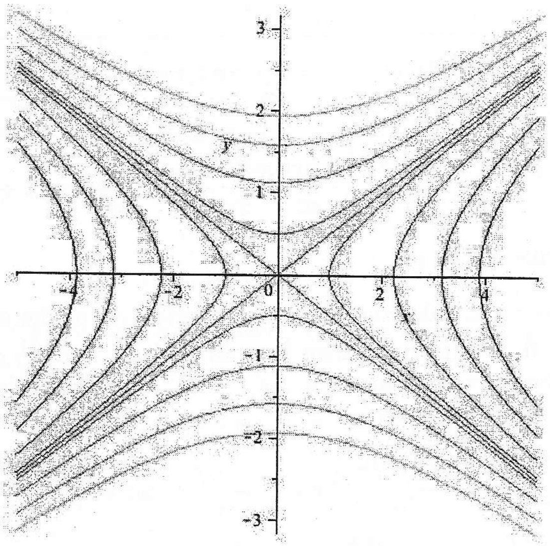
Math 200 - December 2011

Final Exam Solutions

1. Consider the function $f(x,y) = e^{-x^2+4y^2}$.
   (a) Draw a "contour map" of f, showing all types of level curves that occur.
   (b) Find the equation of the tangent plane to the graph $z = f(x,y)$ at the point where $(x,y) = (2,1)$.
   (c) Find the tangent plane approximation to the value of $f(1.99, 1.01)$ using the tangent plane from part (b).

Solution:

(a) Level curves for f are of the form $f(x,y) = e^{-x^2+4y^2} = C$, where C is a positive constant, since an exponential cannot be negative. That is, $-x^2 + 4y^2 = \ln C$. For $C \neq 1$, these are hyperbolas with vertices on the y-axis if $C > 1$ and on the x-axis if $C < 1$. If $C = 1$, $x^2 = 4y^2 \implies |y| = \frac{1}{4}|x|$, i.e. $y = \pm\frac{1}{4}x$. These are two straight lines with slopes $\pm\frac{1}{4}$ crossing at the origin.



(b) Let $F(x,y,z) = f(x,y) - z$. Then the graph of f is the level surface $F(x,y,z) = 0$. Since $f(2,1) = e^{-2^2+4(1)^2} = 1$, we want to find the tangent plane at the point $(2,1,1)$. This is given by the equation

$$\nabla F(2,1,1) \cdot (x-2, y-1, z-1) = 0. \tag{1}$$

We must compute the gradient,

$$\begin{aligned}\nabla F(x,y,z) &= \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y), -1\right)\\ &= \left(-2xe^{-x^2+4y^2}, 8ye^{-x^2+4y^2}, -1\right).\end{aligned}$$

Hence,

$$\nabla F(2,1,1) = \left(-4e^{-2^2+4}, 8e^{-2^2+4}, -1\right) = (-4, 8, -1).$$

Plugging this into (1),

$$\begin{aligned}&(-4,8,-1)\cdot(x-2,y-1,z-1)=0\\ \implies &-4(x-2)+8(y-1)-(z-1)=0\\ \implies &4x-8y+z=1\end{aligned}$$

(c) Plugging in the point into the tangent plane and solving for z, where $z=f(x,y)$:

$$f(1.99,1.01)\approx 1-4(1.99-2)+8(1.01-1)=1+0.04+0.08=1.12.$$

2. Suppose $z=f(x,y)$ has continuous second-order partial derivatives, and $x=r\cos t$, $y=r\sin t$. Express the following partial derivatives in terms of r,t, and partial derivatives of f.
   (a) $\frac{\partial z}{\partial t}$
   (b) $\frac{\partial^2 z}{\partial t^2}$

Solution: Since $z=f(x(r,t),y(r,t))=f(r\cos t,r\sin t)$,

(a)

$$\frac{\partial z}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}=-\frac{\partial f}{\partial x}r\sin t+\frac{\partial f}{\partial y}r\cos t,$$

by the chain rule.

