Math 1225A/B

Unit 1:
Exponential and Logarithmic Functions

(text reference: Sections 5.1 and 5.2 – mixed together

custom text pgs. 3 - 16)

1 Exponential and Logarithmic Functions

We start the course with a review of exponential and logarithmic functions, and their properties. Then in the next unit we will, in the context of these kinds of functions, review what you learned in Introductory Calculus – the various differentiation rules as well as using calculus for graphing and also the concepts of implicit and logarithmic differentiation.

Exponential Functions

Definition 1.1. The function $f(x) = b^x$, with $b > 0$ and $b \neq 1$, is called the **exponential function with base** b.

That is, every positive number other than 1 is the base of an exponential function, and the variable in the function is the exponent, i.e. the power to which the base is raised.

Recall that the **domain** of a function $f(x)$ is the set of all x-values for which that function is defined. For any $b > 0$ with $b \neq 1$, the function $f(x) = b^x$ is defined everywhere, i.e. for all real values of x. Therefore the domain of any exponential function is the set of all real numbers, i.e. the interval $(-\infty, \infty)$.

Similarly, recall that the **range** of a function $f(x)$ is the set of all values which the function has, anywhere in its domain. That is, for $y = f(x)$, the range is the set of y-values which can occur. For any exponential function, when a positive base is raised to any power, no matter whether positive or negative, integer or otherwise, the result is always a positive number. That is, when $b > 0$ and $b \neq 1$, the value of b^x is always positive, no matter what the value of x is. And *any* positive value can be obtained, so the range of any exponential function is the set of all positive real numbers, i.e. the interval $(0, \infty)$.

Graphs of Exponential Functions

For any $b > 1$, the graph of $y = b^x$ looks like the graph in Figure 1, below. Notice that this graph is increasing throughout its domain, is concave upward everywhere and passes through the point (0,1). Also notice that the line $y = 0$ is a horizontal asymptote, with $\lim_{x\to-\infty} b^x = 0$, while $\lim_{x\to\infty} b^x = \infty$.

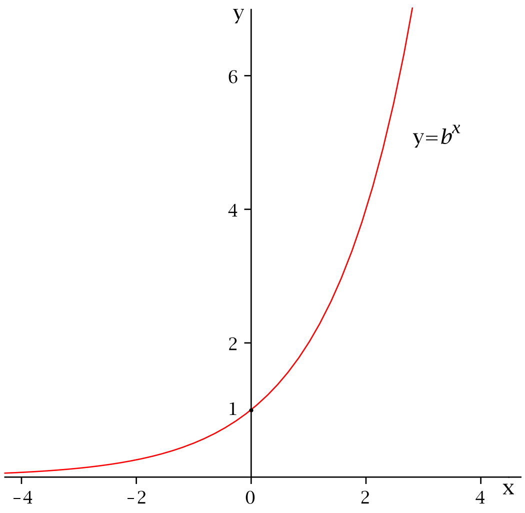


Figure 1: An exponential function with base $b > 1$

For any b with $0 < b < 1$, the graph of $y = b^x$ looks like the graph in Figure 2 (below). It looks just like the previous graph reflected in the y-axis (i.e. flipped over left-to-right). So this time the graph is decreasing throughout $(-\infty, \infty)$, but it is still concave upward everywhere and again passes through the point (0,1). As before, the line $y = 0$ is a horizontal asymptote, but this time it's approached at the other end, i.e. $\lim\limits_{x\to\infty} b^x = 0$, while $\lim\limits_{x\to-\infty} b^x = \infty$.

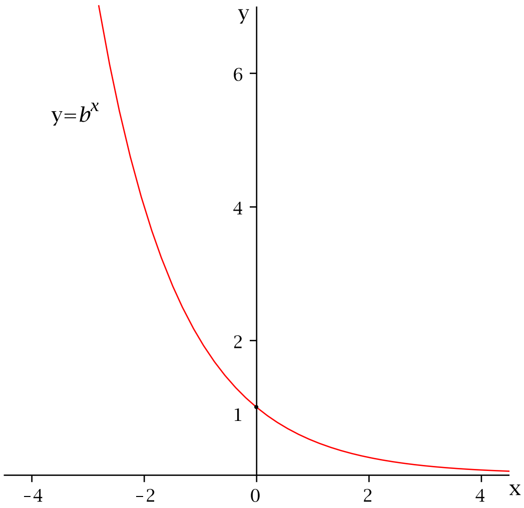


Figure 2: An exponential function with base $b < 1$ (but still, of course, with $b > 0$)

We can summarize the properties we have observed here.

Properties of Exponential Functions

For any function $f(x) = b^x$, with $b > 0$ and $b \neq 1$,

1. The domain of f is $(-\infty, \infty)$.
2. The range of f is $(0, \infty)$.
3. $f(0) = 1$, so the graph of $y = f(x)$ passes through the point $(0, 1)$.
4. f is continuous on $(-\infty, \infty)$.
5. If $b > 1$, then f is increasing throughout $(-\infty, \infty)$, with $\lim\limits_{x\to\infty} f(x) = \infty$, or if $0 < b < 1$, then f is decreasing throughout $(-\infty, \infty)$, with $\lim\limits_{x\to-\infty} f(x) = \infty$.
6. The line $y = 0$ is a horizontal asymptote of the graph of $y = f(x)$, with $\lim\limits_{x\to-\infty} f(x) = 0$ if $b > 1$, or with $\lim\limits_{x\to\infty} f(x) = 0$ if $0 < b < 1$.

If you're going to be working with exponential functions, you need to remember how exponents work. That is, there are certain properties that apply when we work with exponents, but even before we get to those, you need to remember what exponents *mean*.

Recall that for any positive integers m and n:

- b^m means m copies of b multiplied together, i.e.:

$$b^m = \underbrace{b \times b \times ... \times b}_{m \text{ of these}}$$

- $b^{-m} = \dfrac{1}{b^m}$
- $b^{\frac{1}{m}} = \sqrt[m]{b}$, the m^{th} root of b.
  i.e. $b^{\frac{1}{m}}$ is the number that when m copies of it are multiplied together, the value of the product is b, so that $(b^{\frac{1}{m}})^m = b$. And remember: For $m = 2$ we write $\sqrt[2]{b}$ as just $\sqrt{b}$.
- $b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$.

Notice: The last of these tells 2 different ways to interpret something of the form $b^{m/n}$. Which of those you should use to evaluate something of that form depends on the numbers involved. For instance to evaluate $8^{2/3}$ using $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$ is easier than using $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$. On the other hand, for $3^{2/3}$, it is easier to use $3^{2/3} = \sqrt[3]{3^2} = \sqrt[3]{9}$ than to use $3^{2/3} = \left(\sqrt[3]{3}\right)^2 = \left(\sqrt[3]{3}\right)\left(\sqrt[3]{3}\right) = \sqrt[3]{(3)(3)} = \sqrt[3]{9}$.)

Properties of Exponents

1. $b^x b^y = b^{x+y}$

$$\text{e.g. } 2^2 \times 2^3 = \underbrace{(2 \times 2) \times (2 \times 2 \times 2)}_{5 \text{ of them}} = 2^5 = 2^{2+3}$$

2. $\dfrac{b^x}{b^y} = b^{x-y}$

$$\text{e.g. } \frac{2^2}{2^3} = \frac{2 \times 2}{2 \times 2 \times 2} = \frac{1}{2} = 2^{-1} = 2^{2-3}$$

3. $(b^x)^y = b^{xy}$

$$\text{e.g. } (2^3)^2 = \underbrace{(2 \times 2 \times 2) \times (2 \times 2 \times 2)}_{6 \text{ of them}} = 2^6 = 2^{3\times 2}$$

Example 1.1. Express $8\left(2^{x^3-2}\right)$ as simply as possible.

Solution:
Realizing that $8 = 2^3$, and using property 1, we get

$$8\left(2^{x^3-2}\right) = 2^3 \times 2^{x^3-2} = 2^{3+(x^3-2)} = 2^{x^3+1}$$

Example 1.2. Express $\sqrt{\dfrac{b^{3x+2}}{b^3 b^{x+1}}}$ as simply as possible.

Solution:

$$\sqrt{\frac{b^{3x+2}}{b^3 b^{x+1}}} = \left(\frac{b^{3x+2}}{b^{3+x+1}}\right)^{\frac{1}{2}} = \left(\frac{b^{3x+2}}{b^{x+4}}\right)^{\frac{1}{2}} = \left(b^{(3x+2)-(x+4)}\right)^{\frac{1}{2}} = \left(b^{2x-2}\right)^{\frac{1}{2}} = b^{(2x-2)\times\frac{1}{2}} = b^{x-1}$$

Logarithmic Functions

Definition 1.2. For any $b > 0$ and $b \neq 1$, the function $f(x) = \log_b x$ is the **logarithmic funtion with base** b.

Notice that $\log_b x$ is shorthand for, and is pronounced as, "the logarithm to base b of x". You can think of "logarithm to base b of x" as meaning "the power to which the base b would need to be raised, to get the value x". Another way to say this, and *the one and only thing you need to remember*, to understand what $\log_b x$ means, is:

$$y = \log_b x \qquad \text{says the same thing as} \qquad x = b^y$$

(For instance, $y = \log_2 x$ says the same thing as $x = 2^y$, so when $x = 4$ we have $y = \log_2 4$ saying that $4 = 2^y$, and since $4 = 2^2$, we see that this gives $y = 2$. That is, $y = \log_2 4$ gives $y = 2$.)

As with exponential functions, every positive number other than 1 is the base of some logarithmic function. The variable in a logarithmic function is the number whose logarithm is being calculated.

From the relationship between logarithmic functions and exponential functions, we see that the domain of one is the range of the other. That is, the logarithmic function with base b is defined for only those values which can be produced as the value of the exponential function with base b (the x's in the above relationship). And the function values of the logarithmic function with base b can be any y-value for which the exponential function b^y is defined. So whereas, for any $b > 0$ with $b \neq 1$, the exponential function with base b has domain $(-\infty, \infty)$ and range $(0, \infty)$, the logarithmic function with base b has domain $(0, \infty)$ and range $(-\infty, \infty)$. That is, the function $y = \log_b x$ is defined only for positive values of x, but y (the value of the logarithm) can be any real value.

Graphs of Logarithmic Functions

Because of the relationship between exponential and logarithmic functions, the graphs of the functions are related, too. For a base $b > 1$, the graph of $y = \log_b x$ is shown in Figure 3. Notice that it is increasing and concave downward throughout its domain. And it passes through the point (1,0). (Recall that $y = b^x$ passes through (0,1).) We see that the line $x = 0$ is a vertical asymptote, with $\lim\limits_{x \to 0^+} \log_b x = -\infty$, and that $\lim\limits_{x \to \infty} \log_b x = \infty$. (The graph of $y = b^x$ for $b > 1$ also "ran off to infinity" as x approached ∞, but it did so much more quickly than the graph here does.)

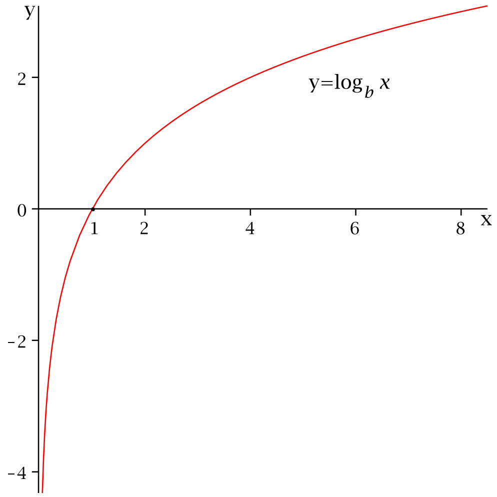


Figure 3: A logarithmic function with base $b > 1$

And for any b with $0 < b < 1$, the graph of $y = \log_b x$ looks like the graph in Figure 4. Again, it looks just like the previous graph reflected, but this time the reflection is in the x-axis (i.e. flipped over top-to-bottom). So this time the graph is decreasing throughout $(0, \infty)$, and is concave upward everywhere, but as before the graph passes through the point (1,0). Once again, the line $x = 0$ is a vertical asymptote, but the graph is "running off" in the opposite direction. That is, we see that $\lim_{x\to 0^+} \log_b x = \infty$, while $\lim_{x\to\infty} \log_b x = -\infty$.

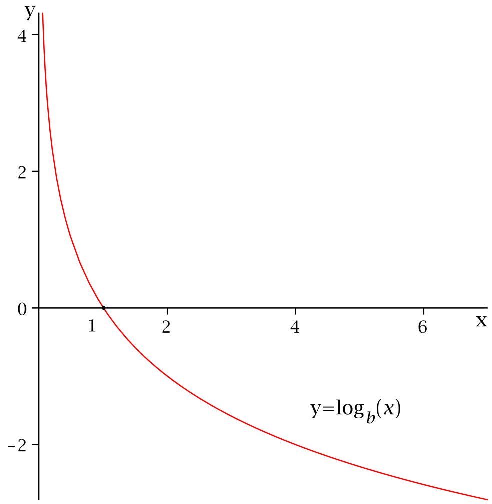


Figure 4: A logarithmic function with base $b < 1$ (but still, of course, with $b > 0$)

You may find it useful to notice that (assuming you've printed out these notes) if you turn the page over (i.e. look *through* the page) and turn it sideways (90 degrees clockwise), figures 3 and 4 look just like figures 1 and 2, respectively.

Once again, we can summarize what we've observed as a list of properties that these kinds of functions always have.

Properties of Logarithmic Functions

For any function $f(x) = \log_b x$, with $b > 0$ and $b \neq 1$,

1. The domain of f is $(0, \infty)$.
2. The range of f is $(-\infty, \infty)$.
3. $f(1) = 0$, so the graph of $y = f(x)$ passes through the point $(1, 0)$.
4. f is continuous on $(0, \infty)$.
5. If $b > 1$, then f is increasing throughout $(0, \infty)$, with $\lim_{x\to\infty} f(x) = \infty$, or if $0 < b < 1$, then f is decreasing throughout $(0, \infty)$, with $\lim_{x\to\infty} f(x) = -\infty$.
6. The line $x = 0$ is a vertical asymptote of the graph of $y = f(x)$, with $\lim_{x\to 0^+} f(x) = -\infty$ if $b > 1$, or with $\lim_{x\to 0^+} f(x) = \infty$ if $0 < b < 1$.

Both from our observations about the domain and range etc., and from looking at the graphs, we see that the functions $y = b^x$ and $y = \log_b x$ just have the roles of x and y reversed, so these functions "undo" each other (when they have the same base). That is, if you raise the base to some value, and then take the logarithm to that base, you just get back to the value you started with (i.e. raised the base to). Likewise, if you take the logarithm of some number, but then raise the base to that new number, you get back to the original number (the one you took the logarithm of). That is, we see that

$$\log_b b^x = x \qquad \text{and} \qquad b^{\log_b x} = x$$

Notice: Using the relationship that we said was the one and only thing you need to remember, we have:

$$\begin{array}{rlll} y = \log_b b^x & \text{says that} & b^x = b^y & \text{so } y = x \\ \text{and } y = b^{\log_b x} & \text{says that} & \log_b x = \log_b y & \text{so again } y = x \end{array}$$

Example 1.3. Evaluate (a) $\log_2 8$ (b) $\log_3 \frac{1}{3}$ (c) $\log_4 (16)^3$

Solution:
(a) We know that $8 = 2^3$, so we have

$$\log_2 8 = \log_2 2^3 = 3$$

That is, "What is the logarithm of 8 to base 2?" really says "What power would 2 need to be raised to, in order to get 8?". And since 8 is 2^3, then the answer is: 3.

Thinking of this another way, let x be the value we're looking for. That is, let $x = \log_2 8$. But $x = \log_2 8$ says the same thing as $8 = 2^x$, and we know that $8 = 2^3$, so we see that $x = 3$.

(b) To find the value of $\log_3 \frac{1}{3}$, we write $\frac{1}{3}$ as a power of 3 and use the fact that $\log_b b^x = x$. That is, we have

$$\log_3 \frac{1}{3} = \log_3 3^{-1} = -1$$

Again, "What is the logarithm of 1/3, to base 3?" simply asks "What power of 3 is 1/3?". And since $\frac{1}{3} = 3^{-1}$, the answer is: -1.

Or, let $x = \log_3 \frac{1}{3}$. Then this says the same thing as $\frac{1}{3} = 3^x$, so we see that $x = -1$.

(c) This time, we use the fact that $16 = 4^2$:

$$\log_4 (16)^3 = \log_4 (4^2)^3 = \log_4 4^{2\times 3} = \log_4 4^6 = 6$$

That is, the question was "What power would you need to raise 4 to, to get the value 16^3?" The answer is: 6. That is, when 4 is raised to the power 6, it gives the same value as 16^3.

Letting $x = \log_4 (16)^3$, we use the "one thing" to see that this says that $4^x = (16)^3$, and since $(16)^3 = (4^2)^3 = 4^6$, then clearly $x = 6$.

Two more useful facts:

$$\log_b b = 1 \qquad \text{and} \qquad \log_b 1 = 0$$

That is, $b = b^1$, so $\log_b b = \log_b b^1 = 1$. Likewise, $1 = b^0$, so $\log_b 1 = \log_b b^0 = 0$.

Properties of Logarithms

These properties follow directly from the properties of exponents.

1. $\log_b(xy) = \log_b x + \log_b y$

   Proof: Let $u = \log_b x$ and $v = \log_b y$.
   Then we know that this means that $x = b^u$ and $y = b^v$, so we have

   $$\log_b(xy) = \log_b[(b^u)(b^v)] = \log_b b^{u+v} = u + v = \log_b x + \log_b y$$

2. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

   Proof: As before, let $u = \log_b x$ and $v = \log_b y$, so that $x = b^u$ and $y = b^v$. This time we get

   $$\log_b\left(\frac{x}{y}\right) = \log_b\left(\frac{b^u}{b^v}\right) = \log_b b^{u-v} = u - v = \log_b x - \log_b y$$

3. $\log_b(x^r) = r\,(\log_b x)$

   Proof: Let $u = \log_b x$, which means that $x = b^u$. Then we have

   $$\log_b(x^r) = \log_b (b^u)^r = \log_b b^{ur} = ur = r\,(\log_b x)$$

Notice: Be careful about using these properties. They say that a product or quotient *inside a logarithm* can be expressed as a sum or difference, respectively, *of* logarithms. And likewise that an exponent *inside a logarithm* corresponds to a multiplier *on the logarithm.* But a product of logarithms, a quotient of logarithms, or an exponent on a logarithm are not the things these properties tell about, and cannot be simplified using these properties. For instance, $(\log_2 3)(\log_2 5)$ and $\frac{\log_2 3}{\log_2 5}$ and $(\log_2 3)^5$ are all expressions which cannot be simplified using the properties listed above.

Example 1.4. Simplify:

(*a*) $\log_2 24 - \log_2 6$ (*b*) $\log_2(8x^2)$ (*c*) $\log 25 + \log 4$

Solution:

(a) $\log_2 24 - \log_2 6 = \log_2\left(\frac{24}{6}\right) = \log_2 4 = \log_2 2^2 = 2$

(b) $\log_2(8x^2) = \log_2 8 + \log_2 x^2 = \log_2 2^3 + \log_2 x^2 = 3 + 2\log_2 x$

(c) We are asked to simplify $\log 25 + \log 4$. There's no base in those logarithms! What's that about? You may have forgotten that:

> **FACT:** Base 10 is so commonly used that a logarithm to base 10 is called a *common* logarithm, and the base is normally omitted.

So whenever you see log without a base, that means logarithm to base 10. Recognizing that, we get:

$$\log 25 + \log 4 = \log(25 \times 4) = \log 100 = \log_{10} 10^2 = 2$$

Note: In each of the above problems, once we get something of the form $\log_b b^a$ for some number a, we can see that this is simply equal to a using any of (i) the definition of logarithm, or (ii) our "rule" that $\log_b b^x = x$, or (iii) the third property above, together with the fact that $\log_b b = 1$, so that we have $\log_b b^a = a(\log_b b) = a(1) = a$. It doesn't matter how you think of getting there, as long as you do get from $\log_b b^a$ to just a.

Example 1.5. If $\log_b x = 3$ and $\log_b y = -2$, find $\log_b x^3y^2$.

Solution:
Using the first and then the third properties of logarithms, we then use the values given, as follows:

$$\log_b x^3y^2 = \log_b x^3 + \log_b y^2 = 3\log_b x + 2\log_b y = 3(3) + 2(-2) = 9 - 4 = 5$$

Example 1.6. Simplify:

(*a*) $3^{-2\log_3 5}$ (*b*) $\log_3 36 + \log_3 2 - \log_3 8$

Solution:
(a) We use the various properties and other facts, as we need them:

$$\begin{aligned} 3^{-2\log_3 5} &= 3^{\log_3 5^{-2}} && \text{using the fact that } r\log_b a = \log_b a^r \\ & && \text{with } r = -2,\ b = 3 \text{ and } a = 5 \\ &= 5^{-2} && \text{using } b^{\log_b a} = a, \text{ with } b = 3 \text{ and } a = 5^{-2} \\ &= \frac{1}{5^2} = \frac{1}{25} \end{aligned}$$

(b) We can express 36 as a product, and use the various properties:

$$\begin{aligned} \log_3 36 + \log_3 2 - \log_3 8 &= \log_3(9\times 4) + \log_3\left(\tfrac{2}{8}\right) && \text{applying Property 2 to re-express the difference} \\ &= \log_3 9 + \log_3 4 + \log_3\left(\tfrac{1}{4}\right) && \text{applying Property 1 to the product} \\ &= \log_3 3^2 + \log_3\left[4\times\left(\tfrac{1}{4}\right)\right] && \text{using Property 1 again} \\ &= 2 + \log_3 1 = 2 && \text{using } \log_3 3^a = a, \text{ and also } \log_3 1 = 0 \end{aligned}$$

(Other approaches work, perhaps even more quickly. For instance,

$$\log_3 36 + \log_3 2 - \log_3 8 = \log_3(36\times 2) - \log_3 8 = \log\left(\frac{36\times 2}{8}\right) = \log_3 9 = \log_3 3^2 = 2)$$

When we have an equation, and we need to solve for some unknown value x which appears in an exponent, it is often useful to take logarithms of both sides of the equation. That is, we take the equation LHS = RHS and re-express it as $\log_b \text{LHS} = \log_b \text{RHS}$ for some base b. Likewise, if the unknown is inside a logarithm to base b, it may be useful to re-express LHS = RHS as $b^{\text{LHS}} = b^{\text{RHS}}$. (We can do this because if LHS = RHS, then if we do the same thing to both LHS and RHS, the equality still holds.)

Example 1.7. Solve for x in each of the following.

