/3 1. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a function and $c, L \in \mathbf{R}$. Provide the definition of $\lim_{x\to c} f(x) = L$. (This requires ϵ and δ.)

Solution: For all $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.

2. Let $b \in \mathbf{R}$ and define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = -x + b$ for all $x \in \mathbf{R}$. Prove
/5 that $\lim_{x\to 1} f(x) = -1 + b$ from the definition above (i.e. using an ϵ-δ argument.)

Solution: Suppose $\epsilon > 0$ and let $\delta = \epsilon$. Then

$$\begin{aligned} 0 < |x-1| < \delta &\Rightarrow |x-1| < \delta \\ &\Rightarrow -\delta < x - 1 < \delta \\ &\Rightarrow \delta > -(x-1) > -\delta \\ &\Rightarrow -\delta < -x + b - (-1+b) < \delta \\ &\Rightarrow |f(x) - (-1+b)| < \delta = \epsilon. \end{aligned}$$

3. Suppose $x \in \mathbf{R}$ and $x < 0$. Provide the definition for the <u>absolute value</u>
/3 $|x|$.

Solution: $|x|$ is equal to x if $x \geq 0$ and is equal to $-x$ if $x < 0$.

/3 4. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a function and $c \in \mathbf{R}$. Provide the definition for f to be <u>continuous at c</u>.

Solution: $\lim_{x\to c} f(x) = f(c)$.

/3 5. Provide an example of a function $f : \mathbf{R} \to \mathbf{R}$ which is **not** continuous at 1. You may use theorems to justify your answer.

Solution: The easiest one I can think of is defined by

$$f(x) \begin{cases} 1, & x \geq 1 \\ 0, & x < 1 \end{cases} .$$

This is discontinuous at 1 since

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 1 = 1 \neq 0 = \lim_{x\to 1^{-1}} 0 = \lim_{x\to 1^-} f(x)$$

and for $\lim_{x\to 1} f(x)$ to exist, one requires the limit from the right to equal the limit from the left.

/3 6. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a function and $c \in \mathbf{R}$. Provide the definition for f to be differentiable at c. (No ϵ and δ necessary.)

Solution: $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists. Another acceptable definition is that $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists.

/5 7. Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} x^2, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}.$$

Prove that f is differentiable at 0 and provide $f'(0)$.

Solution: For $h \neq 0$ we have

$$\begin{aligned} \frac{f(0+h)-f(0)}{h} &= \frac{f(h)}{h} \\ &= \begin{cases} h^2/h, & h \text{ rational} \\ 0/h, & h \text{ irrational} \end{cases} \\ &= \begin{cases} h, & h \text{ rational} \\ 0, & h \text{ irrational} \end{cases}. \end{aligned}$$

Let's prove $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0$. Suppose $\epsilon > 0$. Let $\delta = \epsilon$. Then if h is rational

$$\begin{aligned} 0 < |h-0| < \delta &\Rightarrow |h| < \epsilon \\ &= \left| \frac{f(0+h)-f(0)}{h} - 0 \right| < \epsilon. \end{aligned}$$

On the other hand if h is irrational then

$$\left| \frac{f(0+h)-f(0)}{h} - 0 \right| = 0 < \epsilon.$$

In both cases we obtain

$$\left| \frac{f(0+h)-f(0)}{h} - 0 \right| < \epsilon$$

so we are done.

/3 8. Provide an example of an interior point of the interval $(-2,3) \subset \mathbf{R}$. Justify your example.

Solution: There are many examples. The number 0 is an interior point because $(0-1, 0+1) = (-1,1)$ is an open interval contained in $(-2,3)$.

9. Define $f : [-3,4) \to \mathbf{R}$ by

$$f(x) = \begin{cases} |x+1|, & -3 \le x < 0 \\ x^2 - 4x + 2, & 0 \le x < 3 \\ 2x - 7, & 3 \le x < 4 \end{cases} .$$

/6 (a) Find the critical points of f.

Solution: The critical points are the interior points of $[-3,4)$ at which f' is zero or does not exist. Here they are:

- $x = -1$. Here the derivative does not exist since the limit of the quotient from the left is -1 and the limit from the quotient from the right is 1.
- $x = 0$. Here the function has a jump discontinuity, i.e. the limit from the left is 1 and the limit from the right is 2. Therefore f is not differentiable at 0.
- $x = 2$. For $0 < x < 3$, $f'(x) = 2x - 4$ and so $f'(2) = 0$.

The other points are not critical points. Here is why. For $-3 < x < -1$, $f(x) = -x - 1$ and $f'(x) = -1 \neq 0$. For $-1 < x0$, $f(x) = x + 1$ and $f'(x) = 1 \neq 0$. For $0 < x < 3$ and $x \neq 2$, $f'(x) = 2x - 4 \neq 0$. For $3 < x < 4$, $f'(x) = 2 \neq 0$. For $x = 3$ the limit of the quotient from the left is equal to 2 and this is equal to the limit from the right. So the derivative exists at $x = 3$, but is not equal to zero. $x = -3$ is the only remaining point in the domain of f, but it is not an interior point.

/4 (b) Determine the absolute maximum and the absolute minimum values of f, provided they exist.

Solution: The absolute maxima and minima occur either at critical points or at endpoints. We have $f(-3) = 2$, $f(-1) = 0$, $f(0) = 2$, and $f(2) = -2$. For $3 \le x < 4$ we have $6 \le 2x < 8$ and

$$-1 \le 2x - 1 = f(x) < 1.$$

Therefore f has an absolute maximum $x = -3$ with maximum value $f(-3) = 2$. The absolute minimum is at $x = 2$ and the minimum value is $f(2) = -2$.