(b) To find $\frac{\partial^2 z}{\partial t^2}$, we must take another partial derivative of the expression we found for $\frac{\partial z}{\partial t}$ in part (a). However, it is important here to note that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are actually functions of x and y, which themselves are each functions of r and t. Keeping this in mind, we apply the chain and product rules repeatedly to obtain

$$\begin{aligned}\frac{\partial^2 z}{\partial t^2}&=-\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial t}+\frac{\partial^2 f}{\partial x\partial y}\frac{\partial y}{\partial t}\right)r\sin t-\frac{\partial f}{\partial x}r\cos t+\left(\frac{\partial^2 f}{\partial y\partial x}\frac{\partial x}{\partial t}+\frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial t}\right)r\cos t-\frac{\partial f}{\partial y}r\sin t\\ &=-\left(-\frac{\partial^2 f}{\partial x^2}r\sin t+\frac{\partial^2 f}{\partial x\partial y}r\cos t\right)r\sin t-\frac{\partial f}{\partial x}r\cos t\\ &\quad+\left(-\frac{\partial^2 f}{\partial y\partial x}r\sin t+\frac{\partial^2 f}{\partial y^2}r\cos t\right)r\cos t-\frac{\partial f}{\partial y}r\sin t\\ &=\frac{\partial^2 f}{\partial x^2}r^2\sin^2 t-2\frac{\partial^2 f}{\partial x\partial y}r^2\cos t\sin t+\frac{\partial^2 f}{\partial y^2}r^2\cos^2 t-\frac{\partial f}{\partial x}r\cos t-\frac{\partial f}{\partial y}r\sin t\\ &=\frac{\partial^2 f}{\partial x^2}r^2\sin^2 t-\frac{\partial^2 f}{\partial x\partial y}r^2\sin(2t)+\frac{\partial^2 f}{\partial y^2}r^2\cos^2 t-\frac{\partial f}{\partial x}r\cos t-\frac{\partial f}{\partial y}r\sin t\end{aligned}$$

Note that since f has continuous partial second derivatives, we can say $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$.

3. A bee is flying along the curve of intersection of the surfaces $3z+x^2+y^2=2$ and $z=x^2-y^2$ in the direction for which z is increasing. At time $t=2$, the bee passes through the point $(1,1,0)$ at speed 6.
   (a) Find the velocity (vector) of the bee at time $t=2$.

(b) The temperature T at position (x, y, z) at time t is given by $T = xy - 3x + 2yt + z$. Find the rate of change of the temperature experienced by the bee at time $t = 2$.

Solution:

(a) Since the bee is flying along the curve of intersection of these two surfaces, his velocity vector must be orthogonal to both of their normal vectors, which we compute as follows. Note that

$$\nabla(3z + x^2 + y^2) = (2x, 2y, 3).$$

Hence, at the point $(1, 1, 0)$, the vector $\vec{n}_1 = (2(1), 2(1), 3) = (2, 2, 3)$ is normal to the surface $3z + x^2 + y^2 = 2$. Similarly,

$$\nabla(x^2 - y^2 - z) = (2x, -2y, -1),$$

so the vector $\vec{n}_2 = (2(1), -2(1), -1) = (2, -2, -1)$ is normal to the surface $z = x^2 - y^2$ at the point $(1, 1, 0)$.

The cross product

$$\begin{aligned}\vec{n}_1 \times \vec{n}_2 &= (2, 2, 3) \times (2, -2, -1)\\ &= (2(-1) - (-2)(3), 3(2) - (-1)(2), 2(-2) - 2(2))\\ &= (4, 8, -8)\end{aligned}$$

is orthogonal to both normal vectors, $\vec{n}_1$ and $\vec{n}_2$. Since we are told that the bee's z-coordinate is increasing, we can conclude that he is moving in the direction of $-\vec{n}_1 \times \vec{n}_2 = (-4, -8, 8)$, which is parallel to $(-1, -2, 2)$, which has length $\sqrt{1 + 4 + 4} = 3$. Let $\hat{n}$ be the corresponding unit vector; that is,

$$\hat{n} = \frac{(-1, -2, 2)}{|(-1, -2, 2)|} = \frac{1}{3}(-1, -2, 2).$$

The magnitude of the bee's velocity is his speed, which is equal to 6 (as given in the problem). Hence, the velocity vector is

$$\vec{v} = 6\hat{n} = 6\frac{1}{3}(-1, -2, 2) = (-2, -4, 4).$$

(b) Let $T(t, x, y, z) = xy - 3x + 2yt + z$.