(*a*) $2^{3x-4} = \frac{1}{4}$ (*b*) $\log_3(2x-1) = -1$

Solution:

(a) Here, the unknown is in the exponent (on 2), so we take logarithms, to base 2, of both sides:

$$\begin{aligned} 2^{3x-4} = \frac{1}{4} \quad &\Rightarrow \quad \log_2 2^{3x-4} = \log_2\left(\frac{1}{4}\right) &&\Rightarrow \quad 3x - 4 = \log_2 2^{-2} \\ &\Rightarrow \quad 3x - 4 = -2 &&\Rightarrow \quad 3x = -2 + 4 \\ &\Rightarrow \quad x = \frac{-2+4}{3} = \frac{2}{3} \end{aligned}$$

(b) This time, the x is inside a logarithm to base 3, so we make this an exponent on a 3, i.e. we equate 3^{LHS} to 3^{RHS}:

$$\begin{aligned}\log_3(2x-1) = -1 \quad &\Rightarrow \quad 3^{\log_3(2x-1)} = 3^{-1} \quad \Rightarrow \quad 2x - 1 = \tfrac{1}{3} \\ &\Rightarrow \quad 2x = \tfrac{1}{3} + 1 = \tfrac{4}{3} \qquad \Rightarrow \quad x = \tfrac{4}{3} \times \tfrac{1}{2} = \tfrac{2}{3}\end{aligned}$$

Change of Base

Suppose we have a logarithm to one base, but (for whatever reason) it would be more useful to have the logarithm expressed to another base. We can easily do that using the *change of base formula.*

Consider some logarithm, $\log_b a$. Let's call that y. That is, set $y = \log_b a$. Then we know that this says the same thing as $a = b^y$. Now, suppose we would rather be working with a logarithm to some other base, c. We can take logarithms to base c of both sides of the current equation, and then use properties of logarithms to restate the logarithm on the right side, and finally rearrange the resulting equation:

$$a = b^y \quad \Rightarrow \quad \log_c a = \log_c b^y \quad \Rightarrow \quad \log_c a = y \log_c b \quad \Rightarrow \quad \frac{\log_c a}{\log_c b} = y$$

But remember, we started with $y = \log_b a$. So we get

Change of Base Formula: $$\log_b a = \frac{\log_c a}{\log_c b}$$

That is, we can change the base of a logarithm, simply by taking the logarithm to the new base (of whatever we were taking the logarithm of before) and dividing by the logarithm, to the new base, *of the old base.*

Example 1.8. Express $\log_2 100$ using (a) logarithms to base 5 and (b) common logarithms.
Solution:
For both, we just use the change of base formula above.
(a) To express a log to base 2 as a logarithm to base 5, we change the base, and divide by the logarithm (to this new base) of the original base, 2. So we get

$$\log_2 100 = \frac{\log_5 100}{\log_5 2}$$

(b) Of course a common logarithm is a logarithm to base 10. And since the thing we're taking the log of is a power of 10, we can simplify after we change the base:

$$\log_2 100 = \frac{\log 100}{\log 2} = \frac{\log 10^2}{\log 2} = \frac{2}{\log 2}$$

Example 1.9. Evaluate $\log_8 16$ by changing the base of the logarithm.
Solution:
We see that both the base of the logarithm and the thing we're taking the logarithm of are powers of 2. So it will be useful to change the logarithm to base 2:

$$\log_8 16 = \frac{\log_2 16}{\log_2 8} = \frac{\log_2 2^4}{\log_2 2^3} = \frac{4}{3}$$

That was much easier than realizing, or needing to figure out, that $16 = 8^{4/3}$, or any other way of evaluating the given logarithm.

The Natural Exponential and Logarithmic Functions

You will remember from your introductory calculus course that there's a special number, e, which is often used as the base of exponential and logarithmic functions. Because this number often arises in nature, these functions are referred to as the *natural* exponential and logarithmic functions. There are a couple of different ways to define the number e, but those aren't important to us. For our purposes, we can define e as follows.

Definition 1.3. e is the number such that the graph of the function $f(x) = e^x$ crosses the y-axis with slope 1.

Of course, it's useful to have *some* idea what number this is. Suffice it to say that e is an irrational number between 2 and 3, or more specifically that $e = 2.71828....$ (But you won't be expected to remember that.)

And we have special notation for *natural logarithms*. That is, because this natural base arises so often, it's used *at lot*, and so we have shorthand for the shorthand!

Definition 1.4. $\ln x$ means $\log_e x$.

Notice: In the context of the natural base, the cancellation properties we learned earlier can be expressed as:

$$\ln e^x = x \qquad \text{and} \qquad e^{\ln x} = x$$

Figure 5 shows the graphs of the functions $f(x) = e^x$ and $f(x) = \ln x$. Their shapes are exactly what you would expect for a base larger than 1.

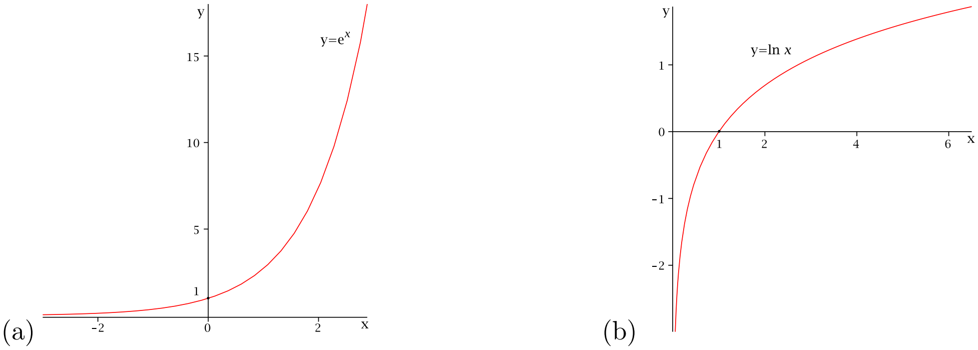


Figure 5: The graphs of (a) the natural exponential function and (b) the natural logarithmic function.

Example 1.10. Let $f(x) = \ln\left(\frac{1}{e^{x^2-3}}\right)$.

(a) Simplify $f(x)$. (b) Evaluate $f(\sqrt{e^3})$.

Solution:
(a) We use the various properties and/or rules of exponents and logarithms.

$$\ln\left(\frac{1}{e^{x^2-3}}\right) = \ln e^{-(x^2-3)} = -(x^2-3) = 3 - x^2$$

We see that $f(x) = 3 - x^2$. (Notice that for any value of x, $e^{x^2-3} > 0$ so (i) the denominator of $\frac{1}{e^{x^2-3}}$ is never 0, and (ii) the value of $\frac{1}{e^{x^2-3}}$ is in the domain of the ln function. So $f(x)$ is defined for all $x \in (-\infty, \infty)$, no matter which way we express it.)

(b) Since $\sqrt{e^3} = e^{3/2}$, we get

$$f(\sqrt{e^3}) = f(e^{3/2}) = 3 - (e^{3/2})^2 = 3 - e^{(3/2)\times 2} = 3 - e^3$$

Transformations of Functions

Next, we look at the effect on the shape of a function of various kinds of transformations, and look at how those effects apply to exponential and logarithmic functions.

Vertical Translation

For any function $y = f(x)$, adding a constant c to the function value simply shifts the graph vertically by c units. That is, for any function $f(x)$, the graph of $y = f(x) + c$ for some constant c looks just like the graph of $y = f(x)$, shifted up by c units if $c > 0$, or shifted down by $|c|$ units if $c < 0$. (That is shifting up by a negative number of units means shifting down. As always, negative movement means movement in the opposite direction.)

Example 1.11. Sketch the graphs of (a) $y = 2^x - 3$ and (b) $y = 1 + \ln x$.

Solution:
(a) For $f(x) = 2^x$, we know that the graph of $y = f(x)$ has the same shape as the graph in Figure 1 on p. 1 (since $2 > 1$). And for $g(x) = 2^x - 3$, we simply subtract 3 from each function value, so that at any particular horizontal position $x = a$, $g(a) = f(a) - 3$, so the height of the graph of $y = g(x)$ is 3 units lower than the height of the graph of $y = f(x)$.

This means that where we have $y = 2^x$ passing through the point $(0, 1)$, and approaching the horizontal asymptote $y = 0$ at the left end, instead we have the graph of $y = 2^x - 3$ passing through the point $(0, -2)$ and approaching the horizontal asymptote $y = -3$ at the left end.

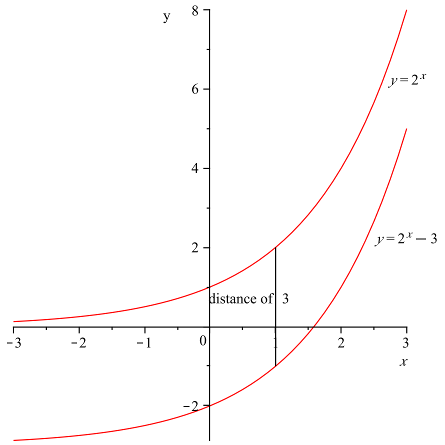


Figure 6: The graph of $y = 2^x - 3$.

(b) We know that $y = \ln x$ is defined only for $x > 0$, so $y = 1 + \ln x$ is, too. And $y = \ln x$ has $x = 0$ as a vertical asymptote – so does $y = 1 + \ln x$, since

$$\lim_{x\to 0^+} (1 + \ln x) = 1 + \left(\lim_{x\to 0^+} \ln x\right) = 1 + (-\infty) = -\infty$$

Of course, $y = \ln x$ passes through $(1, 0)$, and so $y = 1 + \ln x$ passes through $(1, 1)$. That is, for $f(x) = \ln x$ and $g(x) = 1 + \ln x$, we have $f(1) = \ln\ 1 = 0$, and so $g(1) = 1 + f(1) = 1$.

The whole graph of $y = 1 + \ln x$ is just the graph of $y = \ln x$ shifted up 1 unit. (Near the vertical asymptote, it can be hard to see that's what's happening.)

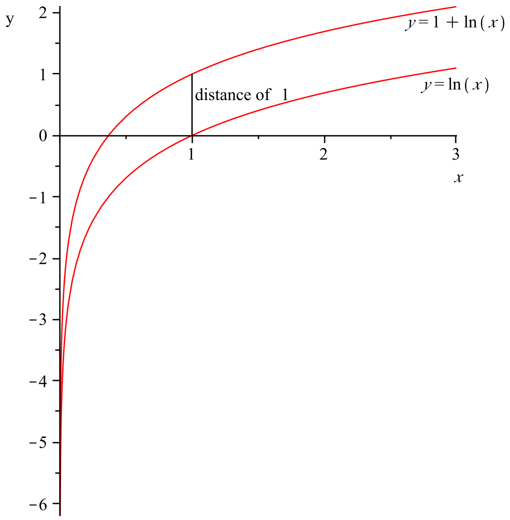


Figure 7: The graph of $y = 1 + \ln x$.

Horizontal Translation

As we saw above, vertical translation, i.e. shifting a function up or down, is accomplished by adding (or subtracting) a constant to the function value, *after* applying the function. If we instead add a constant to the x-value *before* applying the function, we translate the graph horizontally, shifting the graph to the left by the value of the constant (or to the right, if the constant is negative – i.e. moving to the left by a negative number of units means moving to the right).

Notice that the direction of this shift may seem counterintuitive to you. For instance, the graph of $y = f(x + 1)$ lies 1 unit to the *left* of the graph of $y = f(x)$ – because to get the same y-value, you need an x-value that is 1 unit *smaller* with $f(x + 1)$ than with $f(x)$.

So adding a positive constant to the x-value before applying the function shifts the whole graph to the left, while adding a negative constant (i.e. subtracting a positive constant) shifts the whole graph to the right. And the size of the shift is always the magnitude of the constant. (That is, $f(x + k)$ is $f(x)$ shifted by $|k|$ units – to the left if $k > 0$, or to the right if $k < 0$.)

Example 1.12. Sketch the graph of (a) $y = \ln\ (x - 2)$ and (b) $y = 2^{x+1} - 3$.

Solution:
(a) For $y = \ln\ (x - 2)$, we shift the whole of the graph of $y = \ln x$ to the right by 2 units. The domain is $(2, \infty)$, i.e. all x for which $x - 2 > 0$, and the asymptote is $x = 2$ instead of $x = 0$. The graph is shown on the next page (Figure 8(a)).

(b) In Example 1.11(a) we sketched the graph of $y = 2^x - 3$. The graph of $y = 2^{x+1} - 3$ is the same, but shifted one unit to the left. So whereas $y = 2^x - 3$ passes through $(0, -2)$, the graph of $y = 2^{x+1} - 3$ passes through $(-1, -2)$. Both graphs have horizontal asymptote $y = -3$ approached at the left end.

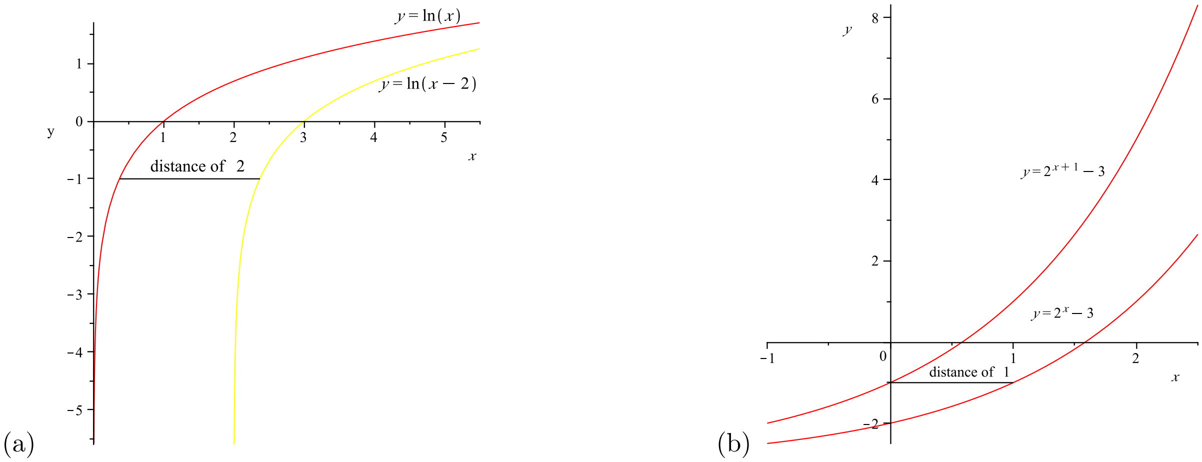


Figure 8: The graphs of (a) $y = \ln\,(x - 2)$ and (b) $y = 2^{x+1} - 3$.

Reflection

Consider any function $f(x)$. For $g(x) = -f(x)$, the graph of $y = g(x)$ looks like the graph of $y = f(x)$ reflected in the x-axis, i.e. flipped upside down (vertical reflection). And for $h(x) = f(-x)$, the graph of $y = h(x)$ looks like the graph of $y = f(x)$ relected in the y-axis, i.e. flipped left-to-right (horizontal reflection).

Consider the effect of vertical reflection for exponential functions. The graph of $y = -b^x$ has the same asymptote as $y = b^x$, but the graph lies entirely below the x-axis, instead of above, and so it approaches the asymptote from below instead of from above. And at the other end, the graph approaches $-\infty$ instead of ∞.

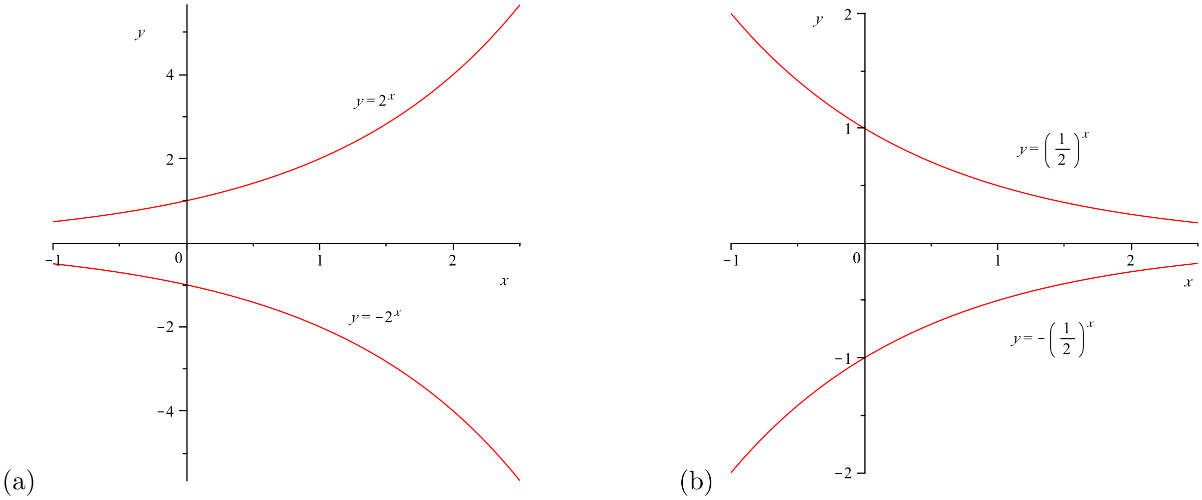
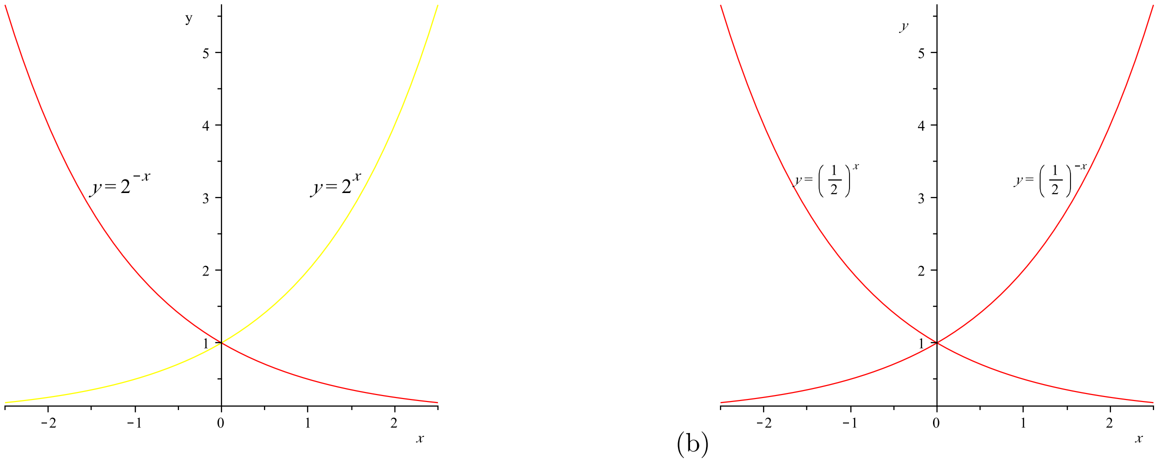


Figure 9: The graphs of $y = -b^x$ for (a) $b = 2$ and (b) $b = \dfrac{1}{2}$.

Horizontal reflection of an exponential function, though, produces a graph which we've already seen. Because

$$b^{-x} = \frac{1}{b^x} = \left(\frac{1}{b}\right)^x$$

(Realizing that 1 raised to any power is just 1, so we can always replace 1 by 1 raised to a power. In this case it is convenient to consider 1 to be 1^x.) Letting $a = \frac{1}{b}$, we see that if $b > 1$, then we have $0 < a < 1$, while if $0 < b < 1$, we have $a > 1$. So the graph of $y = b^{-x} = a^x$ has the shape of the "other" kind of exponential function (from whichever shape the graph of $y = b^x$ has). That is, if $b > 1$, then the graph of $y = b^{-x}$ has the shape of an exponential function whose base is between 0 and 1, whereas if $0 < b < 1$, then the graph of $y = b^{-x}$ has the shape of an exponential function whose base is bigger than 1. This, of course, is because those two possible shapes of exponential functions are just the horizontal reflections of one another.



(a) (b)

Figure 10: The graphs of $y = b^x$ and $y = b^{-x}$ for (a) $b = 2$ and (b) $b = \dfrac{1}{2}$.

For logarithmic functions, since the relationship between exponential and logarithmic functions corresponds to having the roles of x and y interchanged, we get the opposite sort of thing. Vertical reflection produces a graph of the "other" logarithmic form, whereas horizontal reflection gives a new graph, which lies entirely to the left of the y-axis instead of lying entirely to the right. That is, the two possible shapes of graphs of logarithmic functions are vertical reflections of one another, while the function $y = \log_b(-x)$ has domain $(-\infty, 0)$, and has vertical asymptote $x = 0$ approached from the left instead of from the right. Graphs showing the 2 kinds of relection for a logarithmic function with $b > 1$ are shown in Figure 11, on the next page.

Notice: For vertical reflection of a logarithmic function, we can see why it gives a logarithmic function of the "other" shape by considering the effect of the negative. For $f(x) = \log_b x$ and $g(x) = -\log_b x$, we have $g(x) = (-1)[f(x)]$. But then

$$g(x) = -\log_b x = (-1)\log_b x = \log_b x^{-1} = \log_b\left(\frac{1}{x}\right)$$

And of course $y = \log_b\left(\frac{1}{x}\right)$ says that $\frac{1}{x} = b^y$, so that we have

$$\frac{1}{x} = b^y \quad \Rightarrow \quad xb^y = 1 \quad \Rightarrow \quad x = \frac{1}{b^y} = \left(\frac{1}{b}\right)^y$$

Since $x = \left(\frac{1}{b}\right)^y$ says the same thing as $y = \log_{1/b}\, x$, we see that $y = -\log_b x$ is the same function as $y = \log_{1/b}\, x$. (Where, as before, $a = \frac{1}{b}$ is bigger than 1 if $0 < b < 1$, or is between 0 and 1 if $b > 1$, so that this function has the "other" logarithmic shape.)

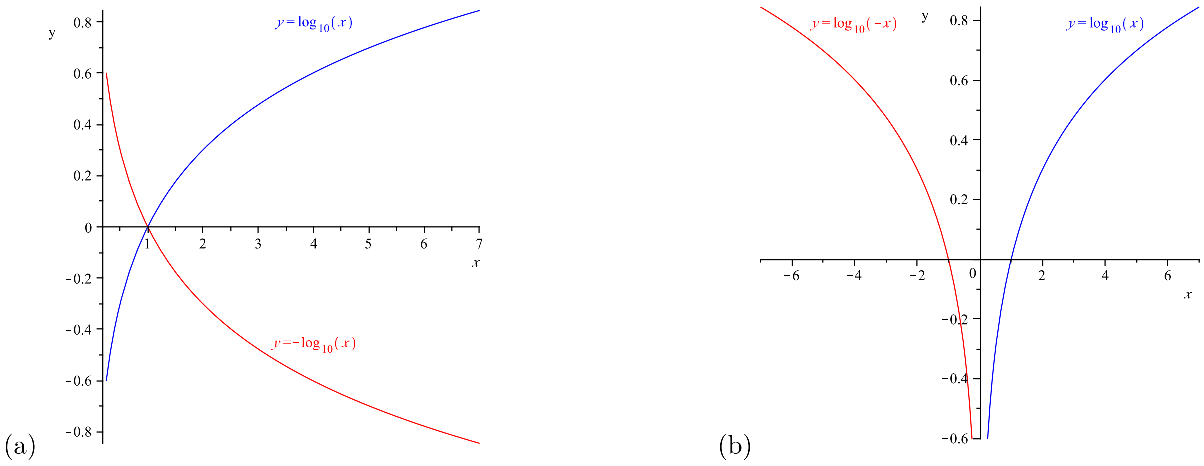


Figure 11: The graphs of $y = \log_b x$ with (a) $y = -\log_b x$ and (b) $y = \log_b(-x)$ (for $b = 10$).

