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Modern Geometries

The History of Geometry
Part three

Contents:

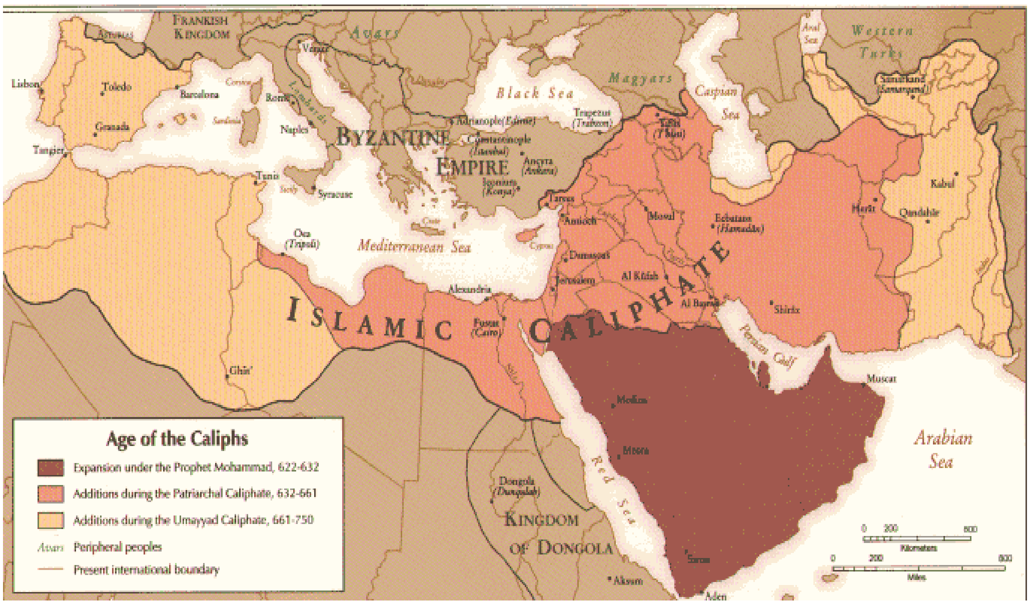
1. Different paths: Geometry development

We dedicated part two to Euclid's Fifth Postulate and the birth of non Euclidean Geometries, but Geometry developed in many other directions.
Let's see an elemental outline of some of them.

2. Islamic Geometry (c. 700 - 1500)

The Arabian domination began with the escape of Mahomet from La Meca to Medina 622 A.D.

The Islamic Caliphate (Islamic Empire) was established across the Middle East, North Africa, Spain, Portugal, Afghanistan and parts of Pakistan. It began around 640 CE.



In 641 Alexandria fell under Arabian power and Baghdad turn into the new Alexandria.

Interesting works from India and Greece were translated into Arabian.

During this period Islamic Mathematics were primarily algebraic rather than geometric, though there were important works on Geometry. Scholarship in Europe declined. Hellenistic works of antiquity were lost to them, and survived only in the Islamic centres of learning.

Let's see some Islamic mathematicians:

- **Thabit ibn Qurra** (born 836) contributed with theorems in **Spherical Trigonometry, Analytic Geometry, and non-Euclidean Geometry.**

- **Ibrahim ibn Sinan** (born 908), who introduced a more general **method of integration** than that of Archimedes, and **al-Quhi** (born 940), who investigated the **optical properties of mirrors made from conic sections**.

■ **Omar Khayyám** (born 1048 in Persia) mathematician and poet was a forerunner of **Algebraic Geometry** which combines Algebra and Geometry.

■ **Sharafeddin Tusi** (born 1135 in Persia) followed Khayyam's application of Algebra to Geometry. He wrote a treatise on cubic equations, which represents an essential contribution to another algebra which aimed to study curves by means of equations, thus **inaugurating the study of Algebraic Geometry.**

3. Geometry development

During the XV century, this development and the maturation of Symbolic Calculus placed to Descartes and Fermat in an advantageous position that they took up to translate classical problems of Greek Geometry into algebraic language and created Analytic Geometry or geometry with coordinates and equations.

4. Pier Fermat (1601-1665)

He is one of the most relevant French Mathematicians
This is his famous last theorem:

The equation:
$x^n + y^n = z^n$
has not integer solutions for n> 2

It was found written on the margin of "The Arithmetic" by Diofanto. He did not give the demonstration because "it did not fit in the book margin". **Andrew Wiles** finally proved Fermat's Last Theorem in 1995.

