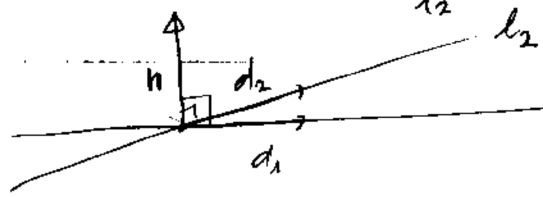
1341A.F06 Diag Solns

1. An equation for the plane which contains the two lines with parametric equations $\underbrace{x+1 = t,\ y-6 = -t,\ z+4 = 3t}_{l_1}$ and $\underbrace{x+3 = -4s,\ y-6 = 2s,\ z-7 = 5s}_{l_2}$, is:



A. $7x - 11y + 2z = 47$
B. $11x - 2y + 9z = 23$
C. $16x - 12y + z = 11$
D. $11x + 17y + 2z = 83$
E. $8x + 9y - 13z = 46$
F. $22x + 19y + 4z = 42$

$d_1 = (1, -1, 3) \qquad d_2 = (-4, 2, 5)$

$$n = d_1 \times d_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ -4 & 2 & 5 \end{vmatrix} = (-11, -17, -2)$$

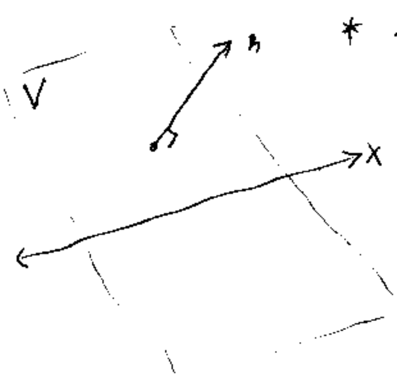
The only plane above with the correct normal $(11, 17, 2)$ is D

2. An equation for the plane passing through the points (0, 0, -3) and (-1, 2, 1), and which is parallel to the x-axis is:

A. $-3x + 7y - 2z = 3$
B. $2x - z = 5$
C. $x + y - z = 4$
D. $x - y = 1$
E. $2y - z = 3$
F. $x + y + z = 2$

A plane $\parallel$ to the x-axis means the coefficient of x in its equation is zero. *

Hence the only possibility is E, and both points above are only the plane with equation $2y - z = 3$.

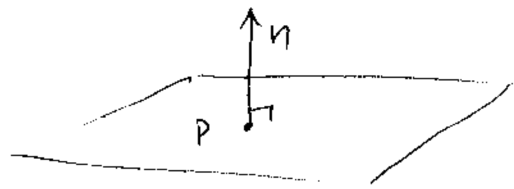


* If V is $\parallel$ to the x-axis, $n \perp (1,0,0)$, so if $n = (a,b,c)$, $n \cdot (1,0,0) = a = 0$.

3. Find an equation of the plane which passes through the point $(1, -7, 8)$ and which is perpendicular to the line whose parametric equations are:

$$x = -6 + 4t,\ y = 12 - 8t,\ z = -73 + 2t;\ t \in \mathbf{R}.$$

A. $2x - 4y + z = -38$
B. $2x - 4y + z = 38$
C. $2x + 7y - 3z = -71$
D. $2x - 4y + z = -28$
E. $-4x + 2y + z = -10$
F. $-4x + 2y + z = 10$



