Mathematics 209
Midterm test, October 21, 2012 1h30min

Only authorized calculators permitted

[75 pts=100%]

1. [10pts] Find the limits

(a) $\lim_{x\to 0} \frac{\sqrt{36+x}-6}{x}$,
(b) $\lim_{x\to +\infty} \frac{4-x^3}{x^2+5}$, (c) $\lim_{x\to 1^-} \frac{c^2}{x-3}$ where c is a real number.

2. [15pts](3+4+4+4) Without simplifying find the derivatives $f'(x)$ of the following

(a) $f(x) = \frac{1}{5}x^{-3} + 8\sqrt{x} + 4$, (b) $f(x) = 4(x^2-6)^5$, (c) $f(x) = \frac{2+x^2}{x^5-3}$,
(d) $f(x) = e^{3x^4+2x}$.

3. [10pts] Find the derivative of the function $3x^2$ from first principles. i.e. using the four-step method.

4. [5pts] Suppose that interest is compounded continuously and that 8000 current dollars have a future value of $12,000$ dollars after five years. What is the effective rate of interest that is being charged?

5. [10pts] The function $t(x)$ is given implicitly by the equation $e^t + 3t - x = 3$. Calculate the slope of the tangent line at the point $(e, 1)$

6. [15pts] Market studies for a new product show that the demand as a function of price p, is $x = 500,000 - 1000p$.

(a) Find the average revenue as a function of p.

(b) Find the marginal average revenue when $p = 50$

7. [10pts] A point is moving along the graph of $2y^2 - e^x = x + 1$. When the point is at $(x, y) = (0, -1)$ its x coordinate has velocity zero. How fast is the y coordinate changing at that moment?

Math209

Midterm Solutions.

Problem 1

a) $$\lim_{x\to 0} \frac{\sqrt{36+x}-6}{x} = \lim_{x\to 0} \frac{\sqrt{36+x}-6}{x} \cdot \frac{(\sqrt{36+x}+6)}{(\sqrt{36+x}+6)}$$

$$= \lim_{x\to 0} \frac{(\sqrt{36+x})^2 - 6^2}{x(\sqrt{36+x}+6)} = \lim_{x\to 0} \frac{\cancel{36}+x-\cancel{36}}{x(\sqrt{36+x}+6)}$$

$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{36+x}+6)} = \lim_{x\to 0} \frac{1}{\sqrt{36+x}+6}$$

$$= \frac{1}{\sqrt{36}+6} = \frac{1}{12}.$$

b) $$\lim_{x\to\infty} \frac{4-x^3}{x^2+5} = \lim_{x\to\infty} \frac{-x^3}{x^2} = \lim_{x\to\infty} -x = -\infty.$$

c) $$\lim_{x\to 1^-} \frac{c^2}{x-3} = \frac{c^2}{1-3} = \frac{-c^2}{2}.$$

Problem 2

a) $$f(x) = \frac{1}{5}x^{-3} + 8\sqrt{x} + 4$$

$$f'(x) = \frac{-3}{5}x^{-4} + \frac{8}{2\sqrt{x}} = \frac{-3}{5x^4} + \frac{4}{\sqrt{x}}$$

b) $$f(x) = 4(x^2-6)^5$$

$$f'(x) = 4\cdot 5\cdot(x^2-6)^4\cdot 2x \quad \text{(Chain Rule)}$$

$$= 40(x^2-6)^4$$

c) $f(x) = \dfrac{2+x^2}{x^5-3}$

$$f'(x) = \frac{2x(x^5-3) - 5x^4(2+x^2)}{(x^5-3)^2} \quad \text{(Quotient Rule)}$$

$$= \frac{2x^6 - 6x - 10x^4 - 5x^6}{(x^5-3)^2}$$

$$= \frac{-3x^6 - 10x^4 - 6x}{(x^5-3)^2} = \frac{x(-3x^5 - 10x^3 - 6)}{(x^5-3)^2}$$