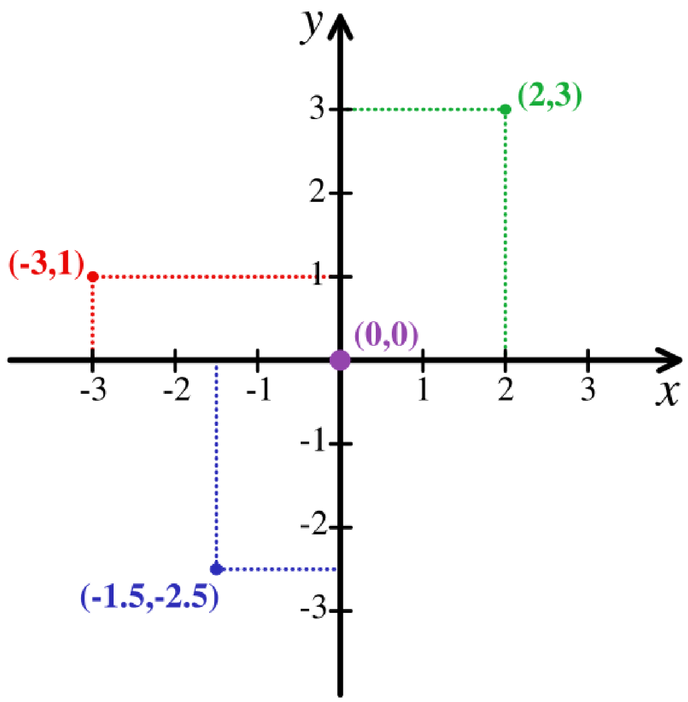
5. René Descartes (1596-1650)

He was notable French philosopher and mathematician. He was one of the key thinkers of the Scientific Revolution in the Western World.
"I think, therefore I exist" is the fundament of his philosophy
He worked to merge Algebra and Euclidean Geometry. He was influential in the development of Analytic Geometry, Calculus, and Cartography.
He introduced his famous coordinates system in part two of his "Discourse on Method" and in "La Géométrie".

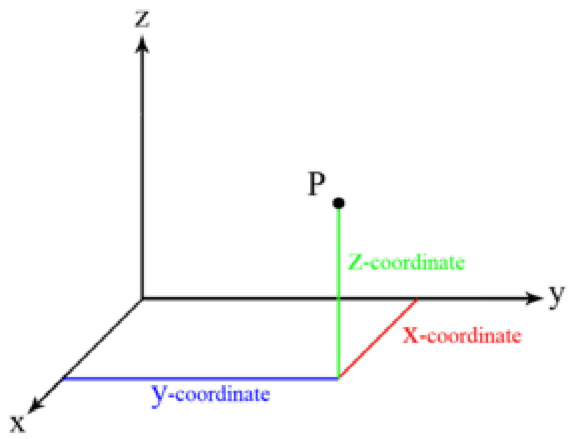
Cartesian coordinate system uniquely determine each point in the plane through two numbers: the *x-coordinate* and the *y-coordinate* of the point, (x, y).

To define the coordinates, we need:

- two perpendicular directed lines (the *x-axis* or *abscissa* and the *y-axis* or *ordinate*)
- the unit length, which is marked off on the two axes



Cartesian coordinate systems are also used in space (where three coordinates are used) and in higher dimensions.



4. The XVII century

A new world of possibilities was opened for Geometry as well as Algebra.
This interaction stimulated strongly the Infinitesimal Analysis.

With the help of Infinitesimal Analysis, created by Newton and Leibnitz, Geometry turned into an essential tool to study Mechanic, Optic and other parts of Physics.

Sir Isaac Newton (1642 –1727)
English mathematician and physicist

Gottfried W. Leibnitz (1646-1716)
German mathematician

This conjunction of topics and people (Fermat, Descartes, Pascal, Newton, Lebnitz, Huygens, The Bernoullis ...) makes the XVII century a privileged one in the development of the mathematical thinking.

7. The Projective Geometry

But Geometry did not stopped here. In the same century, XVII, with Descartes and Pascal a new type of Geometry was growing.
This Geometry began in Art, with the study of the perspective.