We consider the bee's temperature as a function of time,

$$f(t) = T(t, x(t), y(t), z(t)),$$

where $(x(t), y(t), z(t))$ represents the position of the bee as a function of time. Our goal is to compute $f'(2)$. By the chain rule,

$$\begin{aligned}f'(t) &= \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}\\ &= 2y + (y - 3)\frac{dx}{dt} + (x + 2t)\frac{dy}{dt} + \frac{dz}{dt}\end{aligned}$$

At $t = 2$, we know that the position of the bee is $(x, y, z) = (1, 1, 0)$ and in part (a), we found the velocity vector to be $\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (-2, -4, 4)$. Hence,

$$f'(2) = 2(1) + (1 - 3)(-2) + (1 + 2(2))(-4) + 4 = 2 + 4 - 20 + 4 = -10$$

is the rate of change in temperature the bee experiences at $t = 2$.

4. Find the radius of the largest sphere centred at the origin that can be inscribed inside (that is, enclosed inside) the ellipsoid

$$2(x+1)^2 + y^2 + 2(z-1)^2 = 8.$$

Solution: The radius of the largest sphere centred at the origin that can be inscribed inside the given ellipsoid is equal to the minimum distance of all points on the ellipsoid from the origin. We then look for the minimum value of the distance squared,

$$f(x,y,z) = x^2 + y^2 + z^2,$$

subject to the constraint that (x,y,z) lies on the ellipsoid; that is,

$$g(x,y,z) = 2(x+1)^2 + y^2 + 2(z-1)^2 = 8.$$

We use the method of Lagrange multipliers and look for solutions x, y, z and λ to the equations

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 8. \end{cases}$$

Since $\nabla f(x,y,z) = (2x, 2y, 2z)$ and $\nabla g(x,y,z) = (4(x+1), 2y, 4(z-1))$, $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ is equivalent to the following system,

$$\begin{cases} 2x = 4\lambda(x+1) \\ 2y = 2\lambda y \\ 2z = 4\lambda(z-1) \end{cases} \tag{2}$$

Adding the third equation to the first, we have

$$\begin{aligned} &2(x+z) = 4\lambda(x+z) \\ &\implies (x+z) = 2\lambda(x+z) \\ &\implies (2\lambda - 1)(x+z) = 0. \end{aligned}$$

Either $2\lambda - 1 = 0$ or $x + z = 0$.

Consider first the case where $2\lambda - 1 = 0$, that is, $\lambda = 1/2$. From the first equation,

$$\begin{aligned} &2x = 4\lambda(x+1) \\ &\implies 2x = 2(x+1) \\ &\implies 0 = 1, \end{aligned}$$

a contradiction. We must conclude that $2\lambda - 1 \neq 0$ and therefore, $x + z = 0$.

Plugging $z = -x$ into the constraint,

$$\begin{aligned} &2(x+1)^2 + y^2 + 2(z-1)^2 = 8 \\ &\implies 2(x+1)^2 + y^2 + 2(-x-1)^2 = 8 \\ &\implies 4(x+1)^2 + y^2 = 8 \end{aligned} \tag{3}$$

Now, from the second equation in (2),

$$2y = 2\lambda y \implies (\lambda - 1)y = 0$$

and we must conclude that either $y = 0$ or $\lambda = 1$.

If $y = 0$, (3) says

$$4(x+1)^2 = 8 \implies (x+1)^2 = 2 \implies x+1 = \pm\sqrt{2} \implies x = \pm\sqrt{2} - 1.$$

The z-coordinate is then given by

$$z = -x = 1 - (\pm\sqrt{2}).$$

We find that $(x, y, z) = (\sqrt{2} - 1, 0, 1 - \sqrt{2})$ and $(x, y, z) = (-\sqrt{2} - 1, 0, 1 + \sqrt{2})$ are both critical points.

