Section 10.4. Velocity and Acceleration

Definition If a particle moves through space with position vector $\vec{r}(t)$, then the **velocity** of the particle at time t is

$$\vec{v}(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \vec{r}'(t)$$

The **speed** of a particle, $\frac{ds}{dt}$, is the magnitude of the velocity vector. In symbols,

$$\frac{ds}{dt} = |\vec{v}(t)|$$

The **acceleration,** $\vec{a}(t)$, is the derivative of velocity.

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$$

Example The position of a particle in the plane is given by $\vec{r}(t) = t^2\vec{\imath} + t^3\vec{\jmath}$. Find it's velocity, speed and acceleration when $t = -1$ and illustrate geometrically.

Solution: The velocity at time t is

$$\vec{v}(t) = \vec{r}'(t) = 2t\,\vec{\imath} + 3t^2\,\vec{\jmath}$$

and its acceleration at time t is

$$\vec{a}(t) = \vec{v}'(t) = 2\,\vec{\imath} + 6t\,\vec{\jmath}.$$

So at $t = -1$, $\vec{v}(-1) = -2\,\vec{\imath} + 3\,\vec{\jmath}$ and $\vec{a}(-1) = 2\,\vec{\imath} - 6\,\vec{\jmath}$.

Further, the speed at $t = -1$ is

$$\frac{ds}{dt} = |\vec{v}(-1)| = |-2\,\vec{\imath} + 3\,\vec{\jmath}| = \sqrt{13}$$

Example The position of a particle in the plane is given by $\vec{r}(t) = e^{7t}\vec{\imath} + te^{7t}\vec{\jmath} + \ln(t+1)\,\vec{k}$. Find its velocity, speed and acceleration.

Solution: The velocity at time t is

$$\vec{v}(t) = \vec{r}'(t) = 7e^{7t}\vec{\imath} + (e^{7t} + 7te^{7t})\vec{\jmath} + \frac{1}{t+1}\vec{k}$$

The acceleration at time t is

$$\vec{a}(t) = \vec{v}' = 49e^{7t}\vec{\imath} + (14e^{7t} + 49te^{7t})\vec{\jmath} - (t+1)^{-2}\vec{k}$$

The speed at time t is

$$\frac{ds}{dt} = |\vec{v}(t)| = \sqrt{49e^{14t} + (e^{7t} + 7te^{7t})^2 + (t+1)^{-2}}$$

Example A moving particle starts with initial position $\vec{r}(0) = \langle 1,0,0 \rangle$, initial velocity $\vec{v}(0) = \langle 1,2,0 \rangle$, and acceleration $\vec{a}(t) = \langle 6t, \cos t, e^{-t} \rangle$. Find $\vec{r}(t)$.

Solution: Since $\vec{a}(t) = \vec{v}'(t)$, it follows that

$$\vec{v}(t) = \int \vec{a}(t)\, dt = \langle 3t^2, \sin t, -e^{-t} \rangle + \vec{C}$$

Note that the constant of integration, $\vec{C}$, when you integrate a vector function is a constant vector. To determine $\vec{C}$ above, substitute $t = 0$ into both sides of the equality. We get

$$\langle 1,2,0 \rangle = \vec{v}(0) = \langle 0,0,-1 \rangle + \vec{C}$$

Therefore,

$$\vec{C} = \langle 1,2,0 \rangle - \langle 0,0,-1 \rangle = \langle 1,2,1 \rangle$$

Substituting $\vec{C} = \langle 1,2,1 \rangle$ in $\vec{v}(t) = \langle 3t^2, \sin t, -e^{-t} \rangle + \vec{C}$, we get

$$\vec{v}(t) = \langle 3t^2, \sin t, -e^{-t} \rangle + \langle 1,2,1 \rangle$$

Therefore,

$$\vec{v}(t) = \langle 3t^2 + 1, \sin t + 2, -e^{-t} + 1 \rangle$$

To get $\vec{r}(t)$, we note that since $\vec{r}'(t) = \vec{v}(t)$, then

$$\vec{r}(t) = \int \vec{v}(t)\, dt = \int \langle 3t^2 + 1, \sin t + 2, -e^{-t} + 1 \rangle\, dt$$

Therefore,

$$\vec{r}(t) = \langle t^3 + t, -\cos t + 2t, e^{-t} + t \rangle + \vec{C}$$

To determine $\vec{C}$ this time, we again substitute $t = 0$ into both sides of the equality. We get

$$\langle 1,0,0 \rangle = \vec{r}(0) = \langle 0, -1, 1 \rangle + \vec{C}$$

Therefore,

$$\vec{C} = \langle 1,1,-1 \rangle$$

We get,

$$\vec{r}(t) = \langle t^3 + t, -\cos t + 2t, e^{-t} + t \rangle + \langle 1,1,-1 \rangle$$