The Last Dinner by Leonardo Da Vinci

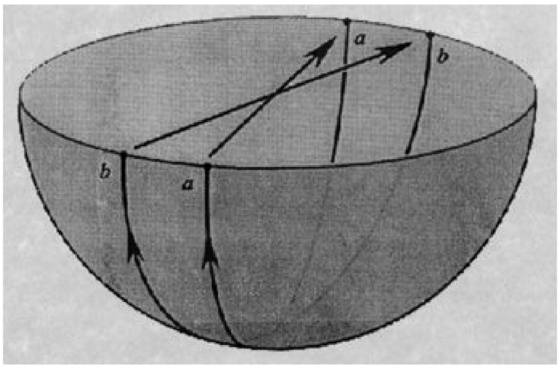
This Geometry **takes apart** the problems in which **measure of distances and angles** plays the main role.

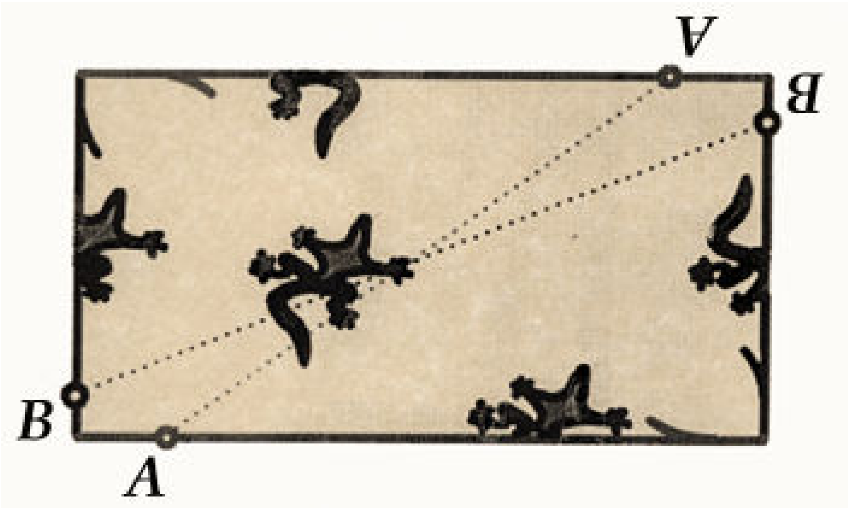
It tries the ones, for instance, in which the **properties of a plane figure are preserved by a projection** from an exterior point to this plane and cut by other plane in the space.

This Projective Geometry reached its top with Poncelet, Steiner, Staudt ... in the XIX century.

8. A model for the real projective plane

- Take the straight lines crossing a point O in the space.
- Each one cut the unit sphere in two antipodal points.
- Identify these two points.





Artistic representation

Lizard moving through a projective plane

9. Geometry posterior evolution

In the XIX century a new idea of what a Geometry is appeared.
It was exposed by Riemann in 1854 in "Over the hypotheses that are under Geometry Fundaments".

Bernhard Riemann (1826-1866)
German Mathematician

10. Riemann's results

To construct a Geometry is necessary to give:

- A manifold of elements
- Its coordinates
- The law that measures the distance between two infinitely nearby elements

This notion can be found in the work of Ricci- Curbastro and Levi-Civita in the XX century.

It is also introduced the Riemannian curvature, k, notion that associates to each tangent plane to a point an scalar.

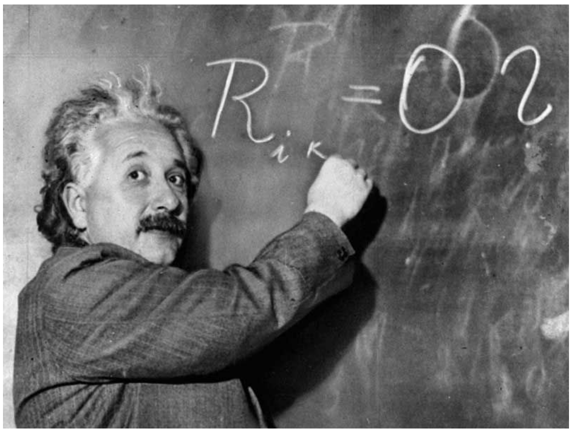
- If this curvature, k, is zero the Riemannian manifold is said to be a parabolic manifold.
- If k is constant and greater than zero it is said to be elliptic manifold.
- If k is constant and less than zero it is said to be hyperbolic manifold.

