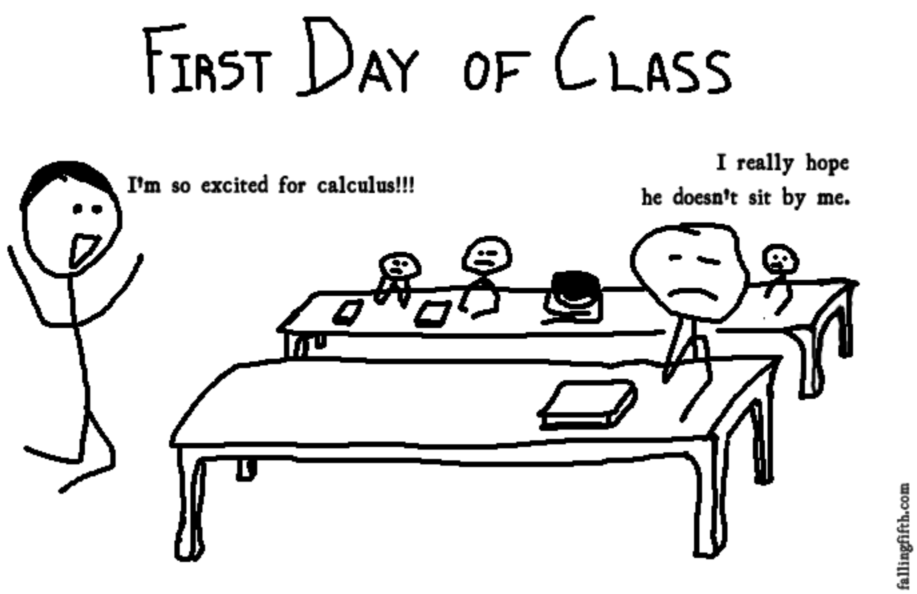
# Precalculus Review

MAT 1300 C

Winter 2013

*Mathematics* provides a language to precisely model a problem and then provides tools to *logically* solve the problem.
*Calculus* is the study of functions with special emphasis on the rate of change of the function.



# 1 Intervals of the Real Number Line

Notation:

- $[100, 300]$ means all the real numbers that are between 100 and 300 inclusively.
- $[100, 300)$ means all the real numbers that are between 100 and 300 including 100 but not including 300.
- $(100, 300)$ means all the real numbers that are strictly between 100 and 300.
- $[100, \infty)$ means all the real numbers that are greater than or equal to 100.

Note: $\infty$ is not a number and so is *never* included in an interval.

Pictorially: Ex: Draw a diagram of the intervals

a) $(-\infty, 12]$

b) and $[-7, -3)$.

# 2 Solving Inequalities

Ex: The price per unit, $p$, of a product is related to the demand, $x$, for the product by

$$p = 120 - 0.03x.$$

Find reasonable bounds on the demand.

Ex: The profit, $P$, of producing a product is related to the price per unit, $u$, by

$$P = -15000u^2 + 105000u - 150000.$$

For what values of $u$ is the profit positive?

# 3 Absolute Value

If $a$ is a real number, then the **absolute value** of $a$ is defined as

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a \leq 0 \end{cases}$$

The absolute value of a number is like the "positive part" or "magnitude" of the number.

If $c \geq 0$ is a positive constant, then

$$\begin{aligned} |x-a| \leq c \quad &\Leftrightarrow \quad -c \leq x-a \leq c \\ &\Leftrightarrow \quad a-c \leq x \leq a+c \end{aligned}$$

The other inequality $|x-a| \geq c$ has two solutions:

$$x - a \geq c \quad \text{or} \quad x - a \leq -c$$

Examples

1. Solve $|25 - x| \geq 20$.

   *solution:* By the second remark above, we have that either

   $$\begin{array}{rcl} 25 - x \geq 20 & \text{or} & 25 - x \leq -20 \\ -x \geq -5 & & -x \leq -45 \\ x \leq 5 & & x \geq 45 \end{array}$$