Vertical Stretching

For any positive constant c, multiplying a function by this constant causes the graph to be stretched vertically, where this stretching expands the graph if $c > 1$, or compresses the graph if $0 < c < 1$.

For exponential functions, the graph of $y = c(b^x)$ with $c > 1$ climbs toward infinity c times as fast as the graph of $y = b^x$, and approaches zero only $\frac{1}{c}$ times as fast at the other end. Or for $0 < c < 1$, the behaviour is the opposite – climbing toward infinity more slowly and approaching 0 more quickly.

Example 1.13. Sketch the graphs of $y = e^x$, $y = 2e^x$ and $y = \frac{1}{2}(e^x)$.

Solution:
We know what the graph of $y = e^x$ looks like. For $y = 2(e^x)$, the height of the function is always twice the height of $y = e^x$. That is, for $f(x) = e^x$ and $g(x) = 2e^x$, at any particular value $x = a$ we have $g(a) = 2f(a)$. So $y = g(x)$ crosses the y-axis at height 2, passing through the point $(0, 2)$ instead of $(0, 1)$. And to the right of this, it climbs twice as fast, reaching height 4 at the same x-value at which $y = e^x$ reaches height 2. (That x-value being $x = \ln 2$.) On the left side, thinking about what happens as we move further left away from the y-axis, the graph is still approaching the horizontal asymptote $y = 0$, but is approaching only half as quickly – with height 1 at the same x-value at which $y = e^x$ has height $\frac{1}{2}$, and with height $\frac{1}{2}$ at the same x-value at which $y = e^x$ has height $\frac{1}{4}$, and so forth. (Those x-values being $x = \ln\frac{1}{2}$ and $x = \ln\frac{1}{4}$, respectively.) The height of $y = 2e^x$ is *always* double the height of $y = e^x$.

For $y = \frac{1}{2}(e^x)$, we have $h(x) = \frac{1}{2}[f(x)] = \frac{f(x)}{2}$, so the graph of $y = h(x)$ always has exactly half the height of $y = f(x)$. Where $y = e^x$ has height 2 (at $x = \ln 2$), $y = h(x)$ has height only 1. Where $y = e^x$ has height $\frac{1}{2}$ (at $x = \ln\frac{1}{2}$), $y = h(x)$ has height only $\frac{1}{4}$, and so forth. Of course, $y = h(x)$ crosses the y-axis at height $\frac{1}{2}$ instead of at height 1. Moving right from there, it climbs only half as fast as $y = f(x)$, whereas moving left it falls twice as fast as $y = f(x)$.

The graph showing these 3 functions is shown on the next page (Fig. 12).

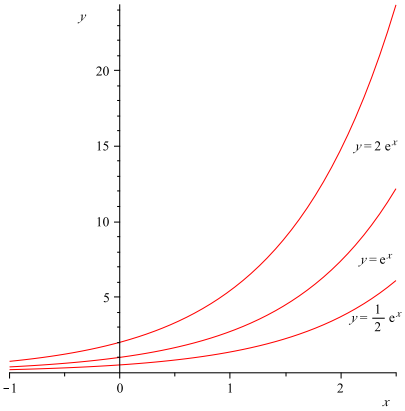


Figure 12: The graphs of $y = e^x$, $y = 2e^x$ and $y = \dfrac{e^x}{2}$.

Notice: Vertical stretching of an exponential function is the same as horizontal translation of the function. That is, because of the shape of the graph, the effect of vertical stretching is to shift the graph horizontally. Let b and c be any positive constants other than 1 and consider the functions $f(x) = b^x$ and $g(x) = c(b^x)$. We can express the constant c as $c = b^{\ln c}$ to see that

$$g(x) = c(b^x) = (b^{\ln c})(b^x) = b^{x+\ln c} = f(x + \ln c)$$

So multiplying b^x by c is the same as adding $\ln c$ to x before applying the function f. That is, vertically stretching $f(x) = b^x$ by a factor of c is the same as shifting the function horizontally by an amount $\ln c$.

Horizontal Stretching

Multiplying the x-value by a positive constant c *before* applying a function has the effect of stretching the function horizontally. The effect is the opposite of what we saw before, insofar as the stretching corresponds to *compression* when $c > 1$ and to *expansion* when $0 < c < 1$. That is, $y = f(2x)$, for instance, attains the same height as $y = f(x)$ when x is only half as big, whereas $y = f\left(\frac{x}{2}\right)$ doesn't reach that height until x is twice as big as for $f(x)$. So $y = f(2x)$ is steeper than $y = f(x)$, while $y = \left(\frac{x}{2}\right)$ is flatter.

For instance, for $f(x) = 3^x$, $g(x) = 3^{2x}$ and $h(x) = 3^{x/2}$, we have $f(x) = 3$ at $x = 1$, whereas $g(x) = 3$ at $x = \frac{1}{2}$, and $h(x) = 3$ at $x = 2$ – so $g(x)$ attains the same height more quickly, while $h(x)$ gets there more slowly. Similarly, the height $\frac{1}{3}$ is attained by $f(x)$ at $x = -1$, while $g(x)$ has that height at $x = -\frac{1}{2}$ but $h(x)$ doesn't get there until $x = -2$ (if you think about moving away from the y-axis). But of course all of these 3 functions pass through $(0, 1)$, since $f(0) = g(0) = h(0) = 3^0 = 1$. So $y = 3^{2x}$ rises more quickly as x increases from 0, and falls more quickly as x decreases from 0, as compared to $y = 3^x$, resulting in a steeper graph everywhere. On the other hand $y = 3^{x/2}$ rises more slowly as x increases from 0, and falls more slowly as x decreases from 0 (as compared to $y = 3^x$), making the graph flatter. (See graphs next page, Figure 13.)

Example 1.14. For each of the following, sketch all 3 graphs on the same axes.
(a) $y = \ln x$, $y = 2\ln x$ and $y = \frac{1}{2}(\ln x)$ (b) $y = \ln x$, $y = \ln 2x$ and $y = \ln\left(\frac{x}{2}\right)$

Solution:
(a) Since $\ln 1 = 0$, then for $f(x) = \ln x$, $g(x) = 2\ln x$ and $h(x) = \frac{1}{2}(\ln x)$ we have $f(1) = 0$,

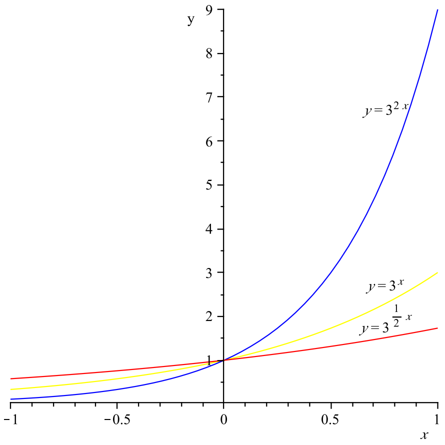


Figure 13: The graphs of $y = 3^x$, $y = 3^{2x}$ and $y = e^{x/2}$.

$g(1) = 2(0) = 0$ and $h(1) = \frac{1}{2}(0) = 0$. That is, the graphs of all of these functions pass through the point $(0, 1)$. But the graph of $y = g(x)$ is more vertically stretched out than the graph of $y = f(x)$, while the graph of $y = h(x)$ is more compressed vertically.

For instance, at the horizontal position $x = e$ we have $f(e) = \ln e = 1$, while $g(e) = 2\ln e = 2(1) = 2$ and $h(e) = \frac{1}{2}(\ln e) = \frac{1}{2}(1) = \frac{1}{2}$. We see that although all 3 graphs cross the x-axis at the same place, at $x = 1$, by $x = e$ the graph of $y = g(x)$ has twice the height that $y = \ln x$ has, while the graph of $y = h(x)$ has only half that height. Or suppose we move closer to 0 than $x = 1$. At $x = \frac{1}{e} = e^{-1}$ we have $f\left(\frac{1}{e}\right) = \ln e^{-1} = -1$, with $g\left(\frac{1}{e}\right) = 2\ln e^{-1} = 2(-1) = -2$ and $h\left(\frac{1}{e}\right) = \left(\frac{1}{2}\right)(\ln e^{-1}) = \frac{1}{2}(-1) = -\frac{1}{2}$. So at this horizontal position, the graph of $y = g(x)$ is twice as far below the x-axis as is the graph of $y = f(x)$, while the graph of $y = h(x)$ is only half that far below. So once again, the graph of $y = 2\ln x$ has changed twice as much in terms of vertical height, compared to $y = \ln x$, while the graph of $y = \frac{1}{2}(\ln x)$ has changed only half as much.

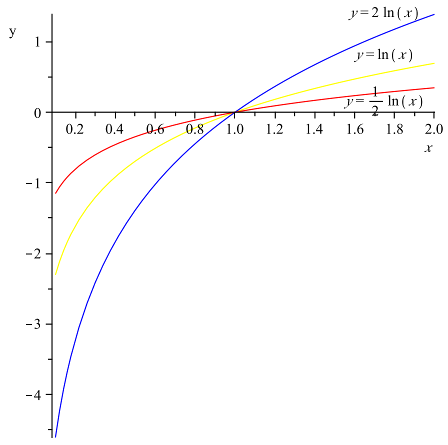


Figure 14: The graphs of $y = \ln x$, $y = 2\ln x$ and $y = \dfrac{\ln x}{2}$.

(b) The graph of $y = \ln 2x$ is stretched horizontally relative to the graph of $y = \ln x$, while the graph of $y = \ln\left(\frac{x}{2}\right)$ is horizontally compressed. But given the shape of the graph of $y = \ln x$, this horizontal stretching and compression has an unusual effect – it's really just a vertical shift of the graph.

We can see this by applying properties of logarithms. For any positive constant c, we see that

$$\ln cx = \ln c + \ln x = k + \ln x$$

where k is just a constant. So $\ln 2x = (\ln x) + \ln 2$, and $\ln\left(\frac{x}{2}\right) = (\ln x) + \ln\left(\frac{1}{2}\right) = (\ln x) + \ln 2^{-1} = (\ln x) + (-1)(\ln 2) = (\ln x) - \ln 2$. That is, the function $y = \ln 2x$ is the same as the function obtained by adding the positive constant $\ln 2$ to the function $y = \ln x$, while the function $y = \ln\left(\frac{x}{2}\right)$ is the same as the function obtained by subtracting this same constant from the function $y = \ln x$. Therefore the graph of $y = \ln 2x$ is just the graph of $y = \ln x$ shifted up by $\ln 2$ units, and the graph of $y = \ln\left(\frac{x}{2}\right)$ is just the same graph, but instead shifted down by $\ln 2$ units. Instead of passing through $(1, 0)$, the graph of $y = \ln 2x$ passes through $(1, \ln 2)$ and the graph of $y = \ln\left(\frac{x}{2}\right)$ passes through $(1, -\ln 2)$. So where do these 2 graphs cross the x-axis? For $y = \ln 2x = (\ln x) + \ln 2$ we see that $y = 0$ when $\ln x = -\ln 2 = \ln 2^{-1} = \ln\left(\frac{1}{2}\right)$, i.e. when $x = \frac{1}{2}$. And for $y = \ln\left(\frac{x}{2}\right) = (\ln x) - \ln 2$ we see that $y = 0$ when $\ln x = \ln 2$, so the x-intercept in this case is $x = 2$. (Remember, we were talking about horizontal expansion or compression, by a factor of 2, so the function that is expanded horizontally is twice as far from the vertical asymptote when it crosses the x-axis as compared to the one which is neither stretched nor compressed, while the one that is compressed is only half as far from the vertical asymptote when it crosses the x-axis.)

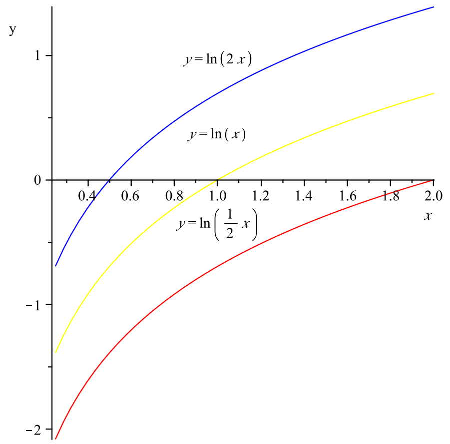


Figure 15: The graphs of $y = \ln x$, $y = \ln 2x$ and $y = \ln \frac{x}{2}$.

Math 1225A/B

Unit 2:

Review of Introductory Calculus

Differentiation Rules and Other Calculus Techniques

(text reference: Sections 5.3 and 5.4 – mixed together

custom text pgs. 19 - 35)

2 Differentiation of Exponential and Logarithmic Functions

In this section we review all the differentiation rules you learned in your Introductory Calculus course, mostly in the context of finding derivatives of functions involving exponentials and logarithms. We start with the derivatives of the natural exponential and logarithmic functions. Then later we derive formulas for the derivatives of more general exponential and logarithmic functions, i.e. with any base b (with $b > 0$ and $b \neq 1$, of course). Along the way, we will also review *implicit differentiation* and *logarithmic differentiation*, as well as reviewing what derivatives tell us about the shape of the graph of a function.

Derivatives of the Natural Exponential and Logarithmic Functions

Recall the *Definition of Derivative.*

Definition 2.1. For any function f, the derivative function $f'(x)$ is the function given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists. (If the limit DNE, we say that f is *not differentiable.*)

Remember: This is the formula for the instantaneous rate of change in $f(x)$ – which is the slope of the tangent line to $y = f(x)$. So if we evaluate $f'(x)$ at $x = a$, we get the slope of the tangent line to $y = f(x)$ at $x = a$.

You probably remember doing things like the following example.

Example 2.1. Use the definition of derivative to find $f'(x)$ if $f(x) = x^2$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} &&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} \frac{h(2x+h)}{h} &&= \lim_{h \to 0} (2x + h) &&= 2x \end{aligned}$$

Unfortunately, applying the definition of derivative to find the derivative of $f(x) = e^x$ or the derivative of $f(x) = \ln x$ requires mathematics that is beyond the scope of this course, to evaluate the limits. Never mind that. We will simply accept the following simple rules:

Rule: $\dfrac{d}{dx}(e^x) = e^x$ and **Rule:** $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

Notice: We said that e was the base such that $y = e^x$ crosses the y-axis with slope 1. For $f(x) = e^x$, the rule says that $f'(x) = e^x$, so when $x = 0$ we get $f'(0) = e^0 = 1$. As promised, the slope (of the tangent line) as the curve crosses the y-axis (i.e. at $x = 0$) is 1.

Now that we know the 2 facts mentioned above (the Rules), we can use them to *recognize* a certain limit as giving the derivative of some particular function, at some value, according to the definition of derivative. That allows us to find the value of the specified limit without having to evaluate it, by instead evaluating the derivative function.

That is, if we recognize that a limit has the form $\lim\limits_{h\to 0}\left[\dfrac{f(a+h)-f(a)}{h}\right]$ for some function $f(x)$ and some value a, then we know that this limit is giving $f'(x)$ when evaluated at $x=a$, so we can find the value of the limit simply by evaluating $f'(a)$. We find the derivative of the function $f(x)$ which we see being used, and then evaluate it at the particular a value we see in the limit.

Example 2.2. Evaluate $\lim\limits_{h\to 0}\left[\dfrac{\ln(2+h)-\ln 2}{h}\right]$.

Solution:
Looking at this limit, we see that we can't evaluate it in the usual way, because we don't know how to get the h out of the logarithm, and we can't cancel the h in the denominator while the h in the numerator is inside a logarithm. However, we do see a limit which has a familiar form. First of all, we are taking the limit as h goes to 0, and the denominator is just h. Also, in the numerator, we see something being done first to $2+h$ and then to 2. The thing that's being done to each of these is "taking the natural logarithm of". And the numerator is the difference of those two natural logarithms.

This tells us that something's being done with the function that takes the natural logarithm of a number. That is, we've got the function $f(x)=\ln x$. Expressed in terms of that function, the numerator of the limit is $f(2+h)-f(2)$. That is, we're evaluating $f(x+h)-f(x)$ at the particular value $x=2$. And in fact using $f(x)=\ln x$, we can express the whole limit as

$$\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] \qquad \text{evaluated at } x=2$$

But we know that $\lim\limits_{h\to 0}\left[\dfrac{f(x+h)-f(x)}{h}\right]=f'(x)$, and so we simply need to evaluate $f'(2)$ to find the value of the given limit. That is, since we know that for $f(x)=\ln x$ the derivative function is $f'(x)=\frac{1}{x}$, we have:

$$\lim_{h\to 0}\left[\frac{\ln(x+h)-\ln x}{h}\right]=f'(x)=\frac{1}{x} \qquad \text{and so} \qquad \lim_{h\to 0}\left[\frac{\ln(2+h)-\ln 2}{h}\right]=f'(2)=\frac{1}{2}$$

Of course, we don't need to use the definition of derivative every time we want to evaluate a derivative. You learned a variety of *differentiation rules*, including: the constant multiplier rule, the sum and difference rules, the power rule, the product rule and the quotient rule. Let's briefly review what those rules say and how they are used.

- The constant multiplier rule:
  For any differentiable function f and any constant c, the derivative of c times $f(x)$ is c times $f'(x)$. For instance, for $f(x)=2e^x$ we get $f'(x)=2\left[\frac{d}{dx}(e^x)\right]=2e^x$.
- The sum and difference rules:
  For any differentiable functions f and g, the derivative of $h(x)=f(x)\pm g(x)$ is $h'(x)=f'(x)\pm g'(x)$. For instance, for $f(x)=x+e^x$ we get $f'(x)=\frac{d}{dx}(x)+\frac{d}{dx}(e^x)=1+e^x$. Similarly, for $f(x)=e^x-\ln x$ we get $f'(x)=\frac{d}{dx}(e^x)-\frac{d}{dx}(\ln x)=e^x-\frac{1}{x}$.
- The power rule:
  For any constant n, the derivative of x^n is nx^{n-1}. For instance, the derivative of x^2 is $2x$ and the derivative of $\frac{1}{x}=x^{-1}$ is $(-1)x^{-1-1}=-x^{-2}=-\frac{1}{x^2}$. Of course, it is this rule which also gives us $\frac{d}{dx}(x)=1$ and $\frac{d}{dx}(c)=0$ for any constant c. We have $\frac{d}{dx}(x)=\frac{d}{dx}(x^1)=1x^0=1(1)=1$. Likewise, for any constant c, $c=cx^0$ and so (using the constant multiplier rule as well as the power rule) we have

$$\frac{d}{dx}(c)=\frac{d}{dx}(cx^0)=c\left[\frac{d}{dx}(x^0)\right]=c\left[0x^{-1}\right]=c(0)=0$$

- The product rule: For any differentiable functions f and g, if $h(x) = [f(x)]\,[g(x)]$ then the derivative of h is $h'(x) = [f(x)]\,[g'(x)] + [f'(x)]\,[g(x)]$. For instance, the product rule tells us that derivative of $f(x) = x \ln x$ is
$$f\,'(x) = [x]\left[\frac{d}{dx}(\ln x)\right] + \left[\frac{d}{dx}(x)\right][\ln x] = x\left(\frac{1}{x}\right) + 1(\ln x) = \frac{x}{x} + \ln x = 1 + \ln x$$
- The quotient rule: For any differentiable functions f and g, the derivative of $h(x) = \dfrac{f(x)}{g(x)}$ is $h'(x) = \dfrac{[f'(x)][g(x)] - [f(x)][g'(x)]}{[g(x)]^2}$. For example, the derivative of $h(x) = \dfrac{\ln x}{x}$ is
$$\frac{\left[\frac{d}{dx}(\ln x)\right][x] - [\ln x]\left[\frac{d}{dx}(x)\right]}{(x)^2} = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

We can look at some more examples of using these rules.

Example 2.3. Find $f'(x)$ where $f(x) = 3e^x + x^3 - 5\ln(x^2)$.

Solution:

$$\begin{aligned}
f'(x) &= \tfrac{d}{dx}[3e^x + x^3 - 5\ln(x^2)] \\
&= \left[\tfrac{d}{dx}(3e^x)\right] + \left[\tfrac{d}{dx}(x^3)\right] - \left[\tfrac{d}{dx}(5\ln(x^2)\right] && \text{(using the sum and difference rules)} \\
&= 3\left[\tfrac{d}{dx}(e^x)\right] + \left[3(x^2)\right] - 5\left[\tfrac{d}{dx}(2\ln x)\right] && \text{(using the constant multiplier and power rules} \\
& && \text{and then simplifying } \ln(x^2)) \\
&= 3(e^x) + 3x^2 - 5(2)\left[\tfrac{d}{dx}(\ln x)\right] && \text{(constant multiplier rule again in the last term)} \\
&= 3e^x + 3x^2 - 10\left(\tfrac{1}{x}\right) \\
&= 3(e^x + x^2) - \tfrac{10}{x}
\end{aligned}$$

Example 2.4. Find y' if $y = \ln x^{(x^2+x+1)}$.

Solution: We first simplify y, using properties of logarithms, to bring the exponent that is inside the logarithm outside as a multiplier on the logarithm: $y = (x^2 + x + 1)\ln x$.
Now, we have a product, so we need the product rule:

$$\begin{aligned}
y\,' &= \tfrac{d}{dx}\left[(x^2 + x + 1)(\ln x)\right] \\
&= (x^2 + x + 1)\left[\tfrac{d}{dx}(\ln x)\right] + \left[\tfrac{d}{dx}(x^2 + x + 1)\right](\ln x) \\
&= (x^2 + x + 1)\left(\tfrac{1}{x}\right) + (2x + 1 + 0)(\ln x) \\
&= \frac{x^2 + x + 1}{x} + (2x + 1)(\ln x) \\
&= x + 1 + \frac{1}{x} + \ln x^{2x+1}
\end{aligned}$$

Example 2.5. For $y = \dfrac{\sqrt{x}}{e^x + \ln x}$, find $\frac{dy}{dx}$.

