MATH 1005B
Test 1 Solutions

January 31, 2012

[Marks]

[6] 1. Solve the initial-value problem $2y' = \dfrac{\cos(x)}{y},\ y(0) = 2$.

Solution:
$2yy' = \cos(x) \ \Rightarrow\ y^2 = \sin(x) + c \ \Rightarrow\ y = \pm\sqrt{\sin(x) + c}$.
$y(0) = 2 \ \Rightarrow\ 2 = \sqrt{c} \ \Rightarrow\ c = 4 \ \Rightarrow\ y = \sqrt{\sin(x) + 4}$.

[6] 2. Find the general solution of $y' = \dfrac{x^2 + y^2}{xy}$.

Solution:
$y' = \dfrac{x^2 + y^2}{xy} = \dfrac{x}{y} + \dfrac{y}{x},\ u = \dfrac{y}{x} \ \Rightarrow\ u + xu' = \dfrac{1}{u} + u \ \Rightarrow\ xu' = \dfrac{1}{u} \ \Rightarrow\ uu' = \dfrac{1}{x} \ \Rightarrow$
$\dfrac{1}{2}u^2 = \ln|x| + c \ \Rightarrow\ u = \pm\sqrt{2\ln|x| + k} \ \Rightarrow\ y = \pm x\sqrt{2\ln|x| + k}$.

[6] 3. Find the general solution of $xy' + 3y = x + 1$.

Solution:
$y' + \dfrac{3}{x}y = \dfrac{x+1}{x} \ \Rightarrow\ I(x) = e^{\int \frac{3}{x}\,dx} = e^{3\ln|x|} = |x|^3 = \pm x^3$, and we may take $I(x) = x^3$.
Then $x^3y' + 3x^2y = x^3 + x^2 \ \Rightarrow\ (x^3y)' = x^3 + x^2 \ \Rightarrow\ x^3y = \dfrac{x^4}{4} + \dfrac{x^3}{3} + c \ \Rightarrow$
$y = \dfrac{x}{4} + \dfrac{1}{3} + \dfrac{c}{x^3}$.

[6] 4. Find an integrating factor which makes the equation $x + y^2 + xyy' = 0$ exact. Do not solve the equation.

Solution:
$P = x + y^2,\ Q = xy,\ P_y = 2y,\ Q_x = y,\ P_y \neq Q_x$, so the equation is not exact. $\dfrac{P_y - Q_x}{Q} = \dfrac{1}{x}$ is a function of x only, so an integrating factor $I(x)$ exists, determined by $\dfrac{I'(x)}{I(x)} = \dfrac{P_y - Q_x}{Q} = \dfrac{1}{x}$, which gives $I(x) = x$.

[6] 5. Find the general solution of $x^2y^3 + (x^3y^2 + 2y)y' = 0$.

Solution:
$P = x^2y^3,\ Q = x^3y^2 + 2y,\ P_y = 3x^2y^2 = Q_x \ \Rightarrow$ the equation is exact.
$f_x = P = x^2y^3 \ \Rightarrow\ f(x,y) = \dfrac{1}{3}x^3y^3 + g(y),\ f_y = Q \ \Rightarrow\ x^3y^2 + g'(y) = x^3y^2 + 2y \ \Rightarrow$
$g'(y) = 2y \ \Rightarrow\ g(y) = y^2 + c \ \Rightarrow\ f(x,y) = \dfrac{1}{3}x^3y^3 + y^2 + c$, and the general solution of the equation is $\dfrac{1}{3}x^3y^3 + y^2 = k$.