Or

$$\vec{r}(t) = \langle t^3 + t + 1, -\cos t + 2t + 1, e^{-t} + t - 1 \rangle$$

Example A projectile is fired with angle of elevation α and initial scalar velocity v_0. Assume that the only external force on the projectile is gravity. (a) Find its position vector $\vec{r}(t)$. (b) What value of α maximizes the horizontal distance the projectile travels?

Solution: We have

$$\vec{a}(t) = -g\,\vec{j}$$

where $g = 9.8 \text{ m/s}^2$ is the acceleration due to gravity.

Therefore,

$$\vec{v}(t) = -gt\vec{j} + \vec{v}_0$$

where $\vec{v}_0$ is the initial velocity vector. Since the initial velocity vector is $v_0 \cos\alpha\,\vec{\imath} + v_0 \sin\alpha\;\vec{\jmath}$, we get

$$\vec{v}(t) = v_0 \cos\alpha\,\vec{\imath} + (v_0 \sin\alpha - gt)\,\vec{\jmath}$$

Therefore, the position vector is

$$\vec{r}(t) = (v_0 \cos\alpha\,)t\,\vec{\imath} + \left((v_0 \sin\alpha\,)t - \frac{1}{2}gt^2\right)\vec{\jmath} + \vec{r}_0$$

Since $\vec{r}_0 = \vec{0}$, we get the following fact.

True Fact: The position vector of a projectile that is fired with angle of elevation α and initial scalar velocity v_0 is given by

$$\vec{r}(t) = (v_0 \cos\alpha\,)t\,\vec{\imath} + \left((v_0 \sin\alpha\,)t - \frac{1}{2}gt^2\right)\vec{\jmath}$$

Or, in parametric form, the position of the object is as follows.

True Fact: The position of a projectile that is fired with angle of elevation α and initial scalar velocity v_0 is given by the parametric equations

$$x = (v_0 \cos\alpha\,)t \qquad y = \left((v_0 \sin\alpha\,)t - \frac{1}{2}gt^2\right)$$

(b) The horizontal distance the projectile travels is the value of x when its height is 0; i.e. when $y = 0$; i.e. when

$$t = \frac{2\, v_0 \sin\alpha}{g}$$

Thus, the horizontal distance the projectile travels is

$$d = \frac{2(v_0)^2 \sin\alpha\cos\alpha}{g} = \frac{(v_0)^2 \sin(2\alpha)}{g}$$

which is maximized when $\sin(2\alpha) = 1$, that is, when $\alpha = \pi/4$.

Note that in the above example we obtained the following additional fact.

True Fact: The horizontal distance traveled by a projectile fired with angle of elevation α and initial scalar velocity v_0 is given by

$$d = \frac{(v_0)^2 \sin(2\alpha)}{g}$$

The time it takes for the projectile to travel this horizontal distance is

$$t = \frac{2\, v_0 \sin\alpha}{g}$$

Example A gun is fired with angle of elevation 30°. What is the muzzle speed if the maximum height of the shell is 500m.

Solution: From the above true fact, the position of the shell at time t is

$$x = (v_0 \cos\alpha)t \qquad y = \left((v_0 \sin\alpha)t - \frac{1}{2}gt^2\right)$$

The height is maximized when

$$v_0 \sin\alpha - gt = \frac{dy}{dt} = 0$$

That is, when

$$t = \frac{v_0 \sin\alpha}{g}$$

Substituting this value of t into the parametric equation for y, we get the maximum height of the shell is

$$h = (v_0 \sin\alpha)\left(\frac{v_0 \sin\alpha}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin\alpha}{g}\right)^2$$

That is,

$$h = \frac{(v_0 \sin\alpha)^2}{2g}$$

Therefore, in this problem

$$500 = \frac{(v_0 \sin 30°)^2}{2g} = \frac{(v_0)^2}{8g}$$

Hence,

$$(v_0)^2 = 4000g = 39{,}200$$

That is, $v_0 = 198\ m/s$.

Note that in this problem we derived another fact,

True Fact: The maximum height reached by a projectile fired with angle of elevation α and initial scalar velocity v_0 is given by

$$h = \frac{(v_0 \sin \alpha)^2}{2g}$$