TEST 4 SOLUTIONS–MATH 1102

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1. Suppose $A \in M_{nn}(\mathbb{C})$ and $\lambda \in \mathbb{C}$ is an eigenvalue of A. Provide the definition of an eigenvector corresponding to λ.

   Solution: A vector $v \in \mathbb{C}^n$ is an eigenvector corresponding to λ if it is non-zero and $Av = \lambda v$.

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2. Suppose $A \in M_{nn}(\mathbb{C})$ and $\lambda_1, \lambda_2 \in \mathbb{C}$ are *distinct* eigenvalues of A. Prove that there is no eigenvector of A that has *both* λ_1 and λ_2 as eigenvalues. (Hint: Try a proof by contradiction.)

   Solution: Suppose by way of contradiction $v \in \mathbb{C}^n$ is an eigenvector of λ_1 and λ_2. This means that v is non-zero, $Av = \lambda_1 v$, and $Av = \lambda_2 v$. Therefore

   $$\lambda_1 v = \lambda_2 v \Rightarrow (\lambda_1 - \lambda_2)v = 0.$$

   Now, $\lambda_1 \neq \lambda_2$ by hypothesis so $\lambda_1 - \lambda_2 \neq 0$. The equation then implies that $v = 0$, which contradicts v being non-zero. In conclusion, v cannot be an eigenvector with distinct eigenvalues λ_1 and λ_2.

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3. Provide the definition of the dimension of a vector space V.

   Solution: The dimension is the number of vectors in a basis for V.

4. Provide clear justification for why the set $\{1-x, 1+x\}$ is a basis for $P_1(\mathbb{R})$.
/3 (*Hint:* There is a good theorem you can use.)

   Solution: It is clear that $1-x$ and $1+x$ are not scalar multiples of each other by looking at their constant terms. Therefore $\{1-x, 1+x\}$ is linearly independent. Since $\dim P_1(\mathbb{R}) = 2$ and there are two polynomials in $\{1-x, 1+x\}$, we conclude that this set is a basis (Theorem G).

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5. Compute the characteristic polynomial and the eigenvalues of

$$A = \begin{bmatrix} 1+i & 14-7i & 407+9i \\ 0 & 77+6i & -99/2 \\ 0 & 0 & 1 \end{bmatrix} \in M_{33}(\mathbb{C})$$

   Solution: The characteristic polynomial is

$$p_A(x) = \det(A - xI) = \det\left(\begin{bmatrix} 1+i-x & 14-7i & 407+9i \\ 0 & 77+6i-x & -99/2 \\ 0 & 0 & 1-x \end{bmatrix}\right)$$

   The matrix on the right is upper-triangular so its determinant is

$$(1+i-x)(77+6i-x)(1-x).$$

   Clearly, $p_A(x) = 0$ if x is equal to one of $1+i$, $77+6i$ or 1, and these are the eigenvalues.

6. The matrix

$$A = \begin{bmatrix} 3 & 2 & -3 \\ 0 & 1 & 0 \\ 2 & 2 & -2 \end{bmatrix} \in M_{33}(\mathbb{C})$$

has exactly two eigenvalues which are 0 and 1.

/6 (a) Compute the eigenspaces $\mathcal{E}_A(0)$ and $\mathcal{E}_A(1)$

Solution: For convenience, I will write column vectors as 3-tuples in this solution. Row reducing $A - 1I_3$ we obtain

$$\begin{bmatrix} 1 & 1 & -3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that

$$\mathcal{E}_A(1) = \mathcal{N}(A-1I_3) = \{c_3(3,0,2)+c_2(-1,1,0) : c_2, c_3 \in \mathbb{C}\} = \operatorname{span}\{(3,0,2),(-1,1,0)\}.$$

Row reducing $A = A - 0I$ we obtain

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that

$$\mathcal{E}_A(0) = \mathcal{N}(A - 0I_3) = \{c_3(1,0,1) : c_3 \in \mathbb{C}\} = \operatorname{span}\{(1,0,1)\}.$$

/3 (b) Determine the geometric multiplicities $\gamma_A(0)$ and $\gamma_A(1)$.

Solution: $\gamma_A(0) = \dim \mathcal{E}_A(0) = 1$ and $\gamma_A(1) = \dim \mathcal{E}_A(1) = 2$.

Grading: 1 for definition of $\gamma_A(\lambda)$, 1 for each correct multiplicity.