$P = (1, -7, 8)$ $\quad n = d = (4, -8\ 2)$

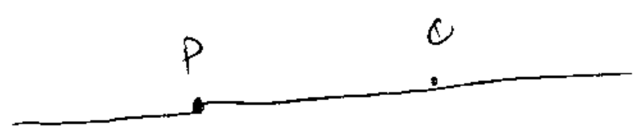
An eqn
$\therefore$ is $4(x-1) - 8(y+7) + 2(z-8) = 0$

or $4x - 8y + 2z = 76$ $\quad$ OR $2x - 4y + z = 38$

P $\qquad$ Q

4. Parametric equations for the line containing $(3, -1, 4)$ and $(-1, 5, 1)$ are:

A. Such a line does not exist.
B. $x = 3 + 4t,\ y = 3 + 6t,\ z = 4 + 3t;\ t \in \mathbf{R}.$
C. $x = 1 - t,\ y = -1 - 6t,\ z = 4 + 3t;\ t \in \mathbf{R}.$
D. $x = 3 + 4t,\ y = -1 - 6t,\ z = 6 + t;\ t \in \mathbf{R}.$
E. $x = 3 + 4t,\ y = -1 - 6t,\ z = 4 + 3t;\ t \in \mathbf{R}.$
F. $x = 1 - t,\ y = 3 + 6t,\ z = 6 + t;\ t \in \mathbf{R}.$



$d = Q - P = (-4, 6, -3)$

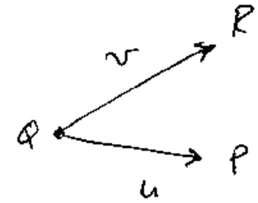
or $P - Q = (4, -6, 3)$

The only line with a direction vector parallel to $(4, -6, 3)$ is E, which does indeed pass through $(3, -1, 4)$.

P Q R

5. An equation of the plane containig the points $(6,\ -1,\ 5)$, $(7,\ 2,\ -4)$ and $(1,\ 1,\ 5)$ is:

A. $12x + 15y + 8z - 46 = 0$
B. $12x + 15y - 8z + 23 = 0$
C. $18x - 45y + 17z + 148 = 0$
D. $18x + 45y + 17z = 148$
E. $6x + 15y + 5z - 46 = 0$
F. $6x + 15y + 5z = -46$



$u = P - Q = (-1, -3, 9)$
$v = R - Q = (-6, -1, 9)$

$$\therefore\ n = u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -3 & 9 \\ -6 & -1 & 9 \end{vmatrix} = (-18, -45, -17)$$

The only plane above with a normal parallel to n is D

(and $18(1) + 45(1) + 17(5) = 63 + 85 = 148$, so R belongs to the plane in D.)

6. Find all vectors in $\mathbf{R}^3$ which are perpendicular to both $(-1,1,5)$ and $(2,1,2)$.

A. $\{(2,\ -8,\ 2)\}$
B. $\{(t+1,\ -8,\ t+1) \mid t \in \mathbf{R}\}$
C. $\{(t,\ -4t,\ t) \mid t \in \mathbf{R}\}$
D. $\{(-t,\ 0,\ t) \mid t \in \mathbf{R}\}$
E. $\{(0,\ 0,\ 0)\}$
F. $\{(3,\ -12,\ 3)\}$

Any multiple of $u \times v$ is perpendicular to u and v, and if a vector is perpendicular to both u & v, it is a multiple of $u \times v$;

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 5 \\ 2 & 1 & 2 \end{vmatrix} = (-3, -(-12), -3)$$

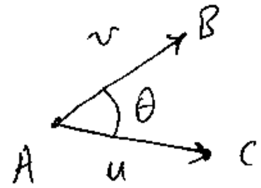
$= (-3, 12, -3)$

$= -3(1, -4, 1).$

Hence C is the correct soln.

7. A triangle has vertices $A = (1,\ 1,\ 1)$, $B = (2,\ 3,\ 1)$ and $C = (1,\ 2,\ 3)$. Find the cosine of the interior angle at A.