Finally, we must also consider the case $\lambda = 1$. From the first equation in (2),

$$2x = 4(x+1) \implies x = 2(x+1) \implies x = -2.$$

Of course, $z = -x = 2$, and the y-coordinate can be obtained from the constraint,

$$\begin{aligned} &2(-2+1)^2 + y^2 + 2(2-1)^2 = 8 \\ &\implies 2 + y^2 + 2 = 8 \\ &\implies y^2 = 4 \\ &\implies y = \pm 2. \end{aligned}$$

We've found two more critical points, $(-2, 2, 2)$ and $(-2, -2, 2)$.

We should also check for points where $\nabla g(x, y, z) = (4(x+1), 2y, 4(z-1)) = 0$. There is only one, namely $(-1, 0, 1)$. However, since $g(-1, 0, 1) = 2(-1+1)^2 + 0^2 + 2(1-1)^2 = 0$, this point does not lie on the ellipsoid.

Comparing values of f at each of the critical points, we have

$$f(-2, \pm 2, 2) = (-2)^2 + (\pm 2)^2 + 2^2 = 12,$$
$$f(\sqrt{2} - 1, 0, 1 - \sqrt{2}) = 2(\sqrt{2} - 1)^2$$

and

$$f(-\sqrt{2} - 1, 0, 1 + \sqrt{2}) = 2(1 + \sqrt{2})^2$$

The smallest of all these is $2(\sqrt{2} - 1)^2$. Therefore, the largest sphere centred upon the origin that can be inscribed within the ellipsoid has radius $\sqrt{2}(\sqrt{2} - 1) = 2 - \sqrt{2}$.

5. (a) Consider the iterated integral

$$\int_{-4}^{0} \int_{\sqrt{-y}}^{2} \cos(x^3)\, dx\, dy$$

   i. Draw the region of integration.
   ii. Evaluate the integral.

   (b) Evaluate the double integral

$$\iint_D y\sqrt{x^2 + y^2}\, dA$$

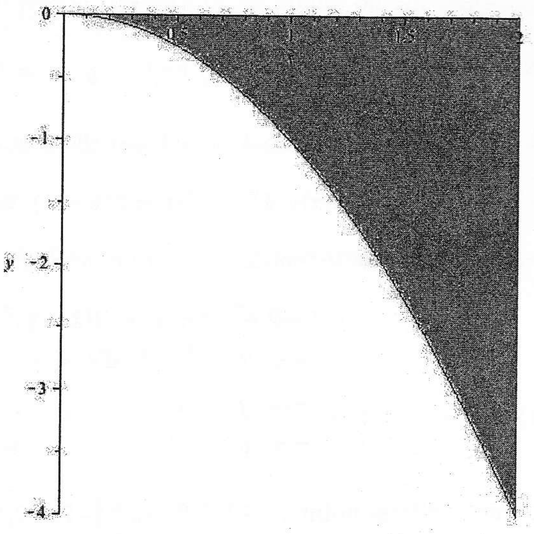
   over the region $D = \{(x, y) | x^2 + y^2 \leq 2 \text{ and } 0 \leq y \leq x\}$.

Solution:

(a) Let

$$I = \int_{-4}^{0}\int_{\sqrt{-y}}^{2} \cos(x^3)\, dx\, dy.$$

i. Draw the region of integration.



ii. Upon first glance, it seems impossible to find an antiderivative for $\cos(x^3)$ (with respect to x) in order to evaluate the inner integral. When faced this type of situation, our first course of action is to reverse the order of integration; with some luck, the resulting integral will be easier to evaluate. Looking at the picture, the domain of integration can also be expressed as

$$\{(x,y) : -x^2 \le y \le 0, 0 \le x \le 2\}.$$

Hence,

$$\begin{aligned} I &= \int_0^2 \int_{-x^2}^{0} \cos(x^3)\, dy\, dx \\ &= \int_0^2 \cos(x^3) y\big|_{-x^2}^{0}\, dx \\ &= \int_0^2 \cos(x^3) x^2\, dx \end{aligned}$$

Indeed, this integral is easy to evaluate using the substitution $u = x^3$. Since $du = 3x^2 dx$,

$$\begin{aligned} I &= \frac{1}{3}\int_0^{2^3} \cos(u)\, du \\ &= \frac{1}{3}\sin(u)\Big|_0^{2^3} \\ &= \frac{1}{3}(\sin(2^3) - \sin 0) \\ &= \frac{1}{3}\sin(2^3). \end{aligned}$$

(b) Let

$$J = \iint_D y\sqrt{x^2+y^2}\, dA$$

where $D = \{(x,y)\,|\,x^2+y^2 \leq 2 \text{ and } 0 \leq y \leq x\}$.