/3 10. Evaluate $\lim_{x\to 0} \frac{2x^2+x}{\sin(x)}$.

Solution:

$$\lim_{x\to 0} \frac{2x^2 + x}{\sin(x)} = \lim_{x\to 0} 2x\frac{x}{\sin(x)} + \frac{x}{\sin(x)}.$$

We know that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Using the reciprocal rule for limits (Theorem 2.3.7), we have $\lim_{x\to 0} \frac{x}{\sin(x)} = 1$. The other rules for limits concerning sums and products (Theorem 2.3.2) then give us

$$\lim_{x\to 0} 2x\frac{x}{\sin(x)} + \frac{x}{\sin(x)} = \lim_{x\to 0} 2x \lim_{x\to 0} \frac{x}{\sin(x)} + \lim_{x\to 0} \frac{x}{\sin(x)} = (0)(1) + (1) = 1.$$

11. Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable and define $g(x) = (f(x))^2$ for all $x \in \mathbf{R}$.

/3 (a) Compute the derivative g'.

Solution: Using the Chain Rule we have

$$g'(x) = 2f(x)\, f'(x),$$

for all $x \in \mathbf{R}$.

/4

(b) Prove that if f is increasing and $f(x) > 0$ for all $x \in \mathbf{R}$ then g is not a decreasing function. (Careful! f increasing does not imply that $f'(x) > 0$ for all $x \in \mathbf{R}$.

Solution: We prove this by contradiction. Assume g is decreasing. Then there exist $x, y \in \mathbf{R}$ such that $x < y$ and $g(x) > g(y)$. Observe that g is differentiable (and so continuous as well). So by the Mean Value theorem there exists $x < c < y$ such that

$$g'(c) = \frac{g(x) - g(y)}{x - y}.$$

The numerator on the right is positive since $g(x) > g(y)$. The denominator on the right is negative since $x < y$. Therefore $g'(c) < 0$. By part (a) and the hypothesis that $f(c) > 0$ we conclude that $f'(c) < 0$. This implies that $f(c) > f(c + h)$ for some small $h > 0$ (Theorem 4.1.2), and this contradicts that f in increasing.

(P.S. A difficult question. There are other ways to prove this. This is the one that is most natural for me personally.)

12. Indicate whether each of the following statements below is true or false. Provide justification for each of your indications.

/3

(a) Suppose $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are both continuous functions. Then the composition $f \circ g$ has a maximum value on $[-1, 5]$.

Solution: True. The composition $f \circ g$ is a continuous function (Theorem 2.4.4). Therefore, by the Extreme Value Theorem, $f \circ g$ has a maximum (and a minimum) on $[-1, 5]$ (in fact on any closed and bounded interval).

/3

(b) Suppose $P(x) = x^3 - 200000x - 10000$ for all $x \in \mathbf{R}$. Then P has an absolute minimum.

Solution: False. In Assignment 10 we proved that $\lim_{x\to-\infty} P(x) = -\infty$. Suppose by way of contradiction that P has an absolute minimum value M. Then there exists $K < 0$ such that $P(x) < M$ for every $x < K$, and this is a contradiction to M being an absolute minimum value.

/3 (c) $\lim_{x\to\pi}(x-\pi)\cos(1/(x-\pi)) = 0$.

Solution: True. Recall that $|\cos(a)| \leq 1$ for all $a \in \mathbf{R}$. In particular $|\cos(1/(x-\pi))| \leq 1$ for every $x \neq \pi$. It follows that $0 \leq |(x-\pi)\cos(1/(x-\pi))| \leq |(x-\pi)|$. The Pinching Theorem tells us that $\lim_{x\to\pi} |(x-\pi)\cos(1/(x-\pi))| = 0$ and a question from Assignment 1 then implies that $\lim_{x\to\pi}(x-\pi)\cos(1/(x-\pi)) = 0$.

/3 (d) Suppose f is a function, $A, B \in \mathbf{R}$, and

$$\lim_{x\to c^-} f(x) = A < B = \lim_{x\to c^+} f(x)$$

for some $c \in \mathbf{R}$. Then there exists $\delta > 0$ such that

$$f(x) < f(y)$$

for all $x < c < y$ satisfying $y - x < \delta$.

Solution: True. Let $\epsilon = \frac{B-A}{2} > 0$. Since $\lim_{x\to c^-} f(x) = A$, there exists $\delta_1 > 0$ such that for all $x \in (c-\delta_1, c)$

$$\begin{aligned} |f(x) - A| < \epsilon \;\; &\Rightarrow \;\; -\frac{B-A}{2} < f(x) - A < \frac{B-A}{2} \\ &\Rightarrow \;\; f(x) < \frac{B+A}{2}. \end{aligned}$$

Similarly, $\lim_{x\to c^+} f(x) = B$ gives us a $\delta_2 > 0$ such that for all $x \in (c, c+\delta_2)$

$$\begin{aligned} |f(x) - B| < \epsilon \;\; &\Rightarrow \;\; -\frac{B-A}{2} < f(x) - B < \frac{B-A}{2} \\ &\Rightarrow \;\; \frac{B+A}{2} < f(x). \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $x < c < y$ and $y - x < \delta$ we have $(y-c)+(c-x) < \delta$, so that both $y-c < \delta \leq \delta_2$ and $c-x < \delta \leq \delta_1$. It follows from the above that

$$f(x) < \frac{B+A}{2} < f(y).$$

(P.S. Draw a picture. A correct answer deserves an A+ on the exam!)