/3 (c) Determine the algebraic multiplicities $\alpha_A(0)$ and $\alpha_A(1)$. (Hint: You don't have to compute $p_A(x)$ for this if you computed the geometric multiplicities correctly.)

Solution: We know that $\alpha_A(0) + \alpha_A(1) = \deg p_A(x) = 3$ and $\alpha_A(0) \geq \gamma_A(0) = 1$, $\alpha_A(1) \geq \gamma_A(1) = 2$. Therefore $\alpha_A(0) = \gamma_A(0) = 1$ and $\alpha_A(1) = \gamma_A(1) = 2$.

/3 (d) Explain why A is singular.

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7. Suppose $A \in M_{nn}(\mathbb{C})$ has an eigenvalue $\lambda \in \mathbb{C}$ with corresponding eigenvector $v \in \mathbb{C}^n$. Prove by induction on $k \geq 1$ that A^k has an eigenvalue λ^k with corresponding eigenvector $v \in \mathbb{C}^n$.

   Solution: Suppose $k = 1$. Then $A^k = A$ and we are done by assumption.

   Suppose the result holds for k.

   Let us prove the result for $k+1$. Using the induction assumption, we compute that

   $$A^{k+1}v = AA^k v = A(\lambda^k v) = \lambda^k Av = \lambda^k \lambda v = \lambda^{k+1} v.$$

   Grading: 3 for form. 2 for use of induction assumption.

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8. Compute the dimension of the subspace $W \subset P_2(\mathbb{R})$ defined by

   $$W = \{f(x) \in P_2(\mathbb{R}) : f(1) = 0\}.$$

   Solution: By definition,

   $$f(x) = a_0 + a_1 x + a_2 x^2 \in W \Leftrightarrow 0 = f(1) = a_0 + a_1 + a_2 \Leftrightarrow a_0 = -a_1 - a_2.$$

   Taking $a_1 = 1$ and $a_2 = 0$ in this equation gives $g_1(x) = -1 + x \in W$. Taking $a_1 = 0$ and $a_2 = 1$ gives $g_2(x) = -1 + x^2 \in W$. By looking at the constant terms of $g_1(x)$ and $g_2(x)$, it is clear that neither is a scalar multiple of the other. Therefore $\{g_1(x), g_2(x)\}$ is linearly independent. It follows that $\dim W \geq 2$. Also, we know that $\dim W \leq \dim P_2(\mathbb{R}) = 3$, and $\dim W = \dim P_2(\mathbb{R})$ only when $W = P_2(\mathbb{R})$. Clearly the constant polynomial 2 belongs to $P_2(\mathbb{R})$, but does not belong to W. Therefore $W \subsetneq P_2(\mathbb{R})$ and $\dim W = 2$.

   Grading: 3 for linearly independent set, 2 for $\dim W \leq \dim P_2(\mathbb{R}) = 3$, 1 for $W \subsetneq P_2(\mathbb{R})$. (Alternatively, 3 for spanning property of linearly independent set.)

9. Indicate whether each of the two statements below is true or false. If you think a statement is true, provide some justification. If you think a statement is false, provide a counterexample to it.

   (a) The converse of the following statement is true.

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   > If $\{v_1, \dots, v_n\}$ is a basis for a vector space V then every element of V may be uniquely expressed as a linear combination of $\{v_1, \dots, v_n\} \subset V$.

   Solution: True. If every element may be expressed uniquely as a linear combination of $\{v_1, \dots, v_n\}$, then by definition $\{v_1, \dots, v_n\}$ spans V. For linear independence, observe that

   $$a_1 v_1 + \cdots + a_n v_n = 0$$

   for scalars $a_1, \dots, a_n$ implies

   $$a_1 v_1 + \cdots + a_n v_n = 0v_1 + \cdots + 0v_n.$$

   By uniqueness, $a_1 = 0, \dots, a_n = 0$.

   (b) Suppose $A \in M_{mn}(\mathbb{C})$. Then the rank $r(A)$ is less than or equal to m and also less than or equal to n.

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   Solution: True.

   $$r(A) = \dim \mathcal{C}(A) \leq \dim \mathbb{C}^m = m$$

   and

   $$r(A) = \dim \mathcal{C}(A) = \dim \mathcal{R}(A) \leq \dim \mathbb{C}^n = n.$$