A. 0
B. 1/5
C. 2/5
D. 3/5
E. 4/5
F. 1



$$\cos\theta = \frac{u \cdot v}{\|u\|\,\|v\|}$$

$$u = C - A = (0, 1, 2)$$
$$v = B - A = (1, 2, 0)$$

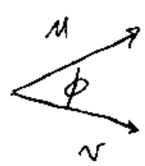
$$\therefore\ u \cdot v = 2$$
$$\|u\| = \sqrt{0^2 + 1^2 + 2^2} = \sqrt{5}$$
$$\|v\| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$$

$$\therefore\ \cos\theta = \frac{2}{\sqrt{5}\cdot\sqrt{5}} = \frac{2}{5}$$

8. Find the angle between the vectors $\overset{u}{(0,\ 3,\ 4)}$ and $\overset{v}{(5\sqrt{2},\ -7,\ -1)}$.

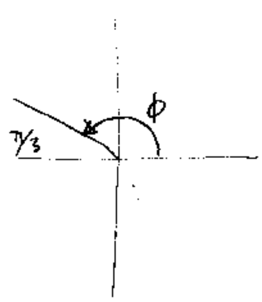
A. $\pi/3$
B. $\pi/6$
C. $2\pi/3$
D. $3\pi/4$
E. $5\pi/6$
F. π

$$\cos\phi = \frac{u \cdot v}{\|u\|\,\|v\|} = \frac{-21-4}{5 \cdot 10} = \frac{-25}{50} = -\frac{1}{2}$$



$$\|u\| = \sqrt{3^2+4^2} = 5$$
$$\|v\| = \sqrt{(5\sqrt{2})^2 + (-7)^2 + (-1)^2}$$
$$= \sqrt{50 + 49 + 1} = 10$$

$$\therefore\ \phi = \pi - \pi/3$$
$$= 2\pi/3$$



9. If $u = (3,\ 3,\ 6)$ and $v = (2,\ -1,\ 1)$ then the length of the projection of u along v is:

A. $(3\sqrt{6})/2$
B. $(3\sqrt{2})/2$
C. 0
D. $\sqrt{6}/2$
E. $(2\sqrt{6})/3$
F. $(2\sqrt{2})/3$

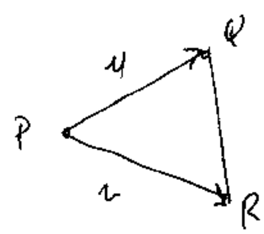
$$w = \text{Proj}_v u = \frac{u \cdot v}{\|v\|^2} \cdot v$$

$$= \frac{9}{\left(\sqrt{4+1+1}\right)^2} v = \frac{9}{6} \cdot v = \frac{3}{2} v$$

$$\therefore \|w\| = \frac{3}{2} \cdot \|v\| = \frac{3}{2}\sqrt{4+1+1} = \frac{3\sqrt{6}}{2}$$

10. Find the area of the triangle whose vertices are the points $P = (3,\ -1,\ 2)$, $Q = (1,\ 1,\ 0)$ and $R = (1,\ 2,\ -1)$.

A. 4
B. $2\sqrt{2}$
C. $\sqrt{2}$
D. 0
E. $4\sqrt{2}$
F. 2



$$\text{Area} = \frac{1}{2}\|u \times v\|$$

$$= \frac{1}{2} \cdot 2\sqrt{2} \quad \text{(from below)}$$

$$u = Q - P = (-2, 2, -2)$$

$$v = R - P = (-2, 3, -3)$$

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & -2 \\ -2 & 3 & -3 \end{vmatrix} = (0, -2, -2)$$

$$\therefore \|u \times v\| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$13+32=45$

11. Write the complex number

$$\frac{(16+13i)(1+2i)}{10+5i} = \frac{((16-26)+45i).(10-5i)}{100+25}$$

in the form a + i b.

