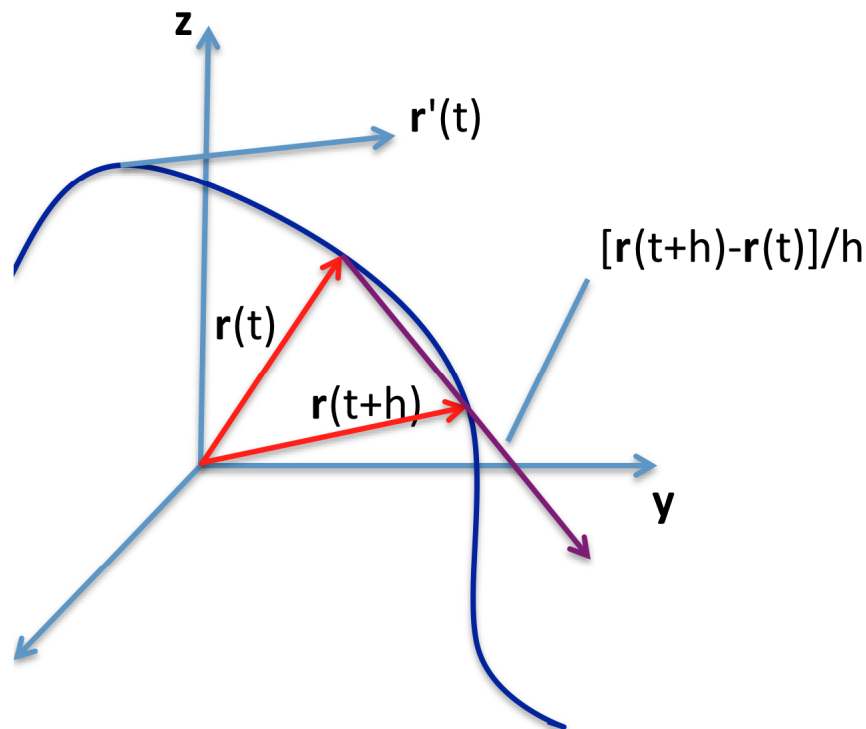


## Section 10.2. Derivatives and Integrals of Vector Functions

The **derivative** of a vector function  $\vec{r}$  is defined by

$$\vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

The geometric significance of the derivative is that the vector  $\vec{r}'(t)$  is tangent to the curve of  $\vec{r}$  at the terminal point of  $\vec{r}(t)$  as shown below.



Since  $\vec{r}'(t)$  (provided  $\vec{r}'(t) \neq \vec{0}$ ) is tangent to the curve defined by  $\vec{r}$ , we get a **unit tangent to the curve** defined by

$$\mathbf{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

The next formula follows from the above definition of the derivative of a vector function and the fact that limits are computed component wise.

**Theorem** If  $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$  and  $f, g, h$  are differentiable, then

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

**Example** Let  $\vec{r}(t) = (\tan t)\vec{i} + e^{t^2}\vec{j} + \ln(t + 1)\vec{k}$ .

- Find the derivative of  $\vec{r}(t)$ .
- Find the unit tangent vector at the point where  $t = 0$ .

**Example** Let  $\vec{r}(t) = \sqrt{t}\vec{i} + (2 + t)\vec{j}$ . Find  $\vec{r}'(t)$  and sketch the position vector  $\vec{r}(1)$  and the tangent vector  $\vec{r}'(1)$ .

**Example.** Find parametric equations of the tangent line to the curve  $x = e^{-t}, y = \sin t, z = (2t + 1)$  at the point  $(e^{-\pi/2}, 1, \pi + 1)$ .

**Definition** A curve defined by a function  $\vec{r}(t)$  is **smooth** on an interval  $I$  if  $\vec{r}(t)$  is continuous and  $\vec{r}'(t) \neq \mathbf{0}$  on  $I$ , except possibly at the endpoints of  $I$ .

**Example** Find the points on the curve defined by  $\vec{r}(t) = (\cos t)\vec{i} + e^{t^2}\vec{j} + \ln(1+t)\vec{k}$  where it is not smooth.

## Differentiation Rules

**Theorem** Let  $c$  be a constant,  $f$  be a differentiable real valued function and  $\vec{u}$  and  $\vec{v}$  differentiable vector functions. Then

$$1. \frac{d}{dt} (\vec{u}(t) + \vec{v}(t)) = \frac{d}{dt} (\vec{u}(t)) + \frac{d}{dt} (\vec{v}(t))$$

$$2. \frac{d}{dt} (c \vec{u}(t)) = c \frac{d}{dt} (\vec{u}(t))$$

$$3. \frac{d}{dt} (f(t) \vec{u}(t)) = f(t) \frac{d}{dt} (\vec{u}(t)) + f'(t) \vec{u}(t)$$

$$4. \frac{d}{dt} (\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}(t) \frac{d}{dt} (\vec{v}(t)) + \vec{v}(t) \frac{d}{dt} (\vec{u}(t))$$

$$5. \frac{d}{dt} (\vec{u}(t) \times \vec{v}(t)) = \vec{u}(t) \times \frac{d}{dt} (\vec{v}(t)) + \frac{d}{dt} (\vec{u}(t)) \times \vec{v}(t)$$

$$6. \frac{d}{dt} (\vec{u}(f(t))) = \vec{u}'(f(t)) \cdot f'(t)$$

**Example.** Show that if  $|\vec{r}(t)| = c$  (a constant), then  $\vec{r}'(t)$  is orthogonal to  $\vec{r}(t)$  for all values of  $t$ .

## Integrals

We can extend the definition of definite integrals via Riemann sums to vector functions. If

$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ , then

$$\int_a^b \vec{r}(t) dt = \int_a^b f(t)dt \vec{i} + \int_a^b g(t)dt \vec{j} + \int_a^b h(t)dt \vec{k}$$

**Example** Let  $\vec{r}(t) = \cos t \vec{i} + \sin t \vec{j} + t \vec{k}$ . Find  $\int_0^{\pi/2} \vec{r}(t) dt$ .