This can be done using Cartesian coordinates. However, drawing a picture (which is always a good idea) shows that the region D is the sector of a circle. That along with the $\sqrt{x^2+y^2}$ strongly suggests that we should use polar coordinates. The region in polar coordinates is

$$D = \{(r,\theta)\,|\,0 \leq r \leq 2,\ 0 \leq \theta \leq \pi/4\}$$

Using $y = r\sin\theta$, $x^2+y^2 = r^2$ and $dA = rdrd\theta$ we get that the integral is

$$\begin{aligned} J &= \iint_D r\sin\theta r(r)\; drd\theta \\ &= \int_0^{\pi/4}\int_0^2 r^3 \sin\theta\; drd\theta \\ &= \int_0^{\pi/4} \sin\theta \frac{x^4}{4}\bigg|_0^2\; d\theta \\ &= \int_0^{\pi/4} \sin\theta\left(\frac{16}{4} - 0\right)\; d\theta \\ &= 4\Big(-\cos\theta|_0^{\pi/4}\Big) \\ &= 4(-\cos(\pi/4) + \cos 0) \\ &= 4\left(-\frac{\sqrt{2}}{2} + 1\right) = 4 - 2\sqrt{2}. \end{aligned}$$

6. Let R be the triangle with vertices $(0,2)$, $(1,0)$, and $(2,0)$. Let R have density $\rho(x,y) = y^2$. Find $\overline{y}$, the y-coordinate of the centre of mass of R. **You do not need to find $\overline{x}$.**

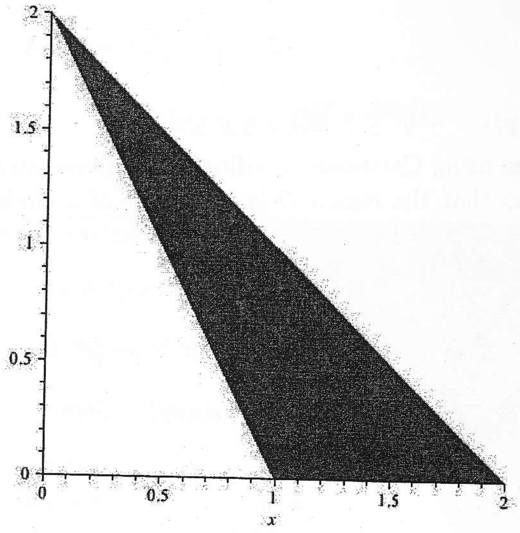
Solution: The y-coordinate of the centre of mass is defined by

$$\overline{y} = \frac{1}{M}\iint_R y\rho(x,y)\, dA,$$

where M is the total mass,

$$M = \iint_R \rho(x,y) dA.$$

We sketch the domain R as follows:



We see that

$$R = \left\{(x,y) \,\middle|\, \frac{2-y}{2} \le x \le 2-y,\, 0 \le y \le 2\right\}.$$

Then

$$M = \iint_R \rho(x,y)dA = \int_0^2 \int_{\frac{2-y}{2}}^{2-y} y^2\,dx\,dy = \int_0^2 y^2 x\Big|_{x=\frac{2-y}{2}}^{x=2-y} dy = \int_0^2 y^2 \frac{1}{2}(2-y)\,dy$$

$$= \frac{1}{2}\int_0^2 (2y^2 - y^3)\,dy = \frac{1}{2}\left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_0^2 = \frac{1}{2}\left[\frac{2}{3}2^3 - \frac{1}{4}2^4\right] = \frac{2^3}{3} - 2 = \frac{8-6}{3} = \frac{2}{3}$$

and

$$\overline{y} = \frac{1}{M}\iint_R y\rho(x,y)\,dA = \frac{3}{2}\int_0^2 \int_{\frac{2-y}{2}}^{2-y} y^3\,dx\,dy = \frac{3}{2}\int_0^2 y^3 \frac{2-y}{2}\,dy = \frac{3}{4}\int_0^2 (2y^3 - y^4)\,dy$$

$$= \frac{3}{4}\left[\frac{y^4}{2} - \frac{y^5}{5}\right]_0^2 = \frac{3}{4}\left(\frac{2^4}{2} - \frac{2^5}{5}\right) = 3\left(2 - \frac{2^3}{5}\right) = \frac{6}{5}.$$