This mathematical model contains most of geometrical models studied until the XX century. The Euclidean spaces can be considered as Riemannian manifolds whose Riemannian curvature is zero. Spheres an projective planes are specials cases of hyperbolic Riemannian manifolds ones. Surfaces and hipersurfaces of Euclidean spaces also admit a Riemannian manifold structure.

The German mathematician Klein gave a model the generalized Riemann's. There are other models that generalized Riemann's and Klein's.

11. The Relativity Theory

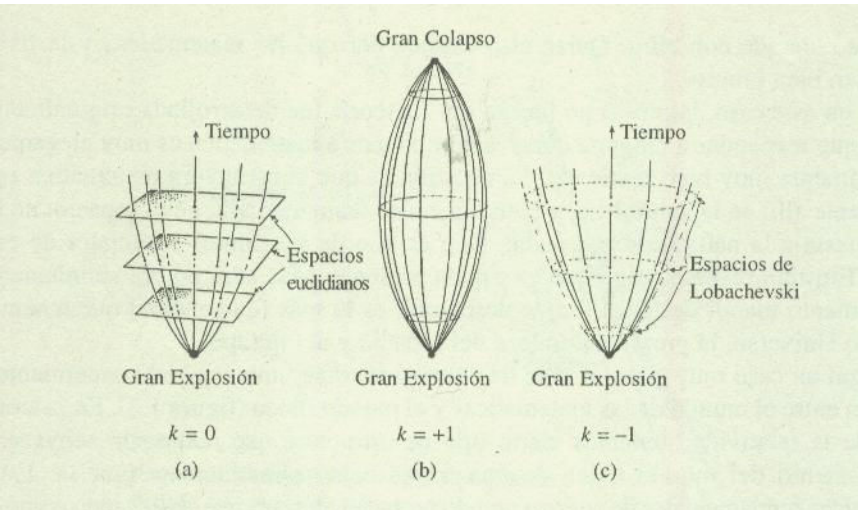
We will point that the Generalized Relativity Theory (Einstein dedicated 8 years of his life to this) is modelled over four dimensional manifolds with constant curvature. This notion is not Riemannian curvature. The fourth dimension is the time and the others are the three dimensions of the space.



Albert Einstein (Ulm 1879 – Princeton 1955)
German physicist

If K=0 (that was called by Einstein cosmological constant) then we can describe the three types of Universe through a parameter k, related to the curvature of the spatial sections of the Universe, with k= -1,0,+1.These values are -1, 0,+1, because the rest of properties are described in a convenient scale (if we talk about the age of Universe then will get a continuous parameter).

- This fact leaves three possibilities opened:



(b)**One Elliptic, where the Universe is finite and closed.** Its spatial sections have positive curvature, then **k=+1.** It has spherical sections, a Big Bang (The Beginning) and a Big Collapse (The End!!).

Two ways of non closed Universe with the Big Bang as The Beginning:

- **Parabolic** or pseudoeuclidean which spatial sections are flat, the curvature is zero and then **k=0.** The sections are Euclidean spaces.

- **Hyperbolic** which spatial sections have negative curvature, then **k=-1.** The sections are Lobachevski's spaces.

12. David Hilbert (1862, Wehlau, Prussia–1943, Göttingen, Germany)

Nowadays the most accepted axiomatic system is the one given by Hilbert in "Geometry Fundaments" in 1899. It uses:

- Three objects that doesn't need definition: point, straight line and plane.

- To express relationship it uses three primitive words: belong, between and congruent.

It do not differentiate between postulates an common notions.

Hilbert was recognized as one of the most influential mathematicians of the 19th and early 20th centuries

13. More modern Geometries

The more recent development of Geometry has followed several ways. At the end of last century were developed several ways of Analytic Geometry evolution:

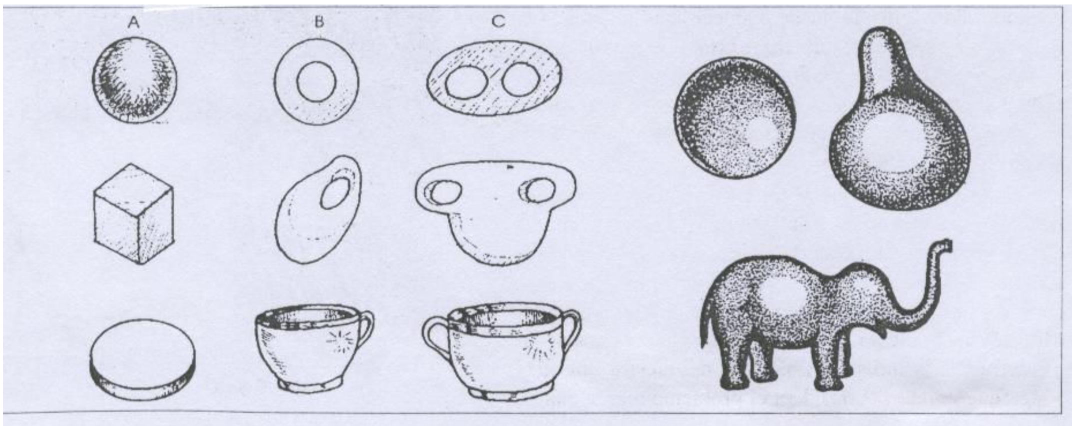
- **Algebraic Geometry** continued its development. As the name suggests, it combines abstract algebra, especially commutative algebra, with geometry. It is implicated in Number Theory.

- **Differntial Geometry**, a fusion of Analysis and Geometry.

- **Topology** is another form of Geometry. In the beginning, this science treated about properties that figures preserved under isometries (transformations that preserves distances between points, this is transformations which deforms figures without breaking them as if they were made of plasticine)

14. Topology

It is called the geometry of plasticine. We will give an example of this. The surfaces before are topologically equivalent in vertical. You can deform them one into other if you think they are made out of plasticine. Try to imagine it!!



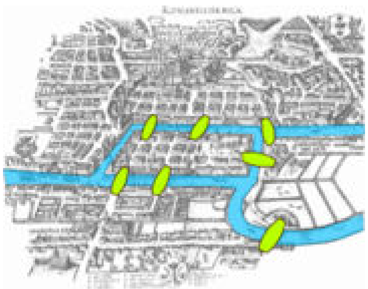
But we can tell many other things about this science. Topology is the geometry of the position in opposition to the usual geometry of the shape.

The motivating insight behind Topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the "way they are connected together".

14.1 The Seven Bridges of Königsberg Problem

The Seven Bridges of Königsberg, one of the most famous problems in topology and is the following:

Could you find a route through the town of Königsberg (now Kaliningrad) that crosses each of its seven bridges exactly once?



One of the first papers in Topology was the demonstration, by Leonhard Euler, that it was impossible. This result did not depend on the length of the bridges, nor on their distance from one another, but only on connectivity properties: which bridges are connected to which islands or riverbanks.

This problem, led to the branch of Mathematics known as Graph Theory.

14.2. The Möbius strip

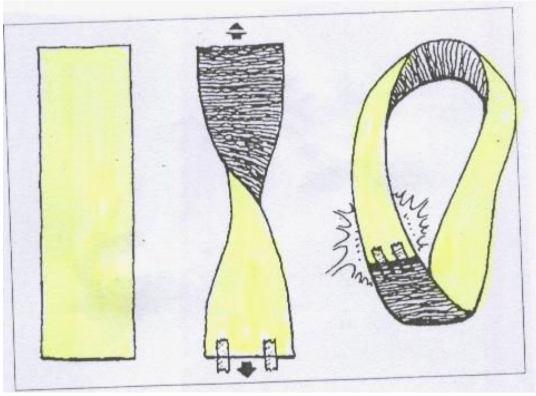
In 1858 Augustus F. Möbius (German mathematician and astronomer 1790-1868) defined the Möbius strip in his "Treatise on Polyhedroms". It was independently discovered by Johann Benedict Listing around the same time.

Möbius strip is a non-orientable two-dimensional surface with only one side when it is embedded in three-dimensional Euclidean space.

14.3. Construct a Möbius strip

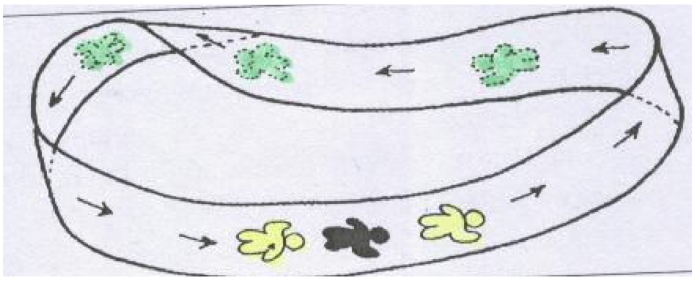
In order to construct a Möbius strip you have only to follow these steps:

- Take a strip of paper.
- Give it a half-twist,
- Then merge the ends of the strip together to form a single strip.