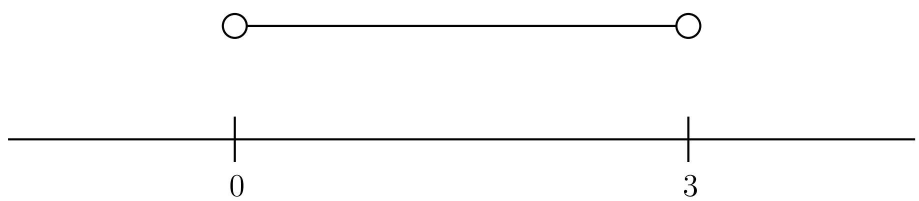
   The set of solutions is represented below:

2. Solve $\left|1 - \frac{2x}{3}\right| < 1$.

   <u>*solution:*</u>

$$\left|1 - \frac{2x}{3}\right| < 1$$
$$-1 < 1 - \frac{2}{3}x < 1$$
$$-2 < -\frac{2}{3}x < 0$$
$$3 > x > 0$$

   Thus the set of solutions is the interval $(0,3)$ and is represented as below:



# 4 Exponents

Rules and Notation:

1. $x^n = x \times x \times \ldots \times x$
2. $x^0 = 1$
3. $x^{-n} = \frac{1}{x^n}$
4. $x^{1/n} = \sqrt[n]{x}$
5. $x^n x^m = x^{n+m}$
6. $\frac{x^n}{x^m} = x^{n-m}$
7. $(xy)^n = x^n y^n$
8. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$
9. $(x^n)^m = x^{nm}$

Ex: Simplify

$$\left(\frac{x^5\sqrt{x}}{\sqrt[3]{x^7}}\right)^{-2}$$

# 5 Polynomials

Tools:

Quadratic Formula for solving $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Difference of Squares for factoring $x^2 - a^2$:

$$x^2 - a^2 = (x - a)(x + a)$$

Ex: Solve for $x$.

$$-12x^2 + 15x - 20 = 3x - 44$$

Ex: Solve for $x$.

$$x^3 - 5x^2 - 5x + 25 = 0$$

# 6 Fractions and Rationalization

To rationalize an expression,

1. If $\sqrt{a}$ is in the denominator, multiply by $\frac{\sqrt{a}}{\sqrt{a}}$.
2. If $\sqrt{a}-\sqrt{b}$ is in the denominator, multiply by $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.
3. If $\sqrt{a}+\sqrt{b}$ is in the denominator, multiply by $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

Note that in each case the term that we are multiplying by is equal to 1, therefore we are not changing the expression, just altering its appearance.

Examples:

1. Simplify $\frac{13}{6+\sqrt{10}}$.
   *solution:*

$$\frac{13}{6+\sqrt{10}} = \frac{13}{6+\sqrt{10}} \cdot \left(\frac{6-\sqrt{10}}{6-\sqrt{10}}\right) = \frac{13(6-\sqrt{10})}{6^2-\sqrt{10}^2}$$

$$= \frac{13(6-\sqrt{10})}{26} = \frac{(6-\sqrt{10})}{2}.$$

2. Simplify $\frac{(x-1)}{\sqrt{x}+x}$.
   *solution:*

$$\frac{(x-1)}{\sqrt{x}+x} = \frac{(x-1)}{\sqrt{x}+x} \cdot \left(\frac{\sqrt{x}-x}{\sqrt{x}-x}\right)$$

$$= \frac{(x-1)(\sqrt{x}-x)}{\sqrt{x}^2-x^2} = \frac{(x-1)(\sqrt{x}-x)}{x-x^2} = \cdots$$

$$\cdots = \frac{(x-1)(\sqrt{x}-x)}{x(x-1)} = \frac{\sqrt{x}-x}{-x}$$

$$= \frac{x-\sqrt{x}}{x} = 1-\frac{\sqrt{x}}{x} = 1-x^{-1/2}.$$

# 7 Functions

A *function* is a mathematical relationship between two variables (called the *independent variable* and the *dependent variable*) such that each value of the independent variable corresponds to a unique value of the dependent variable.

The *domain* of a function is the set of all the possible allowed values of the independent variable.

The *range* of a function is the set of all the values of the dependent variable which correspond to some value of the independent variable in the domain.

Often mathematicians will use $x$ to denote the independent variable, $y$ to denote the dependent variable, and $f$ to denote the function so $y = f(x)$.

Question: Where do functions come from?