(A) $1+4i$
B. 3
C. $1-4i$
D. $4i$
E. $(1/5)+(4/5)i$
F. $-(1/5)+(4/5)i$

$$= \frac{(-10+45i)(10-5i)}{125}$$

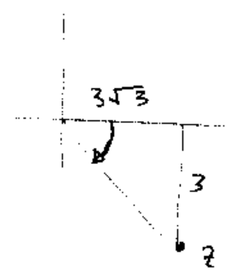
429
5
$145)$

(divide top & bottom by 25)

$$= \frac{(-2+9i)(2-i)}{5}$$

$$= \frac{5+20i}{5} = 1+4i$$

12. What is the polar form of $3\sqrt{3}-3i$? $= re^{i\theta} = z$



A. $36(\cos(\pi/6)+i\sin(\pi/6))$
(B) $6(\cos(-\pi/6)+i\sin(-\pi/6))$
C. $36(\cos(-\pi/6)+i\sin(-\pi/6))$
D. $6(\cos(\pi/6)+i\sin(\pi/6))$
E. $36(\cos(-\pi/3)+i\sin(-\pi/3))$
F. $6(\cos(\pi/3)+i\sin(\pi/3))$

$r = |z| = \sqrt{(3\sqrt{3})^2+9}$

$= \sqrt{36}$

$= 6$

$\cos\theta = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$

$\therefore \theta = -\pi/6$

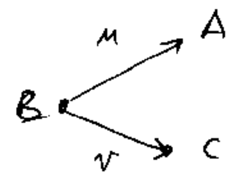
$\sin\theta = \frac{-3}{6} = -\frac{1}{2}$

$\therefore z = 6e^{-i\pi/6} = 6(\cos(-\pi/6) + i\sin(-\pi/6))$

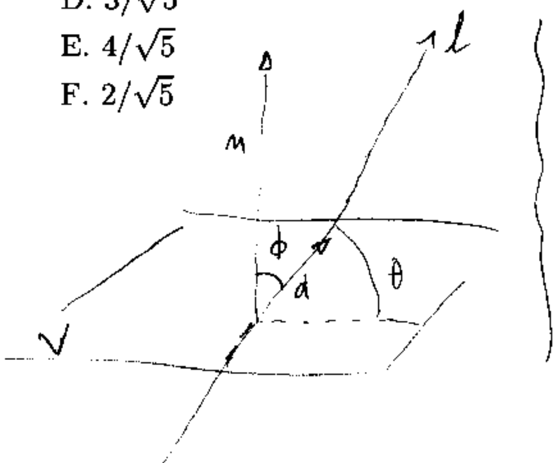
13 (bonus). Suppose $A = (2,\ 4,\ 1)$, $B = (3,\ 0,\ 9)$, $C = (1,\ 4,\ 0)$ and $D = (2,\ 6,\ 2)$. Find the cosine of the angle between the line $\overleftrightarrow{AD}$ and the plane containing A, B and C.

Ⓐ $1/\sqrt{5}$
B. $8/\sqrt{5}$
C. $6/\sqrt{5}$
D. $3/\sqrt{5}$
E. $4/\sqrt{5}$
F. $2/\sqrt{5}$

Plane containing A, B, C : V



$n = u \times v$



Direction vector for l:

$D - A = (0, 2, 1)$

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 4 & -8 \\ -2 & 4 & -9 \end{vmatrix}$$

$$= (-4, -(-7), 4)$$

$$= (-4, 7, 4)$$

$$\cos\phi = n \cdot d / \|n\| \cdot \|d\|$$

$$= \frac{u \times v \cdot d}{\|u \times v\| \cdot \|d\|} \quad (= \sin\theta)$$

$\therefore\ u \times v \cdot d = (-4, 7, 4) \cdot (0, 2, 1) = 18$

$\|u \times v\| = \sqrt{16 + 49 + 16} = \sqrt{81} = 9$

$\|d\| = \sqrt{0 + 4 + 1} = \sqrt{5}$

$$\therefore\ \frac{u \times v \cdot d}{\|u \times v\| . \|d\|} = \frac{18^2}{\sqrt{5}\,9} = \frac{2}{\sqrt{5}}$$

$$\therefore\ \cos\theta = \sqrt{1 - \left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{1 - \frac{4}{5}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} \quad \left(= \frac{\sqrt{5}}{5}\right)$$