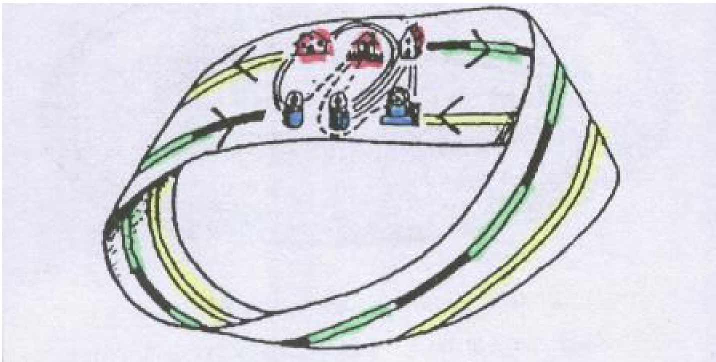


You will obtain this magic strip:

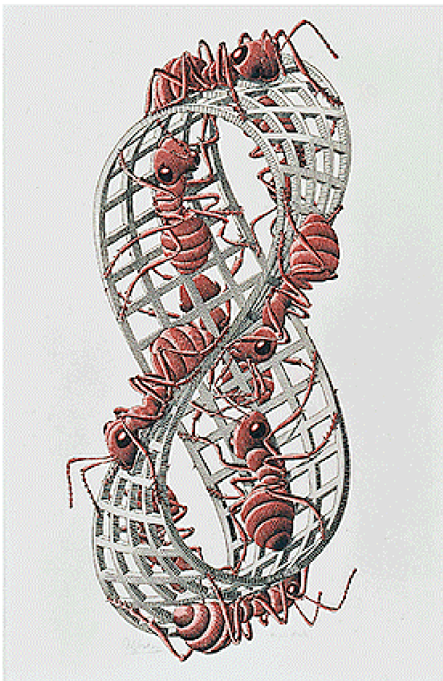
This magic object has amazing properties as a consequence of its non-orientability. For instance, the shape of a right hand turns into the shape of a left hand.



Other example is the following: In the Möbius strip there are paths that come from the backside.



This is a representation of Möbius strip by Esher, a Holland painter, very influenced by Mathematics.

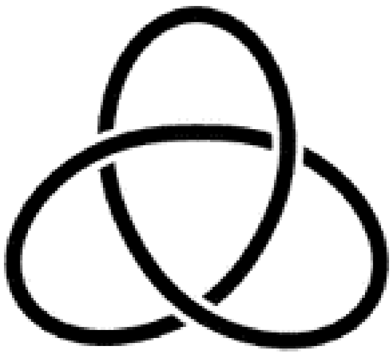


Xilography 1963

Observe the ants. When the ant is beginning to walk along the strip is up but when it comes back it is down. **There is not up and down!!!!!**

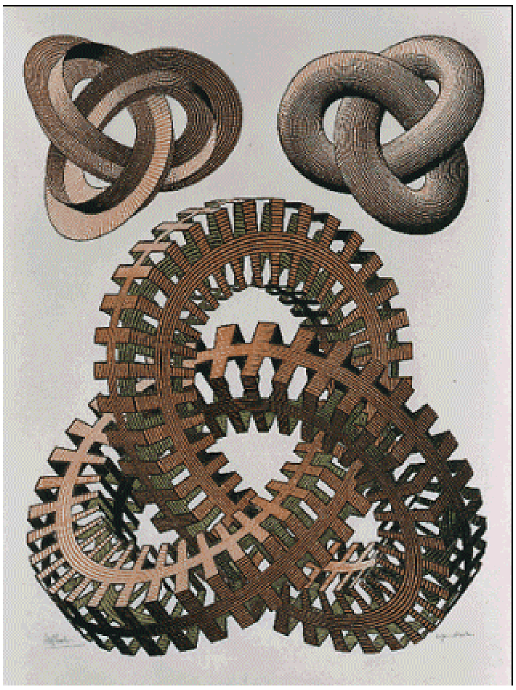
14.5. Knots

Knots are other topological objects: they are embeddings of the circle in three-dimensional space



Trefoil knot, the simplest non-trivial knot.

Here it is another artistic representation of a topological object by M.C. Escher:



There are many other interesting aspects we could see but it is enough for this year. I hope your curiosity about Geometry remain awoke forever.