Section 10.3. Arc Length and Curvature

The arc length formula for a curve in three dimensions

$$x = f(t) \qquad x = g(t) \qquad z = h(t) \qquad a \le t \le b$$

for a curve that is traversed exactly once generalizes the formula obtained in Calculus I for a curve in two dimensions.

Theorem The arc length of the curve $x = f(t), y = g(t), z = h(t), a \le t \le b$, is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$$

In vector form the above formula becomes

$$L = \int_a^b |\vec{r}'(t)|\, dt$$

Example Find the arc length of $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$ from the point $\left(0, 1, \frac{\pi}{2}\right)$ to the point $(1, 0, 2\pi)$.

Curvature

Assume $\vec{r}(t)$ defines a smooth curve on an interval I (i.e. $\vec{r}'(t) \neq \vec{0}$ on I). Then the unit tangent is defined by

$$\mathbf{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

The instantaneous rate of change of the unit tangent with respect to arc length defines the curvature of the curve.

Definition The curvature of a curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

where **T** is the unit tangent.

By the chain rule,

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \cdot \frac{ds}{dt} = \mathbf{T}'(\mathrm{t}) \cdot \vec{r}'(t)$$

Therefore, we have the following formula for computing curvature.

$$\kappa(t) = \frac{|\mathbf{T}'(\mathrm{t})|}{|\vec{r}'(\mathrm{t})|}$$

Example Find the curvature of a circle of radius a.

The following formula is often more convenient for computing curvature.

The curvature of the curve defined by a vector function $\vec{r}$ is given by

$$\kappa(t) = \frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Example Find the curvature of the twisted cubic $\vec{r}(t) = \langle t, t^2, t^3\rangle$ at (0,0,0).

Note that the plane curve $y = f(x)$ has vector equation $\vec{r}(x) = x\vec{\imath} + f(x)\vec{\jmath}$. Then $\vec{r}'(x) = \vec{\imath} + f'(x)\vec{\jmath}$ and $\vec{r}''(x) = f''(x)\vec{\jmath}$. Hence, $\vec{r}(x) \times \vec{r}''^{(x)} = f''(x)\vec{k}$. Hence the curvature of $y = f(x)$ is

$$\kappa(x) = \frac{|f''(x)|}{|1 + (f'(x))^2|^{3/2}}$$

Example Find the curvature of $y = \sin x$ at $(0,0), \left(\pi/4, \sqrt{2}/2\right)$ and $(\pi/2, 1)$.