Solution:
We use the quotient rule.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\left[\frac{d}{dx}(\sqrt{x})\right](e^x + \ln x) - (\sqrt{x})\left[\frac{d}{dx}(e^x + \ln x)\right]}{(e^x + \ln x)^2} \\
&= \frac{\left[\frac{d}{dx}\left(x^{1/2}\right)\right](e^x + \ln x) - (\sqrt{x})\left(e^x + \frac{1}{x}\right)}{(e^x + \ln x)^2} \\
&= \frac{\left[(1/2)x^{-1/2}\right](e^x + \ln x) - (\sqrt{x})\left(e^x + \frac{1}{x}\right)}{(e^x + \ln x)^2} \\
&= \frac{\left[\frac{1}{2\sqrt{x}}\right](e^x + \ln x) - \left(e^x + \frac{1}{x}\right)\sqrt{x}}{(e^x + \ln x)^2}
\end{aligned}$$

We'll leave the answer in that form. (Other presentations are possible.)

Example 2.6. Differentiate $f(x) = \dfrac{\pi e}{2x}$.

Solution:
First, we need to remember that in πe, π is just a constant, and so is e. That is, πe is just a constant. In fact, we can restate $f(x)$ into a form which simply needs the power rule (and the constant multiplier rule). We have

$$f(x) = \frac{\pi e}{2x} = (\pi e)\left(\frac{1}{2}\right)\left(\frac{1}{x}\right) = \left(\frac{\pi e}{2}\right)x^{-1}$$

and so we get

$$f'(x) = \left(\frac{\pi e}{2}\right)\left[\frac{d}{dx}(x^{-1})\right] = \left(\frac{\pi e}{2}\right)[(-1)(x^{-2})] = \left(\frac{\pi e}{2}\right)\left(-\frac{1}{x^2}\right) = -\frac{\pi e}{2x^2}$$

Example 2.7. Find the slope of the tangent line to $y = 2e^x - \ln x$ at the point with $x = \ln 3$.

Solution: For $f(x) = 2e^x - \ln x$ we have $f'(x) = 2e^x - \frac{1}{x}$, and so the slope of the tangent line to $y = 2e^x - \ln x$ at $x = \ln 3$ is

$$f'(\ln 3) = 2e^{\ln 3} - \frac{1}{\ln 3} = 2(3) - \frac{1}{\ln 3} = 6 - \frac{1}{\ln 3}$$

Recall that we can differentiate repeatedly, to find the "higher derivatives" of a function. For instance, the derivative of $f'(x)$ is the second derivative of f, $f''(x)$. And then the derivative of $f''(x)$ is $f'''(x)$, the third derivative of f, and so forth.

Example 2.8. Find the third derivative of $f(x) = xe^x$.

Solution:
For $f'(x)$ we use the product rule:

$$f'(x) = x\left[\frac{d}{dx}(e^x)\right] + \left[\frac{d}{dx}(x)\right](e^x) = xe^x + (1)e^x = (x+1)e^x$$

And then for the second derivative we use the product rule again:

$$f''(x) = \frac{d}{dx}\left[(x+1)e^x\right] = (x+1)\left[\frac{d}{dx}(e^x)\right] + \left[\frac{d}{dx}(x+1)\right](e^x) = (x+1)e^x + (1+0)e^x = (x+2)e^x$$

And we need the product rule one more time for the third derivative:

$$f'''(x) = \frac{d}{dx}\left[(x+2)e^x\right] = (x+2)e^x + (1+0)e^x = (x+3)e^x$$

You've also learnt how the derivative of a function tells us about what the graph of the function looks like. The first derivative gives information about where the function is increasing or decreasing and where it attains relative maxima and minima, while the second derivative gives information about the concavity and inflection points of the curve. Recall that:

- Intervals of Increase and Decrease
  A function $f(x)$ is increasing on an interval (a, b) if $f'(x) > 0$ for all x in (a, b).
  Similarly, $f(x)$ is decreasing on (a, b) if $f'(x) < 0$ for all x in (a, b).
- Relative Maxima and Minima
  A function $f(x)$ has a relative extremum (relative extreme value) $f(a)$ if $f'(a) = 0$ and
  1. $f(x)$ is increasing to the left of $x = a$ and is decreasing to the right of $x = a$
     In this case, f has a relative maximum $f(a)$.
  2. $f(x)$ is decreasing to the left of $x = 1$ and is increasing to the right of $x = a$
     In this case, f has a relative minimum $f(a)$.
- Intervals of Upward and Downward Concavity
  A function $f(x)$ is concave upward on an interval (a, b) if $f''(x) > 0$ for all x in (a, b).
  Similarly, $f(x)$ is concave downward on (a, b) if $f''(x) < 0$ for all x in (a, b).
- Points of Inflection
  A point $(a, f(a))$ on the graph of $y = f(x)$ is an inflection point of the graph if the concavity of the function changes at $x = a$. That is, if f is concave upward to the left of $x = a$ and is concave downward to the right of $x = a$, or vice versa.

- Second Derivative Test
  If $f(x)$ has a relative extremum at $x = a$, this extremum is a relative maximum if $f''(a) < 0$, and is a relative minimum if $f''(a) > 0$.

For instance, consider the functions $f(x) = x^2$, $g(x) = -x^2$ and $h(x) = x^3$. You know what the graphs of these functions look like – sketch them. For $y = f(x)$, we have $f'(x) = 2x$, so $f'(x) > 0$ when $x > 0$, and $f'(x) < 0$ when $x < 0$. As your sketch shows, $y = f(x)$ is decreasing on $(-\infty, 0)$ and is increasing on $(0, \infty)$. That is, $f(x) = x^2$ is decreasing everywhere to the left of $x = 0$, and is increasing everywhere to the right of $x = 0$, with a relative minimum at $x = 0$. This graph is concave upward everywhere, since $f''(x) = 2 > 0$.

For $y = g(x)$ (which looks like the previous graph turned upside down), we have $g'(x) = -2x$, so $g'(x) > 0$ when $x < 0$, and $g'(x) < 0$ when $x > 0$. The function is increasing to the left of $x = 0$ and is decreasing to the right of $x = 0$, attaining a relative maximum at $x = 0$. Since $g''(x) = -2 < 0$, the graph of this function is concave downward everwhere.

And for $y = h(x)$, we have $g'(x) = 3x^2$, so $h'(x) > 0$ for all $x \neq 0$. This graph is increasing everywhere except right at $x = 0$. And $h''(x) = 6x$ is positive for all $x > 0$, but negative for all $x < 0$, so $y = h(x)$ is concave downward to the left of $x = 0$ and concave upward to the right of $x = 0$. That dipsy-doodle (not a technical math term) at $x = 0$, where the concavity changes, is a point of inflection.

Example 2.9. Sketch the graph of $y = x^3 - 3x$.

Solution:
We have $f(x) = x^3 - 3x = x(x^2 - 3) = x(x+\sqrt{3})(x-\sqrt{3})$ so we see that $f(x) = 0$ when $x = 0$, when $x = -\sqrt{3}$ and when $x = \sqrt{3}$. That is, the x-intercepts of $y = x^3 - 3x$ are $-\sqrt{3}$, 0 and $\sqrt{3}$. And of course since $f(0) = 0$ then where the graph passes through the origin is also the y-intercept, as well as being one of the x-intercepts.

Having found the intercepts, we next think about intervals of increase and decrease, and relative extrema. We have $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1)$, so $f'(x) = 0$ when $x = -1$ and when $x = 1$. That is, the *critical numbers* of f are -1 and 1. These numbers break the domain of f into 3 pieces: $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$. We check the sign of f' in each interval to determine whether f is increasing or decreasing on that interval.

	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
sample x	-2	0	2
$x+1$	$-$	$+$	$+$
$x-1$	$-$	$-$	$+$
so f' is	$+$	$-$	$+$
and f is	↗	↘	↗

We see that f is increasing on $(-\infty, -1)$, and then decreasing on $(-1, 1)$, and increasing again on $(1, \infty)$. Since f changes from increasing to decreasing at -1, there is a relative maximum at $x = -1$, with relative extreme value $f(-1) = (-1)^3 - 3(-1) = -1 - (-3) = -1 + 3 = 2$. And then f changes from decreasing to increasing at 1, so there is a relative minimum at $x = 1$, with relative minimum value $f(1) = (1)^3 - 3(1) = 1 - 3 = -2$.

Finally, we use f'' to check the concavity and find the inflection points. We have $f''(x) = 6x$, so $f''(x) = 0$ only when $x = 0$, i.e. 0 is the only critical number of f'. We see that $f''(x)$ is negative on $(-\infty, 0)$ and is positive on $(0, \infty)$, so f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$, with inflection point $(0, f(0)) = (0, 0)$.

Since f is a polynomial function, the function value is running off to $-\infty$ or to ∞ at each end (i.e. as $x \to -\infty$ and as $x \to \infty$). From the increasing/decreasing information we already have, we can tell that $\lim\limits_{x \to -\infty} f(x) = -\infty$ and that $\lim\limits_{x \to \infty} = \infty$.

We use all of this information to sketch the graph of $y = f(x)$, as shown in Figure 16.

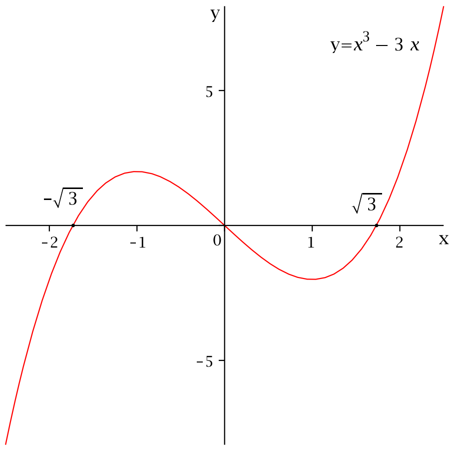


Figure 16: The graph of $y = x^3 - 3x$

Example 2.10. Find all relative extrema and inflection points of $y = xe^x$.

Solution:
We have the graph of $y = f(x)$ for $f(x) = xe^x$. As we saw in Example 2.8, the first derivative is $f'(x) = (x+1)e^x$ and the second derivative is $f''(x) = (x+2)e^x$. Of course, e^x is strictly positive for all values of x, i.e. $e^x > 0$ everywhere in $(-\infty, \infty)$, so $f'(x) = 0$ only when $x + 1 = 0$, and therefore $x = -1$ is the only critical number of f. Using the second derivative test, we see that $f''(-1) = (-1+2)e^{-1} = 1e^{-1} = \frac{1}{e}$, which is positive, so f has relative minimum value $f(-1) = (-1)e^{-1} = -\frac{1}{e}$ at $x = -1$.

Also, $f''(x) = 0$ only when $x + 2 = 0$, so $x = -2$ is the only critical number of f'. We can check the concavity of f on the 2 intervals $(-\infty, -2)$ and $(-2, \infty)$:

	$(-\infty, -2)$	$(-2, \infty)$
sample x	-3	0
$x + 2$	$-$	$+$
e^x	$+$	$+$
so f'' is	$-$	$+$
and f is	$\frown$	$\smile$

We see that the concavity of f does change at this (the only) critical number of f', so the point $(-2, f(-2)) = \left(-2, -2e^{-2}\right)$ is (the only) inflection point of the graph.

In the graph below (Figure 17), we see the relative extremum and concavity results we found here.

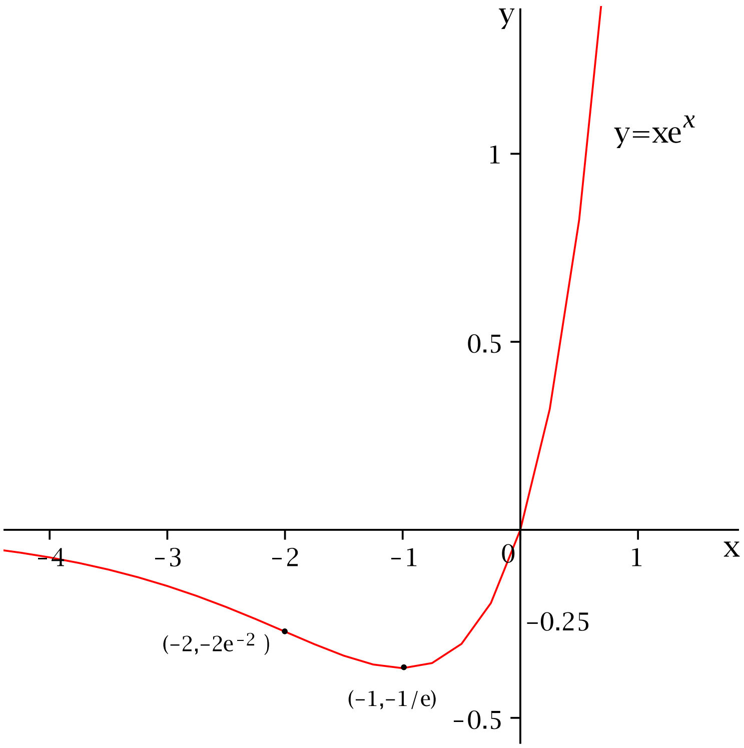


Figure 17: The graph of $y = xe^x$

(*Notice:* If you need more review to follow what was done in these last 2 examples, you should probably get out your Intro Calc text book and look at the relevant chapters/sections.)

The Chain Rule

There is one very important differentiation rule that we haven't reviewed yet, and that's the Chain Rule. Before we can review that, though, you need to remember what a *composite function* is, so that you'll know when to apply the Chain Rule.

> **Recall:** A composite function h has the form $h(x) = f(g(x))$ for some functions f and g. That is, a composite function is a *function of a function* of x. We're doing something not just to x itself, but rather to some function of x.

$h(x) = f(g(x))$ says that to get the function value $h(x)$, you apply the function f to *the output of* the function g. For instance, for $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, we have $f(g(x)) = \sqrt{x^2+1}$. That is, given an x-value, first find $x^2 + 1$ (i.e. find $g(x)$) and *then* take the square root (i.e. apply f to $g(x)$). Likewise, using $f(x) = e^x$ and $g(x) = x^3 - 3x$, we can create two different composite functions $f(g(x)) = e^{x^3-3x}$, or $g(f(x)) = (e^x)^3 - 3(e^x) = e^{3x} - 3e^x$.

Now let's review what the Chain Rule says. There are a couple of different ways to express this rule, and it's worth looking at both.

> **The Chain Rule**
> For $h(x) = f(g(x))$, $h'(x) = f'(g(x))g'(x)$.

That is, we apply f' to $g(x)$, and multiply the whole thing by $g'(x)$.

> **Or, the Chain Rule**
> If $y = f(u)$, where $u = g(x)$, for some functions f and g, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$.

Example 2.11. Find $\dfrac{dy}{dx}$ if $y = e^{x^3-3x}$.

Solution:
We have $y = e^u$ where $u = x^3 - 3x$, so we get

$$\begin{aligned} \frac{dy}{du} &= \frac{d}{du}(e^u) = e^u \\ \text{and } \frac{du}{dx} &= \frac{d}{dx}(x^3 - 3x) = 3x^2 - 3 \end{aligned}$$

So we can find $\frac{dy}{dx}$ using:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (e^u)(3x^2 - 3) = e^{x^3-3x}(3x^2 - 3)$$

We can express a general rule for using the Chain Rule in the context of a function of this type:

General Rule:

$$\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} f'(x) \qquad \text{i.e.} \qquad \frac{d}{dx}\left(e^u\right) = e^u \frac{du}{dx}$$

That is, the derivative of "e raised to some exponent which is a function of x" is *what we started with* times *the derivative of the exponent.* For instance, for $f(x) = e^{5x}$ we get $f'(x) = e^{5x}(5) = 5e^{5x}$, and for $g(x) = e^{1-x^2}$ we get $g'(x) = e^{1-x^2}(0 - 2x) = -2xe^{1-x^2}$.

We have previously stated, just as a rule that we accept as true, that the derivative of $y = \ln x$ is $y' = \frac{1}{x}$. Now that we know how to apply the chain rule to differentiate something of the form $e^{f(x)}$, we can use this to *derive* the other rule.

We know that $u = \ln x$ says that $x = e^u$. That is, for $u = \ln x$ we have $x = e^{\ln x}$. But then, differentiating both sides of this equation, using the fact that $\dfrac{d}{dx}(e^u) = e^u \left[\frac{d}{dx}(u)\right]$, we have

$$x = e^{\ln x} \qquad \Rightarrow \qquad \frac{d}{dx}[x] = \frac{d}{dx}\left[e^{\ln x}\right] \qquad \Rightarrow \qquad 1 = e^{\ln x}\left[\frac{d}{dx}(\ln x)\right]$$

Now, we can rearrange this equation to get

$$\frac{d}{dx}(\ln x) = \frac{1}{e^{\ln x}}$$

And we know that $e^{\ln x} = x$, so we have

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Now, let's look at an example of using the chain rule to find the derivative of the natural logarithm of some function of x.

Example 2.12. Find $h'(x)$ where $h(x) = \ln(x^2 + e^x)$.

Solution:
We have a composite function $h(x) = \ln(x^2 + e^x) = f(g(x))$ where $f(x) = \ln x$ and $g(x) = x^2 + e^x$. And with the sum inside the logarithm, there is no way to simplify the function using the properties of logarithms. We use the Chain Rule. For $f(x) = \ln x$, we have $f'(x) = \frac{1}{x}$. And for $g(x) = x^2 + e^x$ we have $g'(x) = 2x + e^x$. So we get:

$$h'(x) = f'(g(x))g'(x) = f'(x^2 + e^x)g'(x) = \left[\frac{1}{x^2 + e^x}\right](2x + e^x) = \frac{2x + e^x}{x^2 + e^x}$$

Look at what we have this time. We started with a function of the form "natural logarithm of something" and we ended up with a quotient whose denominator is the "something" we were taking the natural logarithm of, and whose numerator is the derivative of that "something" that's in the denominator. And this will happen every time. Whenever we have $y = \ln u$, where u is some function of x, the derivative is

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \left[\frac{d}{du}(\ln u)\right]\frac{d}{dx}(u) = \frac{1}{u} \times u' = \frac{u'}{u}$$

And so once again we can express a general rule.

General Rule:

$$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)} \qquad \text{i.e.} \qquad \frac{d}{dx}(\ln u) = \frac{u'}{u}$$

For instance, for $f(x) = \ln(3x - x^2)$ we get $f'(x) = \frac{3-2x}{3x-x^2}$. Likewise, for $y = \ln(5x)$ we get $y' = \frac{5}{5x} = \frac{1}{x}$ and for $g(x) = \ln(x^2)$ we get $g'(x) = \frac{2x}{x^2} = \frac{2}{x}$.

Notice: For those last two, we could have simplified, using properties of logarithms, before differentiating. We have $y = \ln(5x) = \ln 5 + \ln x$, where of course $\ln 5$ is a constant, so $y' = 0 + \frac{1}{x} = \frac{1}{x}$. Likewise, we have $g(x) = \ln x^2 = 2\ln x$ and so $g'(x) = 2\left(\frac{1}{x}\right) = \frac{2}{x}$. Of course we get the same answer for the derivative whether we use the Chain Rule or use the properties of logarithms to simplify so that the Chain Rule isn't needed.

Also Notice: The derivative of $\ln(5x)$ is just $\frac{1}{x}$. In fact, for any constant c we have $\frac{d}{dx}(\ln cx) = \frac{1}{x}$, either by simplifying first (i.e. $\frac{d}{dx}(\ln cx) = \frac{d}{dx}(\ln c + \ln x) = 0 + \frac{1}{x}$) or by using the Chain Rule (i.e. $\frac{d}{dx}(\ln cx) = \left(\frac{1}{cx}\right)(c) = \frac{c}{cx} = \frac{1}{x}$). So for instance, for $f(x) = \ln(-x)$, the negative is just a constant -1 multiplier, so we get $f'(x) = \frac{1}{x}$.

Most of the uses of the Chain Rule in your Introductory Calculus course were probably instances of the **General Power Rule**, which is just a General Rule for using the Chain Rule in the context of a composite function whose outer function has the form "raise to a constant power".

General Power Rule:

$$\frac{d}{dx}\left[(f(x))^n\right] = n\,(f(x))^{n-1}\,f'(x) \qquad \text{i.e.} \qquad \frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Example 2.13. Find $\frac{dy}{dx}$ where $y = (e^{2x} + \ln(2x))^3$.

Solution:
We have $y = u^3$, where $u = e^{2x} + \ln(2x)$, and so $\frac{dy}{dx} = 3u^2\frac{du}{dx}$. Notice that each term in the sum which gives the function u is itself a composite function. So when we use the sum rule to find $\frac{du}{dx}$, we need to use the Chain Rule for each term.

We see that

$$\frac{du}{dx} = \frac{d}{dx}\left[e^{2x} + \ln(2x)\right] = 2e^{2x} + \frac{1}{x}$$

And therefore the final answer is

$$\frac{dy}{dx} = 3(e^{2x} + \ln(2x))^2\left(2e^{2x} + \frac{1}{x}\right)$$

We know that $\frac{d}{dx}(\ln x) = \frac{1}{x}$, and we saw above that $\frac{d}{dx}[\ln(-x)] = \frac{1}{x}$ as well. Recall that we can only take the logarithm (whether natural or to some other base) of something which is positive. So $\ln x$ is only defined when $x > 0$, and likewise, $\ln(-x)$ is only defined when $-x > 0$, i.e. when $x < 0$. Also recall that the absolute value function can be expressed with a piecewise definition:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

But then we see that we can think of the function $f(x) = \ln|x|$, which is defined everywhere except at $x = 0$, as

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

And since $\ln x$ and $\ln(-x)$ both have the same derivative, $\frac{1}{x}$, we see that $f'(x) = \frac{1}{x}$ whenever $x \neq 0$. That is, on $(-\infty, 0)$ we have $f'(x) = \frac{d}{dx}[\ln(-x)] = \frac{-1}{-x} = \frac{1}{x}$, and on $(0, \infty)$ we have $f'(x) = \frac{d}{dx}[\ln x] = \frac{1}{x}$. Therefore $f'(x) = \frac{1}{x}$ everywhere in the domain of f. It is useful to remember this result.

Special Case:

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

Example 2.14. Differentiate $y = e^{\sqrt{x^2+1}} \ln|x|$.

Solution:
The outer-most thing happening in this function is the product, so we start with the product rule. When we need to differentiate the second term, we simply use our observation from above. But to differentiate the first term we will need to use the Chain Rule, more than once. We get:

$$\begin{aligned}
\frac{dy}{dx} &= \left(e^{\sqrt{x^2+1}}\right)\left[\frac{d}{dx}(\ln|x|)\right] + \left[\frac{d}{dx}\left(e^{\sqrt{x^2+1}}\right)\right](\ln|x|) \\
&= e^{\sqrt{x^2+1}}\left(\frac{1}{x}\right) + e^{\sqrt{x^2+1}}\left[\frac{d}{dx}\left(\sqrt{x^2+1}\right)\right]\ln|x| \\
&= \frac{e^{\sqrt{x^2+1}}}{x} + e^{\sqrt{x^2+1}}(\ln|x|)\left[\frac{d}{dx}(x^2+1)^{1/2}\right] \\
&= \frac{e^{\sqrt{x^2+1}}}{x} + e^{\sqrt{x^2+1}}(\ln|x|)\left[\frac{1}{2}(x^2+1)^{-1/2}\right]\left[\frac{d}{dx}(x^2+1)\right] \\
&= \frac{e^{\sqrt{x^2+1}}}{x} + e^{\sqrt{x^2+1}}(\ln|x|)\left(\frac{1}{2\sqrt{x^2+1}}\right)(2x) \\
&= e^{\sqrt{x^2+1}}\left(\frac{1}{x} + \frac{x\ln|x|}{\sqrt{x^2+1}}\right)
\end{aligned}$$

General Exponential and Logarithmic Functions

So far, we have learnt the derivatives of only the natural exponential and natural logarithmic functions, i.e. those to base e. But we know that an exponential or logarithmic function can have any base b as long as b is a positive number other than 1. How can we find the derivative of an exponential or logarithmic function with another base?