7. Evaluate the triple integral $\iiint_E x\,dV$, where E is the region in the first octant bounded by the parabolic cylinder $y = x^2$ and the planes $y + z = 1$, $x = 0$, and $z = 0$.

Solution: We can express the domain in the following form:

$$E = \{(x,y,z) \,|\, 0 \le z \le 1-y,\, x^2 \le y \le 1,\, 0 \le x \le 1\}$$

The triple integral can then be rewritten as an iterated integral,

$$\begin{aligned}\iiint_E x\, dV &= \int_0^1 \int_{x^2}^1 \int_0^{1-y} x\, dz\, dy\, dx = \int_0^1 \int_{x^2}^1 x(1-y)\, dy\, dx \\ &= \int_0^1 x\left(y - \frac{1}{2}y^2\right)\Big|_{y=x^2}^{y=1} dx = \int_0^1 x\left(1 - \frac{1}{2} - x^2 + \frac{1}{2}x^4\right) dx \\ &= \int_0^1 \left(\frac{1}{2}x - x^3 + \frac{1}{2}x^5\right) dx = \left[\frac{1}{4}x^2 - \frac{1}{4}x^4 + \frac{1}{12}x^6\right]_0^1 \\ &= \frac{1}{12}\end{aligned}$$

8. The body of a snowman is formed by the snowballs $x^2 + y^2 + z^2 = 12$ (this is its body) and $x^2 + y^2 + (z-4)^2 = 4$ (this is its head).
   (a) Find the volume of the snowman by subtracting the intersection of the two snowballs from the sum of the volumes of the snowballs. [Recall that the volume of a sphere of radius r is $\frac{4\pi}{3}r^3$.]
   We'll use spherical coordinates to compute the volume of the intersection.
   (b) We can also calculate the volume of the snowman as a sum of the following triple integrals:
   1.

$$\int_0^{\frac{2\pi}{3}} \int_0^{2\pi} \int_0^2 \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$

   2.

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_{\sqrt{3}r}^{4-\frac{r}{\sqrt{3}}} r\, dz\, dr\, d\theta$$

   3.

$$\int_{\frac{\pi}{6}}^{\pi} \int_0^{2\pi} \int_0^{2\sqrt{3}} \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$

   Circle the right answer from the underlined choices and fill in the blanks in the following descriptions of the region of integration for each integral. [Note: We have translated the axes in order to write down some of the integrals above. The equations you specify should be those *before* the translation is performed.]

   i. The region of integration in (1) is a part of the snowman's body / head / body and head. It is the solid enclosed by the sphere / cone defined by the equation ________________ and the sphere / cone defined by the equation ________________.
   ii. The region of integration in (2) is a part of the snowman's body / head / body and head. It is the solid enclosed by the sphere / cone defined by the equation ________________ and the sphere / cone defined by the equation ________________.
   iii. The region of integration in (3) is a part of the snowman's body / head / body and head. It is the solid enclosed by the sphere / cone defined by the equation ________________ and the sphere / cone defined by the equation ________________.

Solution: The body of a snowman is formed by the snowballs $x^2 + y^2 + z^2 = 12$ (this is its body) and $x^2 + y^2 + (z-4)^2 = 4$ (this is its head).

(a) The intersection of the two snowballs is the region R defined by

$$R = \{(x, y, z) | x^2 + y^2 + z^2 \leq 12 \text{ and } x^2 + y^2 + (z-4)^2 \leq 4\}.$$

(It would be helpful to draw a picture of this region.) We'll take advantage of the symmetry by converting to cylindrical coordinates.