They come from our analysis of the perceived relationship between two variables (such as profit and number of units sold). When a certain relationship is simple and well-studied, we may be able to find a function that exactly describes the relationship between the two variables. If the relationship is complex, then statisticians can collect data on the relationship and create a function that seems to be an acceptable fit to the data. There is often a tradeoff between the simplicity and the accuracy of the function.

Ex: The profit, $P$, of producing $x$ units of a product is given by

$$P(x) = 5x - \sqrt{4x - 100} - 6000.$$

a) What is the domain of this function?

b) What is the profit when 1000 units are produced?

Ex: Find the domain of the function

$$f(x) = \frac{x^2}{x-1},\ x \geq 0.$$

# 8 Composite Functions

Given two functions $f$ and $g$, the *composite*, $f \circ g$, is a new function whose values are $f(g(x))$.

Ex: Let $f(x) = 1 + x^2$ and $g(x) = 2x - 1$. Find $f \circ g$ and $g \circ f$.

Question: Why are composite functions important?

Ex: The profit, $P(x)$, of selling $x$ units of a product is given by

$$P(x) = 5x - \sqrt{4x - 100} - 6000.$$

If the number of units sold, $x$, depends on the price per unit, $q$ as

$$x(q) = \frac{50000}{q},$$

find $P$ as a function of $q$.

# 9 Inverses

Two functions $f$ and $g$ are *inverses* if $f(g(x)) = x$ and $g(f(x)) = x$ for all $x$ in the domain of $g$ and $f$ respectively. Often the inverse of a function $f$ is denoted $f^{-1}$.

(Roughly speaking, $f^{-1}$ undoes what $f$ did to $x$.)

Ex: $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses.

Ex: $f(x) = x^2$ has no inverse but $f(x) = x^2, x \geq 0$ has inverse $f^{-1}(x) = \sqrt{x}$.

We can find the inverse by solving for the independent variable and then appropriately renaming the variables.

Ex: Find the inverse of $f(x) = \frac{2x-1}{x+7}$.

# 10 Graphs

Graphs give a pictorial representation of the relationship described by a function. You should be familiar with the graphs of

1) Linear Functions: $y = mx$ (lines)

2) Quadratic Functions: $y = x^2$ (parabolas)

3) Cubic Functions: $y = x^3$

4) Square Root Functions: $y = \sqrt{x}$

5) Absolute Value Functions: $y = |x|$

6) Hyperbolic Functions: $y = 1/x$

- Replacing $x$ with $x - a$ moves the graph $a$ units to the right.
- Replacing $y$ with $y - b$ moves the graph $b$ units up.

Ex: Graph $y = |x - 3| - 4$.

Soln: Rewriting as $y + 4 = |x - 3|$ we see that this is like $y = |x|$ but shifted 3 units to the right and 4 units down.

# 11 Points of Intersection

$x$**-intercepts:** points at which the graph crosses the $x$-axis.

$\rightarrow$ We find them by solving $f(x) = 0$.

$y$**-intercepts:** points at which the graph crosses the $y$-axis.

$\rightarrow$ We find them by substituting $x = 0$.

Ex: Find the $x$- and $y$-intercepts of $y = |x - 3| - 4$.

The *break-even point* is a number of units that must be sold for the total costs to be equal to the total revenue (i.e. the profit is 0).

Ex: Producing CDs requires an initial investment of \$1980 for studio time and then \$1.80 for each CD made. We sell the CDs for \$9 each. Sketch the graphs of the cost and revenue functions on the same axes. What is the break-even point?

# 12 Lines

A *linear equation* is an equation that can be written in the form $y = mx + b$ (slope-$y$-intercept form) where $m$ is the slope of the line and $b$ is the $y$-intercept.

Given two points $(x_1, y_1)$ and $(x_2, y_2)$ on the line we can calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope is how much $y$ changes for each 1 unit increase in $x$.

Given a point $(x_1, y_1)$ on the line and the slope $m$ we can find the linear equation using

$$y - y_1 = m(x - x_1).$$

Ex: Suppose that the relationship between the demand, $x$, for a product and the price per unit, $p$ is linear. If we can sell 1000 units at \$20 each and 3000 units at \$15 each, find the demand function.