d) $f(x) = e^{3x^4+2x}$

$$f'(x) = e^{3x^4+2x} \cdot (3x^4+2x)' \quad \text{(Chain Rule)}$$

$$= e^{3x^4+2x} \cdot (12x^3+2)$$

Problem 3

$f(x) = 3x^2$

step 1 $\quad f(x+h) = 3(x+h)^2$

step 2 $\quad f(x+h) - f(x) = 3(x+h)^2 - 3x^2$

$$= 3(x^2+2xh+h^2) - 3x^2$$

$$= 3x^2 + 6xh + 3h^2 - 3x^2$$

$$= 6xh + 3h^2$$

step 3 $\quad \dfrac{f(x+h) - f(x)}{h} = \dfrac{6xh+3h^2}{h} = \dfrac{3\cancel{h}(2x+h)}{\cancel{h}} = 3(2x+h)$

step 4 $\quad f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} 3(2x+h)$$

$$= 6x$$

Problem 4

Interest is compounded continuously so

$A = Pe^{rt}$. here $P = 8000\ \$$

$A = 12000\ \$$.

$t = 5$ years.

so $12000 = 8000\, e^{5r}$

$e^{5r} = \frac{12000}{8000}$

$e^{5r} = \frac{3}{2}$.

$\ln e^{5r} = \ln \frac{3}{2}$.

$5r = \ln \frac{3}{2}$.

$r = \frac{1}{5} \ln \frac{3}{2} \simeq 0.081$

Effective rate of interest is 8.1 %.

Problem 5

$e^t + 3t - x = 3$.

Differentiate both sides with respect to x.

$\frac{d}{dx}(e^t + 3t - x) = \frac{d}{dx}(3)$

$e^t . \frac{dt}{dx} + 3\frac{dt}{dx} - 1 = 0$.

$\frac{dt}{dx}(e^t + 3) = 1$.

$\frac{dt}{dx} = \frac{1}{e^t + 3}$

At the point $(\underset{\stackrel{\uparrow}{x}}{e}, \underset{\stackrel{\uparrow}{t}}{1})$ $\left.\frac{dt}{dx}\right|_{(e,1)} = \frac{1}{e^1 + 3} = \frac{1}{e+3}$

Slope of tangent line is $\frac{1}{e+3}$.

Problem 6

$x = 500{,}000 - 1000p.$

a). • Average Revenue $\bar{R}(x) = \frac{R(x)}{x}$ (Divide Revenue by x Not by p!!)

but $R(x) = x \cdot p$

so $\bar{R}(x) = \frac{R(x)}{x} = \frac{x \cdot p}{x} = p.$

So as a function of p, $\bar{R}(p) = p.$

• If we wanted $\bar{R}(x)$ as a function of x, we write p as a function of x

$$p = \frac{-x + 500{,}000}{1000} = \frac{-x}{1000} + 500.$$

$$\bar{R}(x) = \frac{-x}{1000} + 500$$

b) To find the marginal average Revenue we differentiate $\bar{R}(x)$ with respect to x not to p (Marginal is differentiation with respect to demand not to price).

Marginal Average Revenue $= \bar{R}'(x) = \frac{d}{dx}(\bar{R}(x))$
(this is not $\frac{R'(x)}{x}$!!)

$$\bar{R}'(x) = \frac{d}{dx}\left(-\frac{x}{1000} + 500\right)$$

$$= \frac{-1}{1000}$$

When $p = 50$, we also have $\bar{R}'(x) = \frac{-1}{1000}.$

Problem 7

$$2y^2 - e^x = x+1.$$

When $(x,y) = (0,-1)$ $\quad \frac{dx}{dt} = 0.$

$\frac{dy}{dt} = ?$ at $(x,y) = (0,-1)$.

Differentiate both sides of equation with respect to t.

$$\frac{d}{dt}(2y^2 - e^x) = x+1.$$

$$4y\frac{dy}{dt} - e^x \cdot \frac{dx}{dt} = \frac{dx}{dt}$$

$$4y\frac{dy}{dt} = (1+e^x)\frac{dx}{dt}.$$

(Chain Rule. DO NOT FORGET that $x = x(t)$ a function of t !).

Now we plug in values: $x=0$, $y=-1$, $\frac{dx}{dt} = 0$

$$4(-1)\frac{dy}{dt} = (1+e^0)\cdot 0$$

$$-4\frac{dy}{dt} = 0$$

so $\boxed{\frac{dy}{dt} = 0}$ at $(0,-1)$

The y- coordinate is changing at a rate of zero at $(0,-1)$.