The top boundary is given by the equation $x^2 + y^2 + z^2 = 12$; in cylindrical coordinates, this is $r^2 + z^2 = 12$, i.e. $z = \sqrt{12 - r^2}$.

The bottom boundary is $x^2+y^2+(z-4)^2 = 4$; in cylindrical coordinates, this is $r^2+(z-4)^2 = 4$. Since $z < 4$ in this region, $z = 4 - \sqrt{4 - r^2}$.

The ring which defines the outer edge is given by the intersection of the two spheres, $r^2+z^2 = 12$ and $r^2+(z-4)^2 = 4$. Subtracting these two equations gives $z^2 - (z-4)^2 = 8$ which is simplified to $8z - 16 = 8$, so that $z = 3$. Then $r^2 + 9 = 12$ and hence, $r = \sqrt{3}$.

Hence, the volume of intersection is

$$\begin{aligned} V_R &= \iiint_R 1\,dV = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{4-\sqrt{4-r^2}}^{\sqrt{12-r^2}} 1\,r\,dz\,dr\,d\theta \\ &= 2\pi \int_0^{\sqrt{3}} \left(\sqrt{12 - r^2} - 4 + \sqrt{4 - r^2}\right) r\,dr \end{aligned}$$

We make the substitution $u = r^2$, so that $du = 2rdr$ and

$$\begin{aligned} V_R &= \pi \int_0^3 \left(\sqrt{12 - u} - 4 + \sqrt{4 - u}\right) du \\ &= \pi \left[-\frac{2}{3}(12 - u)^{3/2} - 4u - \frac{2}{3}(4 - u)^{3/2}\right]_0^3 \\ &= \pi \left(-\frac{2}{3}(9)^{3/2} - 12 - \frac{2}{3}(1)^{3/2} + \frac{2}{3}(12)^{3/2} + \frac{2}{3}(4)^{3/2}\right) \\ &= \pi \left(-18 - 12 - \frac{2}{3} + \frac{2}{3}(2^2 \cdot 3)^{3/2} + \frac{16}{3}\right) \\ &= \pi \left(-\frac{76}{3} + 16\sqrt{3}\right) \\ &= \frac{4\pi}{3}\left(-19 + 12\sqrt{3}\right) \end{aligned}$$

The total volume of the snowman is equal to the sum of the volumes of the two balls, minus their intersection.

$$\begin{aligned} V &= \frac{4}{3}\pi(\sqrt{12})^3 + \frac{4}{3}\pi(\sqrt{4})^3 - V_R \\ &= \frac{4}{3}\pi(12^{3/2} + 8 + 19 - 12\sqrt{3}) \\ &= \frac{4}{3}\pi(8 \cdot 3\sqrt{3} + 8 + 19 - 12\sqrt{3}) \\ &= \frac{4}{3}\pi(27 + 12\sqrt{3}) \\ &= 4\pi(9 + 4\sqrt{3}) \end{aligned}$$

(b) We can also calculate the volume of the snowman as a sum of the following triple integrals:

1.
$$\int_0^{\frac{2\pi}{3}}\int_0^{2\pi}\int_0^{2} \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

2.
$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_{\sqrt{3}r}^{4-\frac{r}{\sqrt{3}}} r\,dz\,dr\,d\theta$$

3.
$$\int_{\frac{\pi}{6}}^{\pi}\int_0^{2\pi}\int_0^{2\sqrt{3}} \rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$$

  i. The region of integration in (1) is a part of the snowman's *head.*
  It is the solid enclosed by the *sphere* defined by the equation $x^2+y^2+(z-4)^2=4$ and the *cone* defined by the equation $x^2+y^2=3(4-z)^2$.

  ii. The region of integration in (2) is a part of the snowman's *body and head.*
  It is the solid enclosed by the *cone* defined by the equation $x^2+y^2=3(4-z)^2$ and the *cone* defined by the equation $x^2+y^2=\frac{1}{3}z^2$.

  iii. The region of integration in (3) is a part of the snowman's *body.*
  It is the solid enclosed by the *sphere* defined by the equation $x^2+y^2+z^2=12$ and the *cone* defined by the equation $x^2+y^2=\frac{1}{3}z^2$.