 skis/quarter*
- *Inventory carrying cost: $3 per ski/quarter*
- *Regular production cost: $50 per ski*
- *Hiring cost = Firing cost: $500/worker*

Solution: Since no backordering is allowed, if the demand exceeds the supply, it will result in lost sales. Nothing is discussed about options such as subcontracting and overtime.

(a) Level Production Strategy

	Sales Forecast			*Workers*	*Workers*	*Workers*
Qtr.	*Demand*	*Production*	*Inventory*	*Needed*	*Hired*	*Fired*
1	50,000	113, 000	63,000	113	63	
2	150,000	113, 000	26,000	113		
3	200,000	113, 000	negative	113		
4	52,000	113, 000	61,000	113		
Total	452,000	452, 000	150, 000		63	

Production Cost = 452,000*$50 = $ 22,600,000
Inventory Cost = 150,000*$3 = $ 450,000, Hiring Cost = 63*500 = $ 31,500, Firing Cost = 0
Total Cost = $ 23, 081,500

(b) Chase Demand Strategy

	Sales Forecast			*Workers*	*Workers*	*Workers*
Qtr.	*Demand*	*Production*	*Inventory*	*Needed*	*Hired*	*Fired*
1	50,000	50,000	0	50		
2	150,000	150,000	0	150	100	
3	200,000	200,000	0	200	**50**	
4	52,000	52,000	0	52		148
Total	452,000	452, 000	0		**150**	148

Production Cost = 452,000*$50 = $ 22,600,000
Inventory Cost = 0; Hiring Cost = 150*500 = $ 75,000; Firing Cost = 148*500 = $ 74,000
Total Cost = $ 22, 749,000 (LOWEST).

(c) Mixed Strategy

	Sales Forecast			*Workers*	*Workers*	*Workers*
Qtr.	*Demand*	*Production*	*Inventory*	*Needed*	*Hired*	*Fired*
1	50,000	50, 000	0	50		
2	150,000	134, 000	negative	134	84	
3	200,000	134, 000	negative	134		
4	52,000	134, 000	82, 000	134		
Total	452,000	452, 000	82, 000		84	

Production Cost = 452,000*$50 = $ 22,600,000
Inventory Cost = 82,000*3 = $ 246,000
Hiring Cost = 84*500 = $ 42,000
Firing Cost = 0
Total Cost = $ 22, 888,000

Q 6.2: *Formulate a linear programming model for the Wetski Water Ski (WWS) (Refer to the Q6.1 above) that will satisfy demand at minimum cost (Define the variables and write the objective function, and constraints).*

Solution: Let us define the variables as follows:

- $\mathrm{H_t}$ = Number of workers hired for period t, $\mathrm{F_t}$ = Number of workers fired for period t
- $\mathrm{I_t}$ = Number of units in inventory at the end of period t
- $\mathrm{P_t}$ = Units produced in period t, $\mathrm{W_t}$ = Workforce size for period t

$$\text{Minimize } \mathrm{Z} = 50\,(\mathrm{P_1 + P_2 + P_3 + P_4}) + 3\,(\mathrm{I_1 + I_2 + I_3 + I_4}) + 500\,(\mathrm{H_1 + H_2 + H_3 + H_4}) + 500\,(\mathrm{F_1 + F_2 + F_3 + F_4})$$

Subject to

Demand Constraints:
$0 + P_1 - I_1 = 50{,}000$
$I_1 + P_2 - I_2 = 150{,}000$
$I_2 + P_3 - I_3 = 200{,}000$
$I_3 + P_4 - I_4 = 52{,}000$

Production Constraints:
$1000\mathrm{W_1} = \mathrm{P_1}$
$1000\mathrm{W_2} = \mathrm{P_2}$
$1000\mathrm{W_3} = \mathrm{P_3}$
$1000\mathrm{W_4} = \mathrm{P_4}$

Workforce Constraints:
$50 + H_1 - F_1 = W_1$
$W_1 + H_2 - F_2 = W_2$
$W_2 + H_3 - F_3 = W_3$
$W_3 + H_4 - F_4 = W_4$

All variables should be non-negative
$\mathrm{P_t, W_t, H_t, F_t, I_t} \geq 0$

***Q 6.3:** The linear programming model for the above problem was solved using the Excel Solver. The output (with some missing cell entries) is shown below. Determine the total cost of the optimal production plan and compare it with three production strategies of above problem (fill up the relevant missing cells in the Excel Solver output only if required).*

Beginning Workforce:	50	Production Cost:	$50.00	Firing Cost:	$500
Units/Worker	$ 1,000	Inventory Cost:	$3.00	Hiring Cost:	$500
Beg Inv.	0				

				Workers	*Workers*	*Workers*	*Demand*	*Production*	*Workforce*
Qtr.	*Demand*	*Production*	*Inventory*	*Needed*	*Hired*	*Fired*	*Constraint*	*Constraint*	*Constraint*
1	50,000	50,000	0	50			50,000	50,000	50
2	150,000	150,000	0	150	100		150,000	150,000	150
3	200,000	200,000	0	200	50		200,000	200,000	200
4	52,000	52,000	0	52		148	52,000	52,000	52
Total	452,000	452,000			150	148			

Production Cost = 452,000*$50 = $ 22,600,000
Inventory Cost = 0
Hiring Cost = 150*500 = $ 75,000
Firing Cost = 148*500 = $ 74, 000
Total Cost = $ 22,749,000 (Same as the chase strategy in this case).

NOTE:

(1) The columns marked "workforce constraints" and "workers needed" should have the same values because of equality constraints.

(2) The columns marked "demand constraints" and "demand" should have the same values because of equality constraints.

(3) The columns marked "production constraints" and "production" should have the same values because of equality constraints.

(4) Worked hired and fired can be calculated accordingly.

TOPIC: SUPPLEMENT TO LINEAR PROGRAMMING

Q 6.1S: *A jewelry store makes necklaces and bracelets from gold and platinum. The store has developed the following linear programming model for determining the number of necklaces and bracelets (x1 and x2) to make in order to maximize profits.*

$$\text{Maximize } Z = 300x_1 + 700x_2 \quad \text{(Total Profit in \$)}$$

$$\text{Subject to}$$

$$3x_1 + 2x_2 \leq 18 \quad \text{(Gold in ounces)}$$

$$2x_1 + 4x_2 \leq 20 \quad \text{(Platinum in ounces)}$$

$$x_2 \leq 4 \quad \text{(Demand for bracelets)}$$

$$x_1, \quad x_2 \geq 0$$

(a) Solve this model graphically.

(b) The maximum demand for bracelets if 4. If the store produces the optimal number of bracelets and necklaces will the maximum demand for bracelets be met? If not, by how much will it be missed?

Solution:

The feasible region is shown below. The values of the optimal solution is $x_1 = 2$ and $x_2 = 4$. The optimal solution lies at the intersection of $x_2 \leq 4$ and $2x_1 + 4x_2 \leq 20$. The optimal objective function value is $Z = 300 * 2 + 700 * 4 = 3400$. The optimal number of bracelet is 4; hence the maximum demand is met.