For a general logarithmic function, we don't need any special techniques, other than realizing that if the 2 sides of an equation are equal, and are positive, then their natural logarithms are defined and must also be equal. That is, if LHS = RHS, then it is also true that ln LHS = ln RHS (provided these are defined, so only when LHS and RHS are positive). We can use this approach to find another way to express any logarithmic function, which will allow us to easily find the derivative.

Suppose we have any logarithmic function. That is, consider any base b with $b > 0$ and $b \neq 1$ and consider the function $y = \log_b x$. Then we know that this says exactly the same thing as $x = b^y$, so $\ln x = \ln b^y = y(\ln b)$, and rearranging this we have $y = \frac{\ln x}{\ln b} = \left(\frac{1}{\ln b}\right)\ln x$, where $\frac{1}{\ln b}$ is just a constant multiplier. And we know how to differentiate this. That is, we have:

$$y = \log_b x \quad \Rightarrow \quad x = b^y \quad \Rightarrow \quad \ln x = \ln b^y \quad \Rightarrow \quad \ln x = y\ln b \quad \Rightarrow \quad y = \frac{\ln x}{\ln b}$$

and since $\ln b$ is just a constant, we re-write this, and all we need is the constant multiplier rule.

$$y = \frac{\ln x}{\ln b} \quad \Rightarrow \quad y = \left(\frac{1}{\ln b}\right)\ln x \quad \Rightarrow \quad \frac{dy}{dx} = \left(\frac{1}{\ln b}\right)\left[\frac{d}{dx}(\ln x)\right] = \left(\frac{1}{\ln b}\right)\left(\frac{1}{x}\right) = \frac{1}{x\ln b}$$

We see that the following rule holds:

General Rule:
For any base b with $b > 0$ and $b \neq 1$, $\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x\ln b}$.

So for instance, the derivative of $\log_2 x$ is $\frac{1}{x\ln 2}$. And likewise, for the function $f(x) = \log x$, we know that the missing base is 10 (i.e. that when there's no base shown, the base is 10), so we get $f'(x) = \frac{1}{x\ln 10}$.

Now, consider any general exponential function. That is, suppose we have $y = b^x$ for some $b > 0$ with $b \neq 1$. Then as before, we can take natural logarithms of both sides of the equation to get $\ln y = \ln b^x$ so that $\ln y = x \ln b$. The next step is to differentiate both sides. To do this, we need to use the technique called *implicit differentiation*. And what we're doing here (overall) is actually another specific technique, called *logarithmic differentiation*. So let's review how both of those techniques work before coming back to the current problem.

Implicit Differentiation

Suppose we have some equation relating 2 or more variables, more complicated than for instance $y =$ some function of x. So we have some equation LHS = RHS. We can differentiate with respect to any variable as follows. Here, suppose that we want to differentiate with respect to x.

We set $\dfrac{d}{dx}\text{LHS} = \dfrac{d}{dx}\text{RHS}$ and differentiate using the usual differentiation rules, remembering that (here assuming that y is a variable appearing in the original equation):

1. $\dfrac{d}{dx}(y) = \dfrac{dy}{dx}$
   That's what $\frac{dy}{dx}$ means. It's simply a piece of notation which represents "the derivative of y, with respect to x".

2. When y is implicitly a function of x, to find the derivative of some *function of y*, we need the Chain Rule.
   For instance, to differentiate y^2 with respect to x, we use the General Power Rule, considering y to be "the thing in the brackets". So we bring the exponent down as a multiplier, and subtract one from the exponent, and then we *multiply by the derivative of the thing in the brackets*, i.e. by the derivative of y, so we get

$$\frac{d}{dx}(y^2) = 2y^1\left(\frac{dy}{dx}\right) = 2y\frac{dy}{dx}$$

We started with an equation relating 2 or more variables. After differentiating both sides with respect to x we have an equation relating the variables *and their rates of change* (i.e. their derivatives) with respect to x (or whichever variable we needed to differentiate with respect to). We can then rearrange this equation to isolate whatever we needed to find. For instance, if we're trying to "find $\frac{dy}{dx}$", then we simply rearrange the equation to isolate $\frac{dy}{dx}$, i.e. put it into $\frac{dy}{dx} =$ form.

Example 2.15. If $xy^3 = \dfrac{y}{1-xy}$, find $\dfrac{dy}{dx}$.

Solution:

$$xy^3 = \frac{y}{1-xy} \qquad \Rightarrow \qquad \frac{d}{dx}(xy^3) = \frac{d}{dx}\left(\frac{y}{1-xy}\right)$$

We will need the product rule for the LHS and the quotient rule for the RHS.

We get:

$$\begin{aligned}
x\left[\frac{d}{dx}(y^3)\right]+\left[\frac{d}{dx}(x)\right]y^3 &= \frac{\left[\frac{d}{dx}(y)\right](1-xy)-y\left[\frac{d}{dx}(1-xy)\right]}{(1-xy)^2}\\
\Rightarrow\quad x(3y^2)\left(\frac{d}{dx}(y)\right)+1(y^3) &= \frac{\left(\frac{dy}{dx}\right)(1-xy)-y\left[\frac{d}{dx}(1)-\frac{d}{dx}(xy)\right]}{(1-xy)^2} \qquad \text{(product rule for } xy\text{)}\\
\Rightarrow\quad 3xy^2\frac{dy}{dx}+y^3 &= \frac{(1-xy)\frac{dy}{dx}-y\left\{0-\left[x\left(\frac{dy}{dx}\right)+(1)y\right]\right\}}{(1-xy)^2}\\
\Rightarrow\quad \left[3xy^2\frac{dy}{dx}+y^3\right](1-xy)^2 &= (1-xy)\frac{dy}{dx}-y\left(-x\frac{dy}{dx}-y\right)
\end{aligned}$$

At this point, we're finished differentiating, so now we need to isolate $\frac{dy}{dx}$. We untangle the $\frac{dy}{dx}$ and non-$\frac{dy}{dx}$ terms, then collect all the $\frac{dy}{dx}$ terms on the LHS. We continue from where we were, first expanding both the LHS and the last part of the RHS, so that we have some terms which do have $\frac{dy}{dx}$ as a multiplier, and other terms which have no $\frac{dy}{dx}$:

$$\begin{aligned}
& 3xy^2(1-xy)^2\frac{dy}{dx}+y^3(1-xy)^2=(1-xy)\frac{dy}{dx}+xy\frac{dy}{dx}+y^2\\
\Rightarrow\quad & 3xy^2(1-xy)^2\frac{dy}{dx}-(1-xy)\frac{dy}{dx}-xy\frac{dy}{dx}=y^2-y^3(1-xy)^2\\
\Rightarrow\quad & \frac{dy}{dx}\left[3xy^2(1-xy)^2-(1-xy)-xy\right]=y^2-y^3(1-xy)^2\\
\Rightarrow\quad & \frac{dy}{dx}\left[3xy^2(1-xy)^2-1+xy-xy\right]=y^2-y^3(1-xy)^2\\
\Rightarrow\quad & \frac{dy}{dx}=\frac{y^2-y^3(1-xy)^2}{3xy^2(1-xy)^2-1}
\end{aligned}$$

Example 2.16. Consider a rectangular box whose length l, width w and height h are all changing over time. Find the volume of the box, and the rate at which the volume is changing, at an instant at which the length is 10 cm, the width is 5 cm and the height is 3 cm, with the length increasing at a rate of 0.5 cm per minute, the width decreasing at a rate of 0.2 cm per minute and the height growing at a rate of 0.1 cm per minute.

Solution:
We are told that the length, width and height of the box are all changing with respect to time. That is, letting t represent time, l, w and h are all functions of t, although we don't know what those functions are. And their rates of change (with respect to time) are, respectively, $\frac{dl}{dt}$, $\frac{dw}{dt}$ and $\frac{dh}{dt}$.

We know that the volume of a rectangular box with length l, width w and height h is given by $V=l\times w\times h$. This is the equation relating several variables. In this case it expresses V in terms of l, w and h. And since those are all, implicitly, functions of t, then V is also (implicitly) a function of t. And so it is changing with respect to t, just as the others are. The rate of change in volume over time is $\frac{dV}{dt}$. We're asked to find both V and $\frac{dV}{dt}$ at a particular instant in time. To do this we need to find an equation relating $\frac{dV}{dt}$ to the other variables and/or their rates of change. We do this using implicit differentiation. We differentiate both sides of the equation $V=l\times w\times h$, with respect to t. The derivative of the LHS is just $\frac{dV}{dt}$. For the RHS we need to use the product rule twice, since we have a product of 3 terms. (That is, we first consider the product of l with wh.)

$$\frac{dV}{dt} = \frac{d}{dt}\left[l(wh)\right] = l\left[\frac{d}{dt}(wh)\right] + \frac{dl}{dt}(wh) = l\left[w\left(\frac{dh}{dt}\right) + \left(\frac{dw}{dt}\right)h\right] + wh\frac{dl}{dt} = lw\frac{dh}{dt} + lh\frac{dw}{dt} + wh\frac{dl}{dt}$$

Now, we use the values we were given. We need to find the values of V and $\frac{dV}{dt}$ at the moment when $l = 10$, $w = 5$ and $h = 3$ with $\frac{dl}{dt} = 0.5 = \frac{1}{2}$, $\frac{dw}{dt} = -0.2 = -\frac{1}{5}$ and $\frac{dh}{dt} = 0.1 = \frac{1}{10}$. We get

$$\begin{aligned} V &= l \times w \times h = 10 \times 5 \times 3 = 150 \\ \text{and } \frac{dV}{dt} &= lw\frac{dh}{dt} + lh\frac{dw}{dt} + wh\frac{dl}{dt} = (10)(5)\left(\frac{1}{10}\right) + (10)(3)\left(-\frac{1}{5}\right) + (5)(3)\left(\frac{1}{2}\right) = 5 - 6 + \frac{15}{2} = \frac{13}{2} \end{aligned}$$

We see that at that instant, the volume will be 150 cubic centimetres, increasing at a rate of 6.5 cm per minute.

Logarithmic Differentiation

Sometimes even when we have the form $y =$ some function of x, we still need to do something more complicated. And sometimes we don't need to, but choose to, because it makes things easier. What we do is to take the equation $y = f(x)$ and take natural logarithms of both sides, in order to simplify before differentiating. We do this is whenever it's useful, i.e. whenever the RHS of the resulting equation will be something which can be simplified using properties of logarithms and then be differentiated more easily. So the steps are:

1. Given $y = f(x)$, set $\ln y = \ln f(x)$.
2. Use properties of logarithms to simplify $\ln f(x)$.
3. Now use implicit differentiation to find $\frac{dy}{dx}$.

In that last step, if we started out with $y =$ something, we always end up with "and multiply through by y" as the last step in isolating $\frac{dy}{dx}$. Since we were given what y is, in terms of x, then we should be replacing y at this point. That is, if we were given y in terms of x only, then $\frac{dy}{dx}$ should also be expressed in terms of x only.

One of the situations in which logarithmic differentiation is useful is when we have a variable in the exponent and we don't have a rule that allows us to differentiate the function directly. In particular, whenever we have a variable base raised to a variable power, we **must** use logarithmic differentiation.

Example 2.17. If $y = x^{1+x^2}$, find $\dfrac{dy}{dx}$.

Solution:
We start by taking natural logarithms of both sides, and simplifying:

$$y = x^{1+x^2} \qquad \Rightarrow \qquad \ln y = \ln x^{1+x^2} \qquad \Rightarrow \qquad \ln y = (1+x^2)\ln x$$

Now, we differentiate both sides of this last equation, using the chain rule (i.e. implicit differentiation) to differentiate $\ln y$ with respect to x, and using the product rule for the RHS:

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}[(1+x^2)(\ln x)] \\
\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} &= (1+x^2)\left(\frac{1}{x}\right) + (0+2x)(\ln x) \\
\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} &= \frac{1+x^2}{x} + 2x\ln x \\
\Rightarrow \quad \frac{dy}{dx} &= \left[\frac{1}{x} + \frac{x^2}{x} + 2x\ln x\right](y) \\
&= \left[\frac{1}{x} + x + 2x\ln x\right](x^{1+x^2})
\end{aligned}$$

Logarithmic differentiation is also useful when we have a function which is a big mess of products and/or quotients, especially with (constant) exponents on the terms. This is because if we take the natural logarithm of something like that, we can then use the properties of exponents to turn products into sums, quotients into differences and exponents into multipliers. This makes the differentiation much easier, since the sum and difference rules are easier to use than the product and quotient rules, and the constant multiplier rule is easier to use than the general power rule. And finding the derivative of the natural logarithm of a function of x isn't very difficult.

Example 2.18. If $y = \dfrac{(x^3-3)^5(x^2+x)^8}{(e^x-3)^{15}}$, find $\dfrac{dy}{dx}$.

Solution:
Using logarithmic differentiation will be much easier than using the quotient rule, with the product rule and 3 instances of the general power rule:

$$\begin{aligned}
y &= \frac{(x^3-3)^5(x^2+x)^8}{(e^x-3)^{15}} \\
\Rightarrow \quad \ln y &= \ln\left[\frac{(x^3-3)^5(x^2+x)^8}{(e^x-3)^{15}}\right] && \text{(i.e. take ln's)} \\
&= \ln(x^3-3)^5 + \ln(x^2+x)^8 - \ln(e^x-3)^{15} && \text{(start simplifying)} \\
&= 5\ln(x^3-3) + 8\ln(x^2+x) - 15\ln(e^x-3) && \text{(and simplify some more)} \\
\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} &= 5\left(\frac{3x^2}{x^3-3}\right) + 8\left(\frac{2x+1}{x^2+x}\right) - 15\left(\frac{e^x}{e^x-3}\right) && \text{(differentiate, using } \tfrac{d}{dx}(\ln u) = \tfrac{u'}{u}\text{)} \\
\Rightarrow \quad \frac{dy}{dx} &= \left[5\left(\frac{3x^2}{x^3-3}\right) + 8\left(\frac{2x+1}{x^2+x}\right) - 15\left(\frac{e^x}{e^x-3}\right)\right](y) && \text{(but we need to sub for } y\text{)} \\
&= \left[5\left(\frac{3x^2}{x^3-3}\right) + 8\left(\frac{2x+1}{x^2+x}\right) - 15\left(\frac{e^x}{e^x-3}\right)\right]\left[\frac{(x^3-3)^5(x^2+x)^8}{(e^x-3)^{15}}\right]
\end{aligned}$$

At this point, we're ready to go back to what we were working on, finding the derivative of a general exponential function. Suppose we have $y = b^x$ for some $b > 0$ with $b \neq 1$. We want to find

$\frac{dy}{dx}$. We use logarithmic differentiation, to get the x out of the exponent:

$$\begin{array}{rrcll} & y & = & b^x & \text{(first, we take ln of both sides)} \\ \Rightarrow & \ln y & = & \ln b^x & \text{(next, we simplify the new RHS)} \\ \Rightarrow & \ln y & = & x(\ln b) & \text{(now differentiate both sides)} \\ \Rightarrow & \dfrac{1}{y}\dfrac{dy}{dx} & = & (\ln b)(1) & \text{(remember, } \ln b \text{ is just a constant)} \\ \Rightarrow & \dfrac{dy}{dx} & = & y \ln b & \text{(multiply through by } y\text{)} \\ & & = & b^x \ln b & \text{(and then substitute for } y \\ & & & & \text{because } y = b^x \text{ was where we started)} \end{array}$$

We see that the following rule holds:

General Rule: For any base b with $b > 0$ and $b \neq 1$, $\dfrac{d}{dx}(b^x) = b^x \ln b$.

So for instance, the derivative of 2^x is $2^x \ln 2$. And likewise, for the function $f(x) = 5^x$, we get $f'(x) = 5^x \ln 5$.

Now we know how to find the derivative of any exponential or logarithmic function. Of course, for have something more complicated, we need the Chain Rule. If u is some function of x, then the derivative of $y = b^u$ is $y' = (b^u \ln b)u'$, and the derivative of $y = \log_b u$ is $y' = \left(\frac{1}{u \ln b}\right) \times u' = \frac{u'}{u \ln b}$.

Example 2.19. Find the derivative of $g(x) = 3^{1-x^2} - \log_2(x^3 + 5x)$.

Solution:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(3^{1-x^2}\right) - \frac{d}{dx}\left[\log_2(x^3+5x)\right] \\ &= \left(3^{1-x^2}\ln 3\right)\left[\frac{d}{dx}\left(1-x^2\right)\right] - \frac{\left(\frac{d}{dx}(x^3+5x)\right)}{(x^3+5x)\ln 2} \\ &= 3^{1-x^2}(\ln 3)(-2x) - \frac{3x^2+5}{(x^3+5x)\ln 2} \end{aligned}$$

Another approach for deriving the rules for general exponential and logarithmic functions

In the preceding, we restated $y = \log_b x$ as $x = b^y$, and took natural logarithms and rearranged the resulting equation to $y = \left(\frac{1}{\ln b}\right) \ln x$ to derive the rule for finding $\frac{d}{dx}(\log_b x)$. And then we used logarithmic differentation to derive the rule for finding $\frac{d}{dx}(b^x)$. Another way to derive these rules, in the opposite order, is as follows.

For $y = b^x$, we can re-express b as $b = e^{\ln b}$, which gives $y = (e^{\ln b})^x = e^{x \ln b}$. This gives the form $y = e^u$, where $u = x \ln b$, so we just need the Chain Rule. We get

$$\frac{dy}{dx} = e^u \frac{du}{dx} = e^{x \ln b} \left[\frac{d}{dx}(x \ln b) \right] = e^{x \ln b}(\ln b) \left[\frac{d}{dx}(x) \right] = e^{x \ln b}(\ln b)$$

And now, remembering that $e^{x \ln b} = (e^{\ln b})^x = b^x$ (i.e. reversing the first step, where we re-expressed b as $e^{\ln b}$), we get $\frac{dy}{dx} = b^x \ln b$. So, as before, we see that for $y = b^x$ we get $\dfrac{dy}{dx} = b^x \ln b$.

Also, for $y = \log_b x$, we know that this says the same thing as $x = b^y$. So we can use implicit differentiation to find $\frac{dy}{dx}$, using the rule we just derived. That is, we now know that $\frac{d}{dx}(b^y) = \frac{d}{dy}[b^y] \frac{dy}{dx} = (b^y \ln b) \frac{dy}{dx}$, so we see that

$$y = \log_b x \qquad \Rightarrow \qquad x = b^y \qquad \Rightarrow \qquad 1 = (b^y \ln b) \frac{dy}{dx} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{b^y \ln b}$$

And since $b^y = x$, we see (as before) that for $y = \log_b x$ we get $\dfrac{dy}{dx} = \dfrac{1}{x \ln b}$.

Math 1225A/B

Unit 3:

Trigonometric Functions

(text reference: Sections 6.2 and 6.4

custom text pgs. 56 - 76)

3 Trigonometric Functions

In this chapter, our main focus is on doing Calculus with Trigonometric Functions, which is in text section 6.4. The Calculus itself is quite straightforward. There will be a small number of new specific derivatives to remember – two would suffice, but it will probably be easier to memorize a few others, as well. But there aren't any new "differentiation rules", just the same old rules applied to new kinds of functions. So there's really not much that's new here in the way of Calculus. However, it is certainly worthwhile doing some review of what Trigonometric Functions are and how they work. As well, the units in which we measure angles in Calculus may be new to you. All the review you really need us covered in section 6.2 (and these notes), but you may also find it useful to look at text section 6.1, and you should certainly do the assigned homework exercises from section 6.2, which reviews the things you need to know about trig functions, before working on the Trig Derivatives covered in section 6.4.

Perhaps the most important thing for you to get out of the review we do is something that sounds so basic it shouldn't be a problem. *Trig functions are functions.* Well, of course they are! Obviously they are! It says so right there in the name "trig functions". However, forgetting that, or not understanding it in the first place, seems to be the biggest problem that students have in this section of the material. Because with functions come composite functions. That is, if we apply a function to anything more complicated than x, or t, or whatever the variable may happen to be called, then what we have is a composite function. And when you do Calculus with a composite function, then you need the Chain Rule. With trig functions (just like with exponential and logarithmic functions), there's always lots of Chain Rule needed. Keeping in mind that these trig functions *are functions!* will help you to recognize when those rules are needed.

Perhaps a good place to start is reviewing what we mean by a function, to help you see that these "new" functions we're going to be working with *are* functions. Because in previous courses when you've done trigonometry, you perhaps weren't really thinking of these functions as functions. So let's start by recalling *what a function is.*

Definition 3.1. A **function f** is a rule that assigns to each element of the domain exactly one value from the range.

That is, for each value x in the domain of f, f associates the unique value $f(x)$ with the value x. And of course the variable might not be called x. It could be t or y or θ or λ or ... whatever. For no particular reason, we tend to use t as the variable a lot of the time with trig functions. Or θ, the Greek letter "theta", which is often used to represent an angle.

Now, some trig functions. You probably recall that there are a number of them. (Six, in fact.) But there are two main trig functions, and then all the others are defined in terms of those two. So we start with them.

Definition 3.2. For any *angle* t, t has a unique **sine value**, denoted $\boldsymbol{\sin t}$, and also a unique **cosine value**, denoted $\boldsymbol{\cos t}$.

That is, there is a **sine function**, denoted $f(t) = \sin t$, whose domain is the set of all angles, and which associates a particular value with the angle t. (Aha! Maybe that's the problem. Maybe the reason that some students have difficulty realizing that this is a function is because we don't necessarily use the brackets. We normally write functions as $f(t)$ or $g(t)$, but then we write the sine function as $\sin t$, not $\sin(t)$. If that's going to cause you a problem, just imagine that sin is always followed by invisible brackets.) Getting back to what we were saying, this sine function is a rule that associates with every number (angle) t in its domain a unique value, which we call $\sin t$. So that's a function. Similarly, there is another function, the **cosine function**, denoted $g(t) = \cos t$, which has the same domain and associates with each number in its domain, i.e. with each angle, a unique value or number, called $\cos t$. So this is *another* number associated with the same angle.

Notice that just saying "sin" or "cos" by itself has no mathematical meaning, except as the name of a function. It's like just saying "f". We can talk about the function f, but we can't do math with it. We do math with function *values*, either unspecified values, like $f(x)$ or $f(t)$, or specific function values, like $f(1)$ or $f(3)$. Likewise, we can talk about the function sin or the function cos (i.e. the sine function or the cosine function), but we can't do any math with them. We can only do math with function *values*, like $\sin t$ and $\cos t$. This is just like with logarithmic functions. Just saying "log" or "ln" doesn't mean anything by itself, except to name a function. It's only when we put a number with it (whether an unspecified number, i.e. a variable, or a specific number) that we have something meaningful that we can do math with. Like $\log_b x$ or $\ln 5$. So the function name sin or cos must *always* be followed by an angle, whether a specific angle, or an unspecified angle such as t. Or maybe t^2 or $2\pi - 3t$.

In doing trigonometry in High School, you probably mostly, or maybe even exclusively, measured angles in *degrees*. Everybody knows that there are 360 degrees in a circle, right? Sure! And a right angle is 90°. However, as alluded to earlier, in Calculus we generally use a different unit of measurement for angles. (Why? Who knows! That's just the way it is.) Instead of measuring angles in degrees, we measure them in **radians**. What's a radian? Well here's a definition.

Definition 3.3. Consider a segment of a circle. Let r be the radius of the circle and s be the arc length of the segment. Then the angle between the radii producing the segment is ***t* radians**, where $\boldsymbol{t = \frac{s}{r}}$.

For instance, for a full circle, the arc length (i.e. the perimeter of the circle) is $s = 2\pi r$. So we see that there are $t = \dfrac{s}{r} = \dfrac{2\pi r}{r} = 2\pi$ *radians* in a full circle.

Similarly, for a *right angle*, what we have is a circle segment which is one quarter of a full circle. So the arc length is $s = \dfrac{1}{4}(2\pi r) = \dfrac{\pi}{2} r$ and we see that a right angle is $\frac{s}{r} = \frac{\pi}{2}$ *radians.*

Notice: You don't need to remember, or even understand, anything about *arc length.* The only thing you need to remember here is:

There are 2π *radians in a circle, so* 2π *radians* $= 360°$.

And that means that when we measure angles in radians, there is almost always a π in the measurement. That is, an angle measured in radians is generally some multiple of π. So we have angles like 2π, π (that's a semi-circle, i.e. the angle is a straight line), $\dfrac{\pi}{2}$ (we already saw that that's a right angle), or maybe $\dfrac{5\pi}{6}$. Which means that we express specific trig function values as things like $\sin\dfrac{\pi}{6}$ and $\cos\dfrac{3\pi}{4}$.

As you know, we don't use calculators in this course. In your previous study of trigonometry, you may have relied on your calculator to tell you what the sine or cosine value of a particular angle was. Can't do that here. So there are some trig function values that you'll have to **know**, i.e. **memorize**. Just for a few basic angles. The angles whose trig function values you need to know are the multiples of $\dfrac{\pi}{6}$ and $\dfrac{\pi}{4}$. (*Note:* $\dfrac{\pi}{6} = 30°$ and $\dfrac{\pi}{4} = 45°$. These are the same angles you most likely encountered a lot when you did trig before.)

So what are these trig function values? Well, let's start with some *really* basic ones. These values are much easier to just state, and accept, and remember, than they are to derive.

Fact: $\sin 0 = 0$, $\sin\frac{\pi}{2} = 1$, $\cos 0 = 1$ and $\cos\frac{\pi}{2} = 0$.

It's also very useful to remember that:

- The range of both the sine function and the cosine function is $[-1, 1]$. That is, for any angle t, $-1 \leq \sin t \leq 1$ and $-1 \leq \cos t \leq 1$.
- Between 0 radians and 2π radians, the cosine values go from 1 down to -1 and then back to 1 again. Similarly, the sine values start at 0, go up to 1, then down to -1 and back up to 0.

That is, as our fact tells us above, between 0 and $\frac{\pi}{2}$, the value of the sine function goes from 0 up to 1. And then in the next $\frac{\pi}{2}$ radians, i.e. from $\frac{\pi}{2}$ to π, the value goes back down to 0. That is, $\sin\pi = 0$. From there, the same thing happens again, but upside down this time. In the next $\frac{\pi}{2}$ radians, between π and $\frac{3\pi}{2}$, the value of sine goes from 0 down to -1. And then in the next $\frac{\pi}{2}$ radians, from $\frac{3\pi}{2}$ to 2π, the value of sine goes back up from -1 to 0. And that's a full circle, so we're back where we started.

The cosine function follows the same pattern, but starting in a different place. Instead of starting at 0, it starts at 1. In the first $\frac{\pi}{2}$ radians, from 0 to $\frac{\pi}{2}$, (as stated above) the value of cosine goes from 1 down to 0. And then in the next $\frac{\pi}{2}$ radians, i.e. from $\frac{\pi}{2}$ to π, the value keeps going down, to -1. That is, $\cos\pi = -1$. From there, the value goes back up. In the next $\frac{\pi}{2}$ radians, between π and $\frac{3\pi}{2}$, the value of cosine goes from -1 back up to 0. And then in the next $\frac{\pi}{2}$ radians, from $\frac{3\pi}{2}$ to 2π, the value of cosine goes the rest of the way back up, from 0 to 1. Again, that's a full circle, so we're back where we started.

You will probably find it easiest to simply remember what the graphs of these functions look like. They are shown in Figure 1 below.

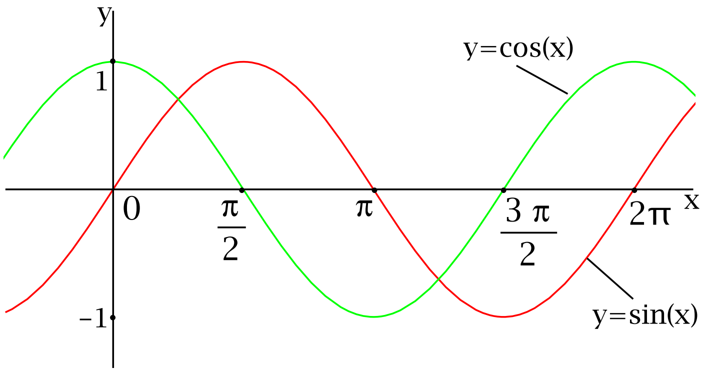
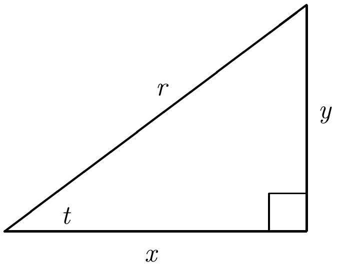


Figure 1: The graphs of $y = \sin x$ and $y = \cos x$.

Although a circle has only 2π radians, we can have angles bigger than that. (Go all the way around the circle, and then keep going around more.) And we can have negative angles, too. (Go backwards around the circle or some part thereof.) So the domain of $\sin t$ and the domain of $\cos t$ are both actually $(-\infty, \infty)$. It's the part from 0 to 2π which is of most interest to us, but the graphs go further than that, on both sides. The graphs of these functions are *cyclic*, with *period* 2π. (That just means that they keep doing the same thing, again and again, ad nauseum, and that it takes 2π

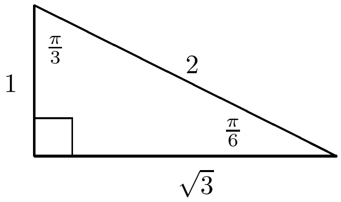
radians to get through the full pattern before it repeats.) The significant points are that the graph of $y = \sin x$ passes through the origin, and is on its way up, whereas the graph of $y = \cos x$ crosses the y-axis at height 1 and is just starting on the way down.

For the other trig function values that you're expected to know, a triangle is useful. Recall that for an acute angle, i.e. an angle smaller than a right angle, the trig function values can be found by considering a right-angled triangle containing that angle. (Remember "soh-cah-toa"?) Consider an angle of t radians, with $0 < t < \frac{\pi}{2}$. Construct a right-angled triangle containing this angle, as shown below. Let the side lengths be: r for the hypotenuse, x for the side adjacent to angle t and y for the side opposite angle t.



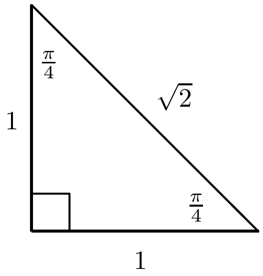
Then the sine value of the angle is given by the ratio of the length of the opposite side to the length of the hypotenuse, i.e., $\sin t = \frac{opp}{hyp} = \frac{y}{r}$. And the cosine value of the angle is given by the ratio of the length of the adjacent side to the length of the hypotenuse, i.e. $\cos t = \frac{adj}{hyp} = \frac{x}{r}$. (That's the "soh" part and the "cah" part. We'll get to the "toa" part later.)

To find the trig function values which you need to know, consider two special triangles. The first you probably know as the 30°-60°-90° triangle. But this is Calculus, so the angles are $\frac{\pi}{6}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$.



In this triangle, the short side is half as long as the hypotenuse. Letting the length of the short side be 1 unit, the hypotenuse has length 2 and the length of the third side is given by the Pythagorean Theorem as $\sqrt{2^2 - 1^2} = \sqrt{3}$. (That is, $1^2 + (\sqrt{3})^2 = 2^2$). From this triangle, and using $\sin t = \frac{opp}{hyp}$ and $\cos t = \frac{adj}{hyp}$, we see that $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos \frac{\pi}{3} = \frac{1}{2}$, and also that $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

The second triangle is an isosceles right-angled triangle in which the hypotenuse is the one non-equal side. Letting the lengths of the two equal sides be 1 unit, we get the length of the hypotenuse as $\sqrt{1^2 + 1^2} = \sqrt{2}$. The 2 equal angles are of course both $\frac{\pi}{4}$ (since the right angle measures $\frac{\pi}{2}$ radians and the angles in a triangle must sum to $180° = \pi$ radians), so using $\sin t = \frac{opp}{hyp}$ and $\cos t = \frac{adj}{hyp}$, we see that $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$. (Our text uses $\frac{\sqrt{2}}{2}$, but that's the same number, and there's no reason to write it that way. $\frac{1}{\sqrt{2}}$ is simpler.)



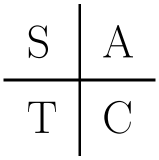
The following table summarizes the trig function values you are expected to know. (The preceding explanations are given only to help you remember them. The only part of this you actually have to remember are these values.)

Angle	**Sine Value**	**Cosine Value**
$t = 0$	$\sin 0 = 0$	$\cos 0 = 1$
$t = \frac{\pi}{6}$	$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$	$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$
$t = \frac{\pi}{4}$	$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$	$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
$t = \frac{\pi}{3}$	$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$	$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
$t = \frac{\pi}{2}$	$\sin\left(\frac{\pi}{2}\right) = 1$	$\cos\left(\frac{\pi}{2}\right) = 0$
$t = \pi$	$\sin \pi = 0$	$\cos \pi = -1$

Notice that the same values occur for different angles, when we compare the sine and cosine functions. How to keep them straight? Well, here are two things that should help. The values we need to keep straight are $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$ and $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. First, there's the relative sizes of these numbers. It's easy enough to realize that $1 < 2 < 3$ and so (since these numbers are all at least 1) their square roots follow the same pattern, that is $\sqrt{1} < \sqrt{2} < \sqrt{3}$, and so if we take half of each of those numbers, we *still* have the same relationships: $\frac{1}{2} < \frac{\sqrt{2}}{2} < \frac{\sqrt{3}}{2}$. And then you just need to remember what the graphs of the sine and cosine functions look like. The sine function is *increasing* on the interval $\left(0, \frac{\pi}{2}\right)$, and so it goes through these values in increasing order. That is, we have $\sin\frac{\pi}{6} < \sin\frac{\pi}{4} < \sin\frac{\pi}{3}$, so we realize that $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. On the other hand, the cosine function is *decreasing* on the interval $\left(0, \frac{\pi}{2}\right)$, so it goes through these values in the opposite order, i.e. in decreasing order. So we have $\cos\frac{\pi}{6} > \cos\frac{\pi}{4} > \cos\frac{\pi}{3}$, and we realize that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$. (That is, for example, for the smallest angle, $\frac{\pi}{6}$, the sine function has the smallest of the 3 values, and the cosine function has the largest of the 3 values.) It also helps to remember that the 2 curves intersect halfway between 0 and $\frac{\pi}{2}$, so the 2 functions have the same value (the middle one of the 3 values we're trying to keep straight, i.e. $\frac{1}{\sqrt{2}}$) at $\frac{\pi}{4}$.

In the above table, the only angle bigger than $\frac{\pi}{2}$ (i.e. 90°) that's mentioned is π (i.e. 180°). That's because for any angle bigger than $\frac{\pi}{2}$, the sine or cosine value of the angle can be found by relating it to the sine or cosine value of a an acute angle.

If we have an angle t outside the "first quadrant" (i.e. bigger than $\frac{\pi}{2}$), we look at the acute angle made with the x-axis (i.e. with a horizontal line). The trig function value of the angle t is the same as the trig function value of this acute angle, except that its sign might be changed. The CAST rule tells us about which of the 3 main trig functions are positive in each quadrant, starting with the fourth quadrant and moving clockwise: only **C**osine, **A**ll, only **S**ine, and only **T**angent.



Notice: Our text refers to the CAST rule as the ASTC rule. (Fine ... whatever. Same idea. If you find the phrase “All Students Take Calculus” easier to remember than the word “CAST”, and/or can’t remember that the C of CAST starts in the fourth quadrant, by all means use the text’s approach instead.)

In applying this rule (whatever you call it), we’re always using the acute angle formed with the horizontal axis. Notice that if the angle is below the axis (in the third or the fourth quadrant) then the sign of the cosine value is the same as for the same angle above the axis (i.e. in the second or first quadrant), but the sign of the sine value is the opposite. (That is, cosine is positive in both the first and fourth quadrants, and is negative in both the second and third quadrants, so the sign of the cosine value doesn’t change when the angle is reflected horizontally. On the other hand, sine is positive in the first quadrant and second quadrants, but negative in the third and fourth quadrant, so that the sign does change when the angle is reflected horizontally.) This gives the following two useful rules:

$$\sin(-x) = -\sin x \qquad \text{and} \qquad \cos(-x) = \cos x$$

Remember: Here, we are applying a function to the value $-x$, instead of to the value x. You can’t “factor out” the negative. What we are observing here is analogous to the observation that for $f(x) = x^2$ and $g(x) = x^3$, $f(-x) = f(x)$, while $g(-x) = -g(x)$. (Those are results which just happen to be true for the functions $f(x) = x^2$ and $g(x) = x^3$, and we have seen here that they also happen to be true for $f(x) = \cos x$ and $g(x) = \sin x$.)

Example 3.1. Find the values of $\sin\theta$ and $\cos\theta$ for (a) $\theta = \dfrac{3\pi}{4}$ and (b) $\theta = -\dfrac{7\pi}{6}$

Solution:
(a) For $\theta = \frac{3\pi}{4}$ we have a second quadrant angle. That is, $\frac{2\pi}{4} < \frac{3\pi}{4} < \frac{4\pi}{4}$, so this angle lies between $\frac{\pi}{2}$ and π. Since only sine is positive in the second quadrant, we know that $\sin\left(\frac{3\pi}{4}\right)$ is positive, but $\cos\left(\frac{3\pi}{4}\right)$ is negative.

Since we have $\theta > \frac{\pi}{2}$, the acute angle formed with the horizontal axis is the angle between the arm of this angle and the left arm of the horizontal axis. And since $\theta < \pi$, this angle is the angle that would take θ the rest of the way to π. So the angle we need to use is $\pi - \theta = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$. (This is what the text calls the *reference angle*, α.) Using what we observed about the signs, we see that

$$\begin{aligned} \sin\theta = \sin\left(\frac{3\pi}{4}\right) &= \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \\ \text{and } \cos\theta = \cos\left(\frac{3\pi}{4}\right) &= -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} \end{aligned}$$

(b) We are asked to find $\sin\left(-\frac{7\pi}{6}\right)$ and $\cos\left(-\frac{7\pi}{6}\right)$. We know that $\sin(-x) = -\sin x$ and that $\cos(-x) = \cos x$ which in this case tells us that

$$\sin\left(-\frac{7\pi}{6}\right) = -\sin\left(\frac{7\pi}{6}\right) \qquad \text{and} \qquad \cos\left(-\frac{7\pi}{6}\right) = \cos\left(\frac{7\pi}{6}\right)$$

The angle $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ is between π and $\frac{3\pi}{2}$, so it is in the third quadrant. This tells us that both the sine and cosine values of this angle are negative, and that the acute angle made with the horizontal axis is the amount by which we have gone beyond π, which is $\frac{\pi}{6}$. So we have

$$\begin{aligned} \sin\left(\frac{7\pi}{6}\right) &= -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2} \\ \text{and } \cos\left(\frac{7\pi}{6}\right) &= -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2} \end{aligned}$$

And therefore we get

$$\begin{aligned}\sin\theta &= \sin\left(-\frac{7\pi}{6}\right) = -\sin\left(\frac{7\pi}{6}\right) = -\left(-\frac{1}{2}\right) = \frac{1}{2} \\ \text{and } \cos\theta &= \cos\left(-\frac{7\pi}{6}\right) = \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}\end{aligned}$$

Another Approach:
$\theta = -\frac{7\pi}{6}$ is obtained by going backwards around the circle by $\frac{7\pi}{6}$ radians, which takes us more than π radians back to the second quadrant. (We go through both the fourth and third quadrants and keep going for another $\frac{\pi}{6}$ radians.) The acute angle made with the horizontal axis is $\frac{\pi}{6}$ (the amount by which we continued backwards past π), and for a second quadrant angle we know that sine is positive and cosine is negative. So we get

$$\begin{aligned}\sin\theta &= \sin\left(-\frac{7\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \\ \text{and } \cos\theta &= \cos\left(-\frac{7\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}\end{aligned}$$

Transformations of the Sine and Cosine functions

In Unit 1 of these notes, we talked about various kinds of transformations of functions, in general, and looked at their effects on exponential and logarithmic functions. Similarly, it's worthwhile to observe the effects of these various transformations on the 2 basic trigonometric functions.

Recall that *vertical translation* of a function is obtained by adding a constant to the function, e.g. $g(x) = f(x)+c$, and has the effect of shifting the graph of the function up or down by $|c|$ units. Since this is true for all functions, it is also true for trigonometric functions. So the graph of $y = c + \sin x$ looks just like the graph of $y = \sin x$, but moved up or down, so that the graph is undulating around the line $y = c$ instead of around the x-axis (the line $y = 0$). Similarly, $y = c + \cos x$ looks just like the graph of $y = \cos x$, but shifted up or down so that it, too, undulates about the line $y = c$.

Also recall that *horizontal translation* of a function is obtained by adding a constant to the x-value, *before* applying the function (e.g. $g(x) = f(x + c)$), and has the effect of shifting the function to the left or to the right by $|c|$ units. So the graph of $y = \sin(x + c)$ looks just like the graph of $y = \sin x$ except that it's been moved to the left or to the right by $|c|$ units. That is, it looks like the axes have been slid over. The graph of $y = \sin(x + c)$ crosses the x-axis on the way up at $x = -c$ (and $x = -c + 2n\pi$) and crosses the x-axis on the way down at $x = \pi - c$ (and $x = (\pi-c)+2n\pi = (2n+1)\pi-c$), instead of at 0 (and $2n\pi$) and at π (and $\pi+2n\pi = (2n+1)\pi$).

Similarly, the graph of $y = \cos(x + c)$ looks just like the graph of $y = \cos x$ except that once again it's been moved to the left or to the right by $|c|$ units. So there is a peak at $x = -c$ (and any $x = -c+2n\pi$) and there is a valley at $x = \pi - c$ (and any $x = \pi - c + 2n\pi = (2n + 1)\pi - c$), instead of a peak at $x = 0$ and a valley at $x = \pi$ (etc.).

Notice that if the cosine function is shifted to the right by $\frac{\pi}{2}$ units, it looks just like the sine function. And if the sine function is shifted to the left by $\frac{\pi}{2}$ units, it looks just like the cosine function. So we see that $\cos\left(x - \frac{\pi}{2}\right) = \sin x$ and similarly $\sin\left(x + \frac{\pi}{2}\right) = \cos x$.

Vertical reflection of a function, obtained by $g(x) = -f(x)$, flips the function upside-down. For both the sine and cosine functions, this means that the peaks become valleys and vice versa. For both, this reflection is exactly the same as a horizontal translation by π units.

Horizontal reflection of a function, obtained by $g(x) = f(-x)$ flips the function left-to-right. Since the graph of $y = \cos x$ is symmetric about the y-axis, horizontal reflection gives the same graph. That is, as we have already noticed, $\cos(-x) = \cos x$, so horizontal reflection has no effect on the graph. And for the sine function, horizontal reflection looks just like vertical reflection – again, the effect is the same as a horizontal shift by π units.

Recall that *vertical stretching* of a function is obtained by multiplying the function by a positive constant (e.g. $g(x) = c[f(x)]$ where $c > 0$) and has the effect of expanding or contracting the graph vertically, by a factor of c. For the trig functions, for any $c > 0$, the graphs of $y = c(\sin x)$ and $y = c(\cos x)$ are vertically stretched so that the height of the peaks is c and the height of the valleys is $-c$ (instead of 1 and -1). The *period* of the function is the same (still 1 full cycle every 2π units), but the *amplitude* is changed (taller, skinnier waves if $c > 1$, or shorter, fatter waves if $0 < c < 1$).

Finally, recall that *horizontal stretching*, expanding or contracting the function horizontally, is achieved by multiplying the x-value by a positive constant before applying the function (e.g. $g(x) = f(cx)$ with $c > 0$). So the graphs of $y = \sin(cx)$ and $y = \cos(cx)$ for any $c > 0$ are horizontally stretched by a factor of c, so that a full cylcle is acheived more or less quickly. The amplitude is unchanged (the peaks and valleys are still at heights 1 and -1, respectively), but the distance between them is expanded or contracted. The period changes to $\frac{2\pi}{c}$, that is it takes only $\frac{2\pi}{c}$ units to get through a full cycle. For instance, the function $y = \sin(2x)$ goes through a full cycle between 0 and π, and has gone through 2 cycles by $x = 2\pi$, whereas the function $y = \cos\left(\frac{x}{2}\right)$ has only gone through half a cycle between 0 and 2π and doesn't finish a full cycle until $x = 4\pi$.

For all of the above results, you should play with some graphs to see that these observations are correct, and to understand why.

The OTHER Trigonometric Functions

Now, we said earlier, and you probably (vaguely) remember from when you've encountered trigonometry in previous courses, that there are *other* trig functions. That is, there are more than just the two values already mentioned associated with a particular angle. There are 4 more trigonometric functions — one more of what we could call the primary trig functions, and then also the reciprocals of those 3 functions. The first is the **tangent** function.

Definition 3.4. For any angle t with $\cos t \neq 0$, the **tangent value** of t, denoted $\boldsymbol{\tan t}$ is given by

$$\tan t = \frac{\sin t}{\cos t}$$

Notice that since $\sin t = \frac{\text{opp}}{\text{hyp}}$ and $\cos t = \frac{\text{adj}}{\text{hyp}}$, this gives $\tan t = \frac{\text{opp}}{\text{adj}}$. (That's the "toa" part of "soh-cah-toa".) Also notice that whenever exactly one of sine and cosine is negative, tangent is also negative. But when both are positive, so is tangent, and when both are negative, the negatives cancel out so that tangent is again positive. This is why in the third quadrant, when both sine and cosine values are negative, tangent is (the only one of the 3 which is) positive, as the CAST rule told us.

Definition 3.5. The **cosecant**, **secant** and **cotangent** functions associate with the angle t the values $\mathbf{csc}\,\boldsymbol{t}$, $\mathbf{sec}\,\boldsymbol{t}$ and $\mathbf{cot}\,\boldsymbol{t}$, respectively, defined as:

$$\csc t = \frac{1}{\sin t}, \qquad \sec t = \frac{1}{\cos t} \qquad \text{and } \cot t = \frac{1}{\tan t}$$

Notice: It will help you to keep these straight if you remember that only one of a primary trig function and its reciprocal starts with c (or "co"). That is, sine doesn't start with "co", so its reciprocal cosecant does. Likewise for tangent and cotangent. On the other hand, cosine *does* start with "co", so its reciprocal secant doesn't.

Example 3.2. Find all trig function values of the angle $\dfrac{11\pi}{4}$.

Solution:
First, we need to realize that $\dfrac{11\pi}{4} > 2\pi$. To form this angle, we go all the way around the circle, and then keep going some more, part way around again. But the full circle that we do doesn't affect the trig function values. All that matters is the part circle we do. So for any of the trig functions, the function value at $\frac{11\pi}{4}$ is the same as the function value at $\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} = \frac{3\pi}{4}$. So this is the angle we need to think about. This angle is in the second quadrant. And in Example 3.1(a) on pg. 41 we found that $\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$. So we know that

$$\sin\left(\frac{11\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos\left(\frac{11\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

We use these values to find the other trig function values:

$$\begin{aligned}
\tan\left(\frac{11\pi}{4}\right) &= \frac{\sin\left(\frac{11\pi}{4}\right)}{\cos\left(\frac{11\pi}{4}\right)} = \frac{1/\sqrt{2}}{-1/\sqrt{2}} = -1 \\
\csc\left(\frac{11\pi}{4}\right) &= \frac{1}{\sin\left(\frac{11\pi}{4}\right)} = \frac{1}{1/\sqrt{2}} = \sqrt{2} \\
\sec\left(\frac{11\pi}{4}\right) &= \frac{1}{\cos\left(\frac{11\pi}{4}\right)} = \frac{1}{-1/\sqrt{2}} = -\sqrt{2} \\
\cot\left(\frac{11\pi}{4}\right) &= \frac{1}{\tan\left(\frac{11\pi}{4}\right)} = \frac{1}{-1} = -1
\end{aligned}$$

Notice that we could instead have used $\cot\left(\dfrac{11\pi}{4}\right) = \dfrac{\cos\left(\frac{11\pi}{4}\right)}{\sin\left(\frac{11\pi}{4}\right)}$ to find that the cotangent value of the angle is -1.

Derivatives of Trigonometric Functions

Now that we've reviewed the trig functions, as well as their graphs and their values, it's time to do some Calculus with these functions. We'll simply accept as true, i.e. we'll define, the following derivatives.

Definition 3.6.

$$\frac{d}{dt}(\sin t) = \cos t \qquad \text{and} \qquad \frac{d}{dt}(\cos t) = -\sin t$$

That is, we won't worry about *why* these derivatives are what they are. The proofs are beyond the scope of this course. However, you may find it interesting to look again at the graphs of these functions, and think about their slopes, that is, about their tangent lines and the slopes of those tangent lines. Because of course the derivative of a function tells us the slope of the tangent line. Look at where the slopes are 0, are positive, and are negative. And look at where the slopes are increasing and where they are decreasing (i.e. where the functions are concave up and are concave down). Think about the slope of the sine function throughout its cyclic period of 2π and compare what you observe there with the corresponding values of the cosine function. (Most obviously, $\cos t$ is 0 at the same t-values at which $\sin t$ has horizontal tangent lines. But there is more going on than just that.)

Now we get to the part where it's important to remember that these functions are functions. Because the derivatives of $\sin t$ and $\cos t$ by themselves aren't very interesting. It's the *composite* functions which are more interesting. Trig functions of something more complicated than just t. And for composite functions, we need the Chain Rule. (As stated previously, with trig functions we *always* need *lots of* Chain Rule.)

For instance, if we have $f(t) = \sin u$, where u is some function of t, then we use the Chain Rule to get

$$f'(t) = \frac{d}{dt}(\sin u) = \left[\frac{d}{du}(\sin u)\right]\frac{du}{dt} = (\cos u)\frac{du}{dt}$$

Similarly, for $g(x) = \cos u$ where u is some function of x, we have

$$g'(x) = \frac{d}{dx}(\cos u) = \left[\frac{d}{du}(\cos u)\right]\frac{du}{dx} = (-\sin u)\frac{du}{dx}$$

That is, we always have to remember to multiply by *the deriviative of the angle function.*

Example 3.3. Find $\frac{dy}{dt}$, where $y = \sin(t^2 + t)$.

Solution:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d}{dt}\left[\sin(t^2+t)\right] \\
&= \frac{d}{dt}\left[\sin(u)\right] && \text{where } u = t^2 + t \\
&= \frac{d}{du}\left[\sin u\right]\frac{du}{dt} && \text{with } \frac{du}{dt} = 2t+1 \\
&= [\cos(u)]\left(\frac{du}{dt}\right) && \text{since } \frac{d}{du}[\sin u] = \cos u \\
&= [\cos(t^2+t)](2t+1) \\
&= (2t+1)\cos(t^2+t)
\end{aligned}$$

Notice: We always write the non-trig terms in a product in front of the trig function, so that it's clear that it's the *function*, not just the *angle* that's being multiplied by that term. $[\cos(t^2+t)](2t+1) = (2t+1)\cos(t^2+t)$ is very different from $\cos\left[(t^2+t)(2t+2)\right] = \cos(2t^3 + 4t^2 + 2t)$. (This is just like what we've always done with logarithms. We write the multiplier in front to avoid needing extra brackets.)

Example 3.4. If $f(x) = e^{\cos(2x^3 - 5x)}$, find $f'(x)$.

Solution:
We have the form e^u where $u = \cos(2x^3 - 5x)$, so we know that $f'(x) = e^u \frac{du}{dx}$.

For $\frac{du}{dx}$ we get:

$$\begin{aligned} \frac{du}{dx} &= \frac{d}{dx}\left[\cos(2x^3 - 5x)\right] = \left[-\sin(2x^3 - 5x)\right]\left[\frac{d}{dx}\left(2x^3 - 5x\right)\right] \\ &= -[\sin(2x^3 - 5x)](6x^2 - 5) = (5 - 6x^2)\sin(2x^3 - 5x) \end{aligned}$$

and so for $f'(x)$ we have

$$f'(x) = e^u \frac{du}{dx} = e^{\cos(2x^3 - 5x)}(5 - 6x^2)\sin(2x^3 - 5x)$$

There is a quirk in the notation used with trig functions. Maybe you've seen it before. Instead of writing something like $(\sin t)^2$, we write $\sin^2 t$. Likewise, we would write $\cos^3(1 - \sqrt{t})$ instead of $[\cos(1 - \sqrt{t})]^3$. We do this partly to avoid extra brackets, but also because it helps us to remember that the exponent applies to the whole function value, *not* to the angle. Notice that, for instance,

$$\cos^2 \frac{\pi}{4} = \left(\cos \frac{\pi}{4}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

In that calculation, the angle, $\frac{\pi}{4}$ is never squared. $\cos\left(\frac{\pi}{4}\right)^2 = \cos\frac{\pi^2}{16}$ is something very different, whose value we don't know without a calculator.

Example 3.5. Find $\dfrac{d}{dt}(\cos^4 2t)$.

Solution:
We use the power rule. And the chain rule. That is, we use the general power rule (the "power-chain" rule), but we also need the normal chain rule, too, because of the $2t$. That is, we have a function (the exponent) of a function (the cosine function) of a function of t (i.e. $2t$), and so we need to use the chain rule twice.

$$\begin{aligned} \frac{d}{dt}\left(\cos^4 2t\right) &= \frac{d}{dt}\left[(\cos 2t)^4\right] = 4\left(\cos 2t\right)^3\left[\frac{d}{dt}\left(\cos 2t\right)\right] = 4\left(\cos 2t\right)^3\left(-\sin 2t\right)\left[\frac{d}{dt}(2t)\right] \\ &= 4(\cos^3 2t)(-\sin 2t)(2) = -8\sin 2t \cos^3 2t \end{aligned}$$

Notice: When we have one trig function multiplying another, we often don't put brackets around them. We understand that *by convention*, $\sin 2t \cos^3 2t$ says the same thing as $(\sin 2t)(\cos^3 2t)$. If we actually meant the sine value of $2t\cos^3 3t$, rather than the sine value of $2t$, times the cube of the cosine value of $2t$, we would *have to* use brackets and write $\sin(2t\cos^3 2t)$.

Example 3.6. Find $\frac{dw}{dt}$ where $w = \sin(\cos(\ln t))$.

Solution:
The first challenge here is to understand what w is equal to. It says "the value of w is obtained by finding the sine value of the angle whose value is the cosine value of $\ln t$". Remember, neither "sin" nor "cos" means anything by itself that we can do arithmetic with, so the sin out in front is *not* a

multiplier, nor is the cos in the middle. Neither of them has an angle with it. That is, the angle that's with the cos *is* $\ln t$, and the angle that's with the sin *is* all of what follows it.

So what we have here is a function (the sine function) of a function (the cosine function) of a function (the natural logarithm function) of t. So that's going to require the chain rule twice, as in the previous example. We get:

$$\begin{aligned}
\frac{dw}{dt} &= \frac{d}{dt}[\sin(\cos(\ln t))] &&= \frac{d}{dt}[\sin u] &&\text{(where } u = \cos(\ln t)\text{)}\\
&= \frac{d}{du}[\sin u]\frac{du}{dt} && && \\
&= [\cos u]\left[\frac{d}{dt}(\cos(\ln t))\right] &&= [\cos u]\left[\frac{d}{dt}(\cos v)\right] &&\text{(where } v = \ln t\text{)}\\
&= [\cos u](-\sin v)\frac{dv}{dt} &&= -[\cos u](\sin v)\left[\frac{d}{dt}(\ln t)\right] && \\
&= -[\cos u](\sin v)\left(\frac{1}{t}\right) &&= -\frac{[\cos u](\sin v)}{t} &&\text{(now sub back for } u \text{ and for } v\text{)}\\
&= -\frac{[\cos(\cos(\ln t))](\sin(\ln t))}{t} &&= -\frac{\sin(\ln t)\cos(\cos(\ln t))}{t} &&
\end{aligned}$$

Notice: We generally write the less complicated trig function before the more complicated trig function. Here, $\sin(\ln t)$ is less complicated than $\cos(\cos(\ln t))$, so we write it first.

Also Note: Of course, $\cos(\cos(\ln t))$, which says the cosine of the cosine of the natural logarithm of t, is not a product. This is *not* the same as $\cos^2(\ln t)$, which says the square of the cosine of the natural logarithm of t. For instance, when $t = e^{\pi/4}$, we have

$$\cos\left(\cos\left(\ln e^{\pi/4}\right)\right) = \cos\left(\cos\left(\frac{\pi}{4}\right)\right) = \cos\left(\frac{1}{\sqrt{2}}\right)$$

which is not a trig function value we know, whereas

$$\cos^2\left(\ln e^{\pi/4}\right) = \cos^2\left(\frac{\pi}{4}\right) = \left(\cos\frac{\pi}{4}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

Knowing the derivatives of $\sin t$ and $\cos t$ also allows us to find the derivatives of all the other trig functions, since each of them can be expressed in terms of the sine and/or cosine function. To express the derivative of $\tan t$ in its simplest and most easily remembered form, we'll need to use a *trig identity*, i.e. a relationship involving trig functions which is true for all angles. You've probably seen this trig identity before, in your previous study of trigonometry. We won't prove that it's true, although it's not hard to do so — it follows directly from the Pythagorean Theorem.

Theorem 3.1. *For any angle t,*

$$\sin^2 t + \cos^2 t = 1$$

Example 3.7. Find the derivative of $\tan t$.

Solution:
We know that $\tan t = \dfrac{\sin t}{\cos t}$, so we use the quotient rule:

$$\begin{aligned}\frac{d}{dt}(\tan t) = \frac{d}{dt}\left(\frac{\sin t}{\cos t}\right) &= \frac{\left[\frac{d}{dt}(\sin t)\right](\cos t) - \left[\frac{d}{dt}(\cos t)\right](\sin t)}{(\cos t)^2} = \frac{(\cos t)(\cos t) - (-\sin t)(\sin t)}{\cos^2 t} \\ &= \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} = \left(\frac{1}{\cos t}\right)^2 = (\sec t)^2 = \sec^2 t\end{aligned}$$

The fact that the derivative of $\tan t$ is $\sec^2 t$ will certainly be worth remembering.

Example 3.8. Find the derivative of $\sec t$.

Solution:
We use the fact that $\sec t = \dfrac{1}{\cos t} = (\cos t)^{-1}$. We can use the general power rule:

$$\begin{aligned}\frac{d}{dt}(\sec t) &= \frac{d}{dt}\left((\cos t)^{-1}\right) = (-1)(\cos t)^{-2}\left[\frac{d}{dt}(\cos t)\right] = -\left(\tfrac{1}{\cos^2 t}\right)(-\sin t) \\ &= \frac{\sin t}{\cos^2 t} = \left(\frac{1}{\cos t}\right)\left(\frac{\sin t}{\cos t}\right) = \sec t \tan t\end{aligned}$$

The fact that the derivative of $\sec t$ is $\sec t \tan t$ is worth remembering too. Because of course you don't want to have to do that everytime you need one of those derivatives. And you'll also want to remember the derivatives of $\csc t$ and $\cot t$. We won't prove them here, but you should derive them, for practice.

Theorem 3.2. *For any angle t in the domain of the following functions, their derivatives are:*

$$\frac{d}{dt}(\tan t) = \sec^2 t, \qquad \frac{d}{dt}(\sec t) = \sec t \tan t, \qquad \frac{d}{dt}(\csc t) = -\csc t \cot t, \qquad \frac{d}{dt}(\cot t) = -\csc^2 t$$

Notice: Since all of these functions are quotients, there are angles for which each is not defined, i.e. angles which are not in the domain of that function, and of course the derivative of the function is only defined for angles in the domain of the function.

Example 3.9. Find $\dfrac{dy}{dt}$ if $y = \cot(t^3 + e^t)$.

Solution:
We need the chain rule. Letting $u = t^3 + e^t$ we have $\dfrac{du}{dt} = 3t^2 + e^t$ and so we get

$$\frac{dy}{dt} = \frac{d}{dt}[\cot(t^3 + e^t)] = \frac{d}{dt}[\cot u] = \frac{d}{du}[\cot u]\frac{du}{dt} = (-\csc^2 u)(3t^2 + e^t) = -(3t^2 + e^t)\csc^2(t^3 + e^t)$$

Example 3.10. Find $f'\left(\frac{\pi}{6}\right)$ if $f(x) = x \csc x$.

Solution:
We need to find $f'(x)$, for which we use the product rule. We get:

$$f'(x) = \left[\frac{d}{dx}(x)\right](\csc x) + (x)\left[\frac{d}{dx}(\csc x)\right] = (1)(\csc x) + x(-\csc x \cot x) = \csc x - x \csc x \cot x$$

Now, we evaluate this function at $x = \frac{\pi}{6}$. We know that $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and so we have

$$\csc\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{1/2} = 2$$

and

$$\cot\frac{\pi}{6} = \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} = \frac{\sqrt{3}/2}{1/2} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Therefore for $f'(x) = \csc x - x\csc x\cot x$ we have

$$f'\left(\frac{\pi}{6}\right) = \csc\frac{\pi}{6} - \frac{\pi}{6}\left(\csc\frac{\pi}{6}\cot\frac{\pi}{6}\right) = 2 - \left(\frac{\pi}{6}\right)(2)(\sqrt{3}) = 2 - \frac{\pi\sqrt{3}}{3} = 2 - \frac{\pi}{\sqrt{3}}$$

Example 3.11. Find $f''(x)$ if $f(x) = \tan e^{3x}$.

Solution:
Of course, to find the second derivative, we must find the first derivative. And as usual, we need the chain rule.

$$\begin{aligned} f'(x) &= \frac{d}{dt}\left(\tan e^{3x}\right) \\ &= (\sec^2 e^{3x})\left[\frac{d}{dx}(e^{3x})\right] \\ &= (\sec^2 e^{3x})(3e^{3x}) \\ &= 3e^{3x}\sec^2 e^{3x} \end{aligned}$$

Note that of course we used the chain rule again to find $\frac{d}{dt}(e^{3x})$. Now for the second derivative we need the product rule:

$$\begin{aligned} f''(x) &= \left(\frac{d}{dx}(3e^{3x})\right)(\sec^2 e^{3x}) + \left[\frac{d}{dx}(\sec^2 e^{3x})\right](3e^{3x}) \\ &= 3\left[\frac{d}{dx}(e^{3x})\right](\sec^2 e^{3x}) + 3e^{3x}\left[\frac{d}{dx}\left(\sec e^{3x}\right)^2\right] \\ &= 3(3e^{3x})\sec^2 e^{3x} + 3e^{3x}\left[2(\sec e^{3x})\left(\frac{d}{dx}(\sec e^{3x})\right)\right] \\ &= 9e^{3x}\sec^2 e^{3x} + 6e^{3x}(\sec e^{3x})(\sec e^{3x}\tan e^{3x})\left[\frac{d}{dx}(e^{3x})\right] \\ &= 9e^{3x}\sec^2 e^{3x} + 6e^{3x}(\sec^2 e^{3x}\tan e^{3x})(3e^{3x}) \\ &= 9e^{3x}\sec^2 e^{3x} + 18(e^{3x})^2\sec^2 e^{3x}\tan e^{3x} \\ &= 9e^{3x}\sec^2 e^{3x}(1 + 2e^{3x}\tan e^{3x}) \end{aligned}$$

Example 3.12. Find an equation of the tangent line to the curve $y = \tan\left(\frac{x}{2}\right)$ at the point with $x = \frac{\pi}{3}$.

Solution:
To find the equation of the tangent line to the curve $y = f(x)$ at the point with $x = a$, we need to find that point on the curve, i.e. find the point $(a, f(a))$, and also find the slope of the tangent line, which of course is $f'(a)$. Here, we have $f(x) = \tan\left(\frac{x}{2}\right)$ and $a = \frac{\pi}{3}$. When $x = \frac{\pi}{3}$ the value of y is

$$y = f\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi/3}{2}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

so the tangent line passes through the point $(x_0, y_0) = \left(\frac{\pi}{3}, \frac{1}{\sqrt{3}}\right)$. Next, we find $f'(x)$:

$$f'(x) = \frac{d}{dx}\left[\tan\left(\frac{x}{2}\right)\right] = \left[\sec^2\left(\frac{x}{2}\right)\right]\left[\frac{d}{dx}\left(\frac{x}{2}\right)\right] = \left[\sec^2\left(\frac{x}{2}\right)\right]\left(\frac{1}{2}\right) = \frac{\sec^2\left(\frac{x}{2}\right)}{2}$$

Therefore the slope of the tangent line to $y = f(x)$ at $x = \frac{\pi}{3}$ is

$$f'\left(\frac{\pi}{3}\right) = \frac{\sec^2\left(\frac{\pi/3}{2}\right)}{2} = \frac{\left(\sec\frac{\pi}{6}\right)^2}{2} = \frac{\left(\frac{1}{\cos(\pi/6)}\right)^2}{2} = \frac{\left(\frac{1}{\sqrt{3}/2}\right)^2}{2} = \frac{\left(\frac{2}{\sqrt{3}}\right)^2}{2} = \frac{4/3}{2} = \frac{2}{3}$$

Now using $y - y_0 = m(x - x_0)$ with $m = f'\left(\frac{\pi}{3}\right) = \frac{2}{3}$ and $(x_0, y_0) = \left(\frac{\pi}{3}, \frac{1}{\sqrt{3}}\right)$ we see that an equation of the tangent line to $y = \tan\left(\frac{x}{2}\right)$ at the point with $x = \frac{\pi}{3}$ is

$$y - \frac{1}{\sqrt{3}} = \frac{2}{3}\left(x - \frac{\pi}{3}\right) \qquad \text{or} \qquad y = \frac{2x}{3} - \frac{2\pi}{9} + \frac{1}{\sqrt{3}}$$

Example 3.13. Find the slope of the tangent line to the curve described by

$$\sin(2y) + \cos\left(x^2\right) + x^3 y = (x + y + 1)^3$$

at the point $(0, 0)$.

Solution:
We know that the slope of the tangent line is given by $\frac{dy}{dx}$. Here, our equation relating x and y implicitly defines y to be a function of x, so we use implicit differentiation to find $\frac{dy}{dx}$. That is, we differentiate with respect to x, using the chain rule when we differentiate an expression involving y, so we use the fact that $\frac{d}{dx}(f(y)) = f'(y)\frac{dy}{dx}$.

Differentiating both sides of the equation, we have:

$$\frac{d}{dx}(\sin(2y)) + \frac{d}{dx}\left(\cos\left(x^2\right)\right) + \frac{d}{dx}\left(x^3 y\right) = \frac{d}{dx}\left[(x + y + 1)^3\right]$$

For the left hand side, we have:

$$\begin{aligned}
\frac{d}{dx}(\sin(2y)) + \frac{d}{dx}\left(\cos\left(x^2\right)\right) + \frac{d}{dx}\left(x^3 y\right) &= [\cos(2y)]\left[\frac{d}{dx}(2y)\right] + \left[-\sin\left(x^2\right)\right]\left[\frac{d}{dx}\left(x^2\right)\right] + \left(3x^2 y + x^3\frac{dy}{dx}\right) \\
&= [\cos 2y]\left(2\frac{dy}{dx}\right) - \left[\sin x^2\right](2x) + 3x^2 y + x^3\frac{dy}{dx} \\
&= (2\cos 2y)\frac{dy}{dx} - 2x\sin x^2 + 3x^2 y + x^3\frac{dy}{dx}
\end{aligned}$$

and for the right hand side we have:

$$\frac{d}{dx}\left[(x + y + 1)^3\right] = 3(x + y + 1)^2\left[\frac{d}{dx}(x + y + 1)\right] = 3(x + y + 1)^2\left(1 + \frac{dy}{dx}\right)$$

so equating the derivatives of the 2 sides we get

$$(2\cos 2y)\frac{dy}{dx} - 2x\sin x^2 + 3x^2 y + x^3\frac{dy}{dx} = 3(x + y + 1)^2\left(1 + \frac{dy}{dx}\right)$$

To find $\frac{dy}{dx}$ at the point $(x, y) = (0, 0)$, we simply substitute in $x = 0$, $y = 0$ and solve for $\frac{dy}{dx}$. We get

$$\begin{aligned} & (2\cos 0)\tfrac{dy}{dx} - (0)(\sin 0) + 0 + 0 &=& \ 3(0+0+1)^2\left(1 + \tfrac{dy}{dx}\right) \\ \Rightarrow \quad & 2(1)\tfrac{dy}{dx} - 0 &=& \ 3 + 3\tfrac{dy}{dx} \\ \Rightarrow \quad & 2\tfrac{dy}{dx} - 3\tfrac{dy}{dx} &=& \ 3 \\ \Rightarrow \quad & -\tfrac{dy}{dx} &=& \ 3 \\ \Rightarrow \quad & \tfrac{dy}{dx} &=& \ -3 \end{aligned}$$

We see that the slope of the tangent line to the curve $\sin(2y) + \cos\left(x^2\right) + x^3y = (x + y + 1)^3$ at the point $(x, y) = (0, 0)$ is -3.

Math 1225A/B

Unit 4:

Antiderivatives and Integration

(text reference: Sections 8.1 and 8.2

custom text pgs. 81 - 110)

4 Antiderivatives and Integration

Now we start getting into some new calculus, not just review of things you've learnt before. But we ease into it by realizing that you probably already know more than you think you do.

Question 1: If $f'(x) = 2x$, what was $f(x)$?
We recognize that $2x = \frac{d}{dx}(x^2)$. Aha! But wait a minute. Actually, $f(x)$ might have been any one of *many* different functions. Maybe $f(x) = x^2$. But then again, maybe $f(x) = x^2 + 1$. Or maybe $f(x) = x^2 - \frac{37\pi e}{62}$, or ... But certainly $f(x)$ must have been a function of the form $f(x) = x^2 + C$, for some constant C.

Question 2: If $f(x) = e^x$, and $\frac{d}{dx}[F(x)] = f(x)$, what is $F(x)$?
This is just the same type of thing expressed a bit differently. We're told that $\frac{d}{dx}[F(x)] = e^x$, and we know that $e^x = \frac{d}{dx}(e^x)$, so $F(x) = e^x$. Or maybe $F(x) = e^x - \ln 2$, or $F(x) = e^x + 2^{4/5}$, or ... But certainly $F(x) = e^x + C$ for some constant C.

In these questions, we used our knowledge that $2x$ is the derivative of x^2, and that e^x is the derivative of e^x, to realize that x^2 is an *antiderivative* of $2x$, and that e^x is an *antiderivative* of e^x. When we say *antiderivative*, we just mean a function whose derivative is whatever function we're talking about.

Definition 4.1. $F(x)$ is an **antiderivative** of $f(x)$ on some interval if $\frac{d}{dx}(F(x)) = f(x)$ for all x in that interval.

But of course $F(x)$ is not unique, because it's always true that there could have been a constant term that disappeared. That is, $\frac{d}{dx}(F(x) + C) = f(x)$ too, for any constant C. For instance, x^2, x^2+1, $x^2 - \frac{37\pi e}{62}$, ... are all antiderivatives of $2x$. We encapsulate this whole family of antiderivatives of $f(x)$ with $F(x) + C$, which we call the *indefinite integral of $f(x)$ with respect to x*, which is also sometimes referred to as the *general antiderivative of $f(x)$*.

Definition 4.2. If $F(x)$ is any antiderivative of $f(x)$, then the **indefinite integral of $f(x)$ with respect to x** is $F(x) + C$, where C is a nonspecific, i.e. arbitrary, constant called the **constant of integration**.

We usually use the antiderivative with constant term 0 as $F(x)$. For instance, rather than saying "well, $x^2 - \frac{37\pi e}{62}$ is an antiderivative of $2x$, so the indefinite integral is $x^2 - \frac{37\pi e}{62} + C$", we just say that the indefinite integral of $2x$ is $x^2 + C$. We have some notation to express "the indefinite integral of $f(x)$ with respect to x". We write $\displaystyle\int f(x)dx$.

Definition 4.3. $\displaystyle\int f(x)\,dx = F(x) + C$, where F is any antiderivative of f.

$\displaystyle\int$ is called an **integral sign**.

The function $f(x)$ is called the **integrand** function.

The process of finding $\int f(x)dx$ is referred to as **integration** or **antidifferentiation**.
dx is (for now) just a symbol to tell us that x is the **variable of integration**,
i.e. we're integrating or antidifferentiating *with respect to x*.

Examples: From what we determined earlier, in answering the 2 questions above, we see that $\int 2x dx = x^2 + C$ and $\int e^x dx = e^x + C$.

Note: In Unit 5 we'll learn why we call $\displaystyle\int f(x)\,dx$ and *indefinite* integral.

Also Note: You can **never** have an integral sign without a dx (or maybe a du or a dy or ...). Think of $\displaystyle\int$ and dx as *bookends* – they only come as a pair, one at each end of the integrand function.

Just as there were many differentiation rules to learn, we'll learn many antidifferentiation rules, or rules of integration. (The words antidifferentiation and integration are used somewhat interchangeably.) In fact, we've already learnt one of these rules. Remember that we previously stated as a differentiation rule that $\frac{d}{dx}(e^x) = e^x$. Now, we have a corresponding antidifferentiation rule:

Rule: $\displaystyle\int e^x dx = e^x + C.$

Notice: This new "antidifferentiation rule" isn't hard to remember. It follows directly from the corresponding "differentiation rule". And that will be the case with many of the new rules we'll be learning. Most of them follow directly from the differentiation rules that you already know. So when we say that we'll learn many rules of integration, that *doesn't* mean that you'll have a whole slew of complicated new rules to remember, in addition to the many rules you're already trying to remember. There will only be a few that are entirely new things to memorize.

Example 4.1. Find: (a) $\displaystyle\int x dx$ (b) $\displaystyle\int 3x^2 dx$ (c) $\displaystyle\int x^2 dx$

Solution:
(a) We know that $\int x dx = F(x) + C$ where $F(x)$ is an antiderivative of x, i.e. where $F'(x) = x$. Also, we know that x^2 is an antiderivative of $2x$, and we see that $x = \frac{1}{2}(2x)$, so it seems likely that $\frac{1}{2}(x^2)$ might be an antiderivative of $\frac{1}{2}(2x)$, i.e. of x. We can check whether this is right:

$$\frac{d}{dx}\left[\frac{1}{2}(x^2)\right] = \frac{1}{2}\left[\frac{d}{dx}(x^2)\right] = \frac{1}{2}(2x) = x$$

Indeed it's true that $\frac{1}{2}(x^2) = \frac{x^2}{2}$ is an antiderivative of x. So we get

$$\int x dx = \frac{x^2}{2} + C$$

(b) For $\int 3x^2 dx$ we recognize that $3x^2$ is the derivative of x^3, and so x^3 is an antiderivative of $3x^2$. Therefore

$$\int 3x^2 dx = x^3 + C$$

(c) We have $\int x^2 dx = F(x) + C$ where $F(x)$ is an antiderivative of x^2. Hmm. Since x^3 is an antiderivative of $3x^2$, then maybe $\frac{1}{3}(x^3)$ is an antiderivative of $\frac{1}{3}(3x^2) = x^2$. As before, since we're just guessing, we need to check whether this is right:

$$\frac{d}{dx}\left[\frac{1}{3}(x^3)\right] = \frac{1}{3}\left[\frac{d}{dx}(x^3)\right] = \frac{1}{3}(3x^2) = x^2$$

Again, it worked! Since $\frac{d}{dx}\left[\frac{1}{3}(x^3)\right] = x^2$ then $\frac{1}{3}(x^3) = \frac{x^3}{3}$ is an antiderivative of x^2, so we get:

$$\int x^2 dx = \frac{x^3}{3} + C$$

Notice what we found in parts (a) and (c) of that example:

$$\int x dx = \int x^1 dx = \frac{x^2}{2} + C = \frac{x^{1+1}}{1+1} + C$$

$$\int x^2 dx = \int x^2 dx = \frac{x^3}{3} + C = \frac{x^{2+1}}{2+1} + C$$

In general, $\frac{x^{n+1}}{n+1}$ is an antiderivative of x^n, since

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}\left[\frac{d}{dx}(x^{n+1})\right] = \frac{1}{n+1}\left[(n+1)x^n\right] = x^n$$

This isn't true for *all* values of n, though, because for $n = -1$, $n + 1 = 0$ so we have $\frac{x^0}{0}$, which is undefined. It's true for all real n other than 0, though. That's our next rule.

Power Rule for Integration: For any $n \neq -1$, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

Example 4.2. Find: (a) $\int \sqrt[3]{x} dx$ (b) $\int x^{-3} dx$ (c) $\int 1 dx$ (d) $\int x^{\pi} dx$

Solution:
(a) We know that $\sqrt[3]{x} = x^{1/3}$, so the integrand has the form x^n for some $n \neq -1$. We use the Power Rule.

$$\int \sqrt[3]{x} dx = \int x^{1/3} dx = \frac{x^{(1/3)+1}}{\frac{1}{3}+1} + C = \frac{x^{4/3}}{\frac{4}{3}} + C = \frac{3}{4}x^{4/3} + C = \frac{3x^{4/3}}{4} + C$$

(b) For $\int x^{-3} dx$ we recognize the form x^n with $n \neq -1$ and so again we use the Power Rule:

$$\int x^{-3} dx = \frac{x^{-3+1}}{-3+1} + C = \frac{x^{-2}}{-2} + C = \left(-\frac{1}{2}\right)\left(\frac{1}{x^2}\right) + C = -\frac{1}{2x^2} + C$$

(c) What do we have here? What is an antiderivative of 1? Well, maybe we just recognize it, but on the other hand, we can use the Power Rule, because $1 = x^0$. That is, we do have the form x^n for some $n \neq -1$. We get

$$\int 1 dx = \int x^0 dx = \frac{x^{0+1}}{0+1} + C = \frac{x^1}{1} + C = x + C$$

(or we could have just realized that 1 is the derivative of x, so x is an antiderivative of 1).

(d) For $\int x^{\pi} dx$, we again have the form x^n with $n \neq -1$ so we just use the Power Rule:

$$\int x^{\pi} dx = \frac{x^{\pi+1}}{\pi+1} + C$$

Okay, so now we know we can use the power rule for any integrand of the form x^n with $n \neq -1$. What if we do have $n = -1$? What then? That is, how do we find an antiderivative of x^{-1}? Do we know one? Do we know a function $F(x)$ for which $F'(x) = x^{-1}$? Well ... sure we do! Because $x^{-1} = \frac{1}{x}$, and we do know a function whose derivative is $\frac{1}{x}$. But we need to be a little careful here.

The integrand function $f(x) = \frac{1}{x}$ is defined for all $x \neq 0$. We want an antiderivative which is also defined on that same domain. We know that $\frac{d}{dx}(\ln x) = \frac{1}{x}$, but $\ln x$ is only defined for $x > 0$. So $\ln x$ is an antiderivative of $f(x) = \frac{1}{x}$, but only on *part* of the domain of f. However, we know that it's also true that $\frac{d}{dx}(\ln |x|) = \frac{1}{x}$, so $\ln |x|$ is also an antiderivative of f. And $\ln |x|$ is defined *for all* $x \neq 0$, i.e. has the same domain as f. Therefore *this* is the antiderivative we want, to express the indefinite integral of $\dfrac{1}{x}$. That is, we have

Rule: $\displaystyle\int x^{-1}dx = C + \ln |x|$

Note: We can write this as $C + \ln |x|$ instead of as $(\ln |x|) + C$ to make it absolutely clear, without brackets, that the integration constant is added to the natural logarithm, not to the thing we're taking the natural logarithm of.

We know that $\frac{d}{dx}(F(x) \pm G(x)) = F'(x) \pm G'(x)$. And this means that $F(x) \pm G(x)$ is an antiderivative of $F'(x) + G'(x)$. That is, because of the sum/difference rule for differentiation, we have a similar sum/difference rule for integration.

Likewise, we know that for any constant k, $\frac{d}{dx}(k(F(x)) = k(F'(x))$, so $k(F(x))$ is an antiderivative of $k(F'(x))$. Again, *because of* the constant multiplier rule for differentiation, a corresponding rule applies for antidifferentiation.

Rules:

$$\int (F'(x) \pm G'(x))dx = \int F'(x)dx \pm \int G'(x)dx = F(x) \pm G(x) + C$$

and $$\int kF'(x)dx = k\int F'(x)dx = kF(x) + C$$

Notice that:

(1) For $\int f(x)dx + \int g(x)dx$, although there are two integrals, we only need one "$+C$" – it incorporates both integration constants. That is, we could write $\int F'(x)dx + \int G'(x)dx = F(x) + C_1 + G(x) + C_2$, but then we could just let $C = C_1 + C_2$ to get the antiderivative as shown above. Because the sum of two arbitrary constants is just an arbitrary constant.

(2) Likewise, we don't need to worry about the fact that since the constant k is multiplying the whole integral, then it should be multiplying all of $F(x) + C$. That is, we should have written $k(F(x) + C)$, but we just wrote $kF(x) + C$, which is not the same thing. But the C is just an arbitrary constant. And an arbitrary constant times some other constant is just a different arbitrary constant. So there's no need to put the k multiplier on the integration constant. It gets absorbed into the constant, and for simplicity we call that new constant C.

(3) We already observed the constant multiplier rule when we found that

$$\int x^n\, dx = \left(\frac{1}{n+1}\right)\int (n+1)x^n dx = \frac{x^{n+1}}{n+1} + C$$

Example 4.3. Evaluate the following integrals:

(a) $\displaystyle\int (7x^3 - 3x^2 + x)dx$ (b) $\displaystyle\int \left(3\sqrt{x} + \frac{5}{x} + 2e^x\right) dx$

Solution:
(a) We have:

$$\begin{aligned}
\int (7x^3 - 3x^2 + x)dx &= \int 7x^3 dx - \int 3x^2 dx + \int x dx \quad \text{(sum/difference rule)} \\
&= 7\int x^3 dx - 3\int x^2 dx + \int x dx \quad \text{(constant multiplier rule)} \\
&= 7\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right) + \frac{x^2}{2} + C \quad \text{(power rule for each)} \\
&= \frac{7x^4}{4} - x^3 + \frac{x^2}{2} + C
\end{aligned}$$

Notice: In this case, for the middle term it would have been easier to leave the 3 multiplier inside the integral and just realize that x^3 is an antiderivative of $3x^2$.

(b) Similarily,

$$\begin{aligned}
\int \left(3\sqrt{x} + \frac{5}{x} + 2e^x\right) dx &= \int 3\sqrt{x}dx + \int \frac{5}{x}dx + \int 2e^x dx = 3\int \sqrt{x}dx + 5\int \frac{1}{x}dx + 2\int e^x dx \\
&= 3\int x^{1/2}dx + 5\int x^{-1}dx + 2e^x = 3\left(\frac{2}{3}x^{3/2}\right) + 5\ln|x| + 2e^x + C \\
&= 2x^{3/2} + 5\ln|x| + 2e^x + C
\end{aligned}$$

Note: We don't write the $+C$ until we do the last integration. That is, it's only as the last integral symbol goes away that we need to add the integration constant.

Example 4.4. Find $\int \left(x^2 - 2\cos x\right)\, dx$.

Solution: Using the difference rule and constant multiplier rule we get

$$\int \left(x^2 - 2\cos x\right)\, dx = \int x^2\, dx - 2\int \cos x\, dx$$

We have a rule for finding an antiderivative of x^2, but what about $\cos x$? Well, do we recognize that as the derivative of some function? Of course we do! We know that $\frac{d}{dx}(\sin x) = \cos x$, and that means that $\sin x$ is an antiderivative of $\cos x$. So we get

$$\int \left(x^2 - 2\cos x\right)\, dx = \int x^2\, dx - 2\int \cos x\, dx = \frac{x^3}{3} - 2\sin x + C$$

We have already seen that the fact that we know the derivative of a particular function means that we know an antiderivative of that derivative function, which give us new "rules" for integration. (For instance, knowing that $\frac{d}{dx}(e^x) = e^x$ gave us the "rule" that $\int e^x\, dx = e^x + C$.) Similarly, since we know the derivatives of the trig functions, we simply need to recognize that these derivatives also tell us antiderivatives of certain trigonometric functions, i.e. give us new rules of integration. Like the one we just used in the last example. We can summarize these rules in a table, shown on the next page.

Antiderivatives of Trigonometric Functions

Derivative		General Antiderivative
$\frac{d}{dt}(\sin t) = \cos t$	$\Rightarrow$	$\int \cos t \; dt = C + \sin t$
$\frac{d}{dt}(\cos t) = -\sin t$	$\Rightarrow$	$\int \sin t \; dt = C - \cos t$
$\frac{d}{dt}(\tan t) = \sec^2 t$	$\Rightarrow$	$\int \sec^2 t \; dt = C + \tan t$
$\frac{d}{dt}(\csc t) = -\csc t \cot t$	$\Rightarrow$	$\int \csc t \cot t \; dt = C - \csc t$
$\frac{d}{dt}(\sec t) = \sec t \tan t$	$\Rightarrow$	$\int \sec t \tan t \; dt = C + \sec t$
$\frac{d}{dt}(\cot t) = -\csc^2 t$	$\Rightarrow$	$\int \csc^2 t \; dt = C - \cot t$

Note: For the ones with the negatives, we're using the constant multiplier rule. For instance,

$$\int \sin t \; dt = -\int (-\sin t) \; dt = -\cos t + C$$

Example 4.5. Find $\int (2\cos x - 3\sin x) \; dx$.

Solution:

$$\int (2\cos x - 3\sin x) \; dx = 2\int \cos x \; dx - 3\int \sin x \; dx = 2(\sin x) - 3(-\cos x) + C = C + 2\sin x + 3\cos x$$

Example 4.6. Find $\int \frac{\sec x + \tan x}{\cos x} \; dx$.

Solution:
Here, we don't recognize the integrand as a derivative we know – it's not in the table. Be we can restate the integrand function. We try to get it to a form we *do* recognize.

$$\begin{aligned}
\int \frac{\sec x + \tan x}{\cos x} \; dx &= \int \left(\frac{1}{\cos x}\right)(\sec x + \tan x) \; dx \\
&= \int (\sec x)(\sec x + \tan x) \; dx \\
&= \int (\sec^2 x + \sec x \tan x) \; dx \\
&= \int \sec^2 x \; dx + \int \sec x \tan x \; dx \\
&= \tan x + \sec x + C
\end{aligned}$$

Indefinite Integrals with Side Conditions
(also called Initial Value Problems)

Sometimes, we know some additional information which allows us to determine which one of the family of antiderivatives is the particular antiderivative function we need. That is, we know some *side condition* or *initial value* which must be satisfied, which allows us to find the specific value for the constant term of the antiderivative.

Example 4.7. If $\frac{d}{dx}(F(x)) = e^x$ and $F(0) = 2$, find $F(x)$.

Solution:
$\frac{d}{dx}(F(x)) = e^x$ tells us that $F(x) = \int e^x dx$, so $F(x) = e^x + C$. But this function must satisfy the condition that $F(0) = 2$. We have

$$F(0) = e^0 + C = 1 + C$$

and so we need $1 + C = 2$, which gives $C = 2 - 1 = 1$. Therefore the specific antiderivative function in this case is $F(x) = e^x + 1$.

Example 4.8. Find the antiderivative of $f(x) = 3x^2$ for which the graph of $y = f(x)$ passes through the point (1,4).

Solution: The fact that our solution must be a function whose graph passes through the point (1,4) is a side condition. We must find an antiderivative of $f(x) = 3x^2$, say $F(x)$, which has $F(1) = 4$. First, we find the general antiderivative, then we use the side condition to find the value of C. We see that $F(x) = \int 3x^2 dx = x^3 + C$. And now, using the side condition, we get: $F(1) = 4 \Rightarrow (1)^3 + C = 4 \quad \Rightarrow C = 4 - 1 = 3$, so we see that $F(x) = x^3 + 3$ is the required function.

Example 4.9. It is known that the slope of the function $f(x)$ is given by $\frac{(x+1)^2}{x}$, and that $f(1) = 2$. What is the function $f(x)$?

Solution: We know that the slope of a function f is given by its derivative, f'. Thus, we are told that $f'(x) = \dfrac{(x+1)^2}{x}$, so that $f(x) = \displaystyle\int f'(x)dx = \int \frac{(x+1)^2}{x}dx$.

To find $\int \frac{(x+1)^2}{x}dx$, we need to restate the integrand:

$$\frac{(x+1)^2}{x} = \frac{x^2+2x+1}{x} = \frac{x^2}{x} + \frac{2x}{x} + \frac{1}{x} = x + 2 + \frac{1}{x}$$

We see that

$$f(x) = \int \frac{(x+1)^2}{x}dx = \int \left(x + 2 + \frac{1}{x}\right) dx = \frac{x^2}{2} + 2x + \ln|x| + C$$

We find the value of C using the side condition that $f(1) = 2$. We have:

$$f(1) = \frac{1^2}{2} + 2(1) + \ln 1 + C = \frac{1}{2} + 2 + 0 + C = \frac{5}{2} + C$$

Since $f(1) = 2$, then we have $\frac{5}{2} + C = 2$, so $C = 2 - \frac{5}{2} = -\frac{1}{2}$. Therefore the required function is $f(x) = \frac{x^2}{2} + 2x + \ln|x| - \frac{1}{2}$.

Example 4.10. If $f'(x) = \sin x$ and it is known that the graph of $y = f(x)$ passes through the origin, find the function f.

Solution: The fact that $f'(x) = \sin x$ tells us that

$$f(x) = \int \sin x \, dx = C - \cos x$$

And since the graph of $y = f(x)$ passes through the origin, then we must have $y = 0$ when $x = 0$, i.e. $f(0) = 0$. We see that $f(0) = C - \cos 0 = C - 1$ so we need $C - 1 = 0$, i.e. $C = 1$. Therefore $f(x) = 1 - \cos x$.

The Substitution Rule

We have learnt rules for finding antiderivatives of some basic functions (e.g. x^n, e^x and $\cos x$), and also some rules for finding antiderivatives of more complicated functions (e.g. functions involving constant multiples and/or sums/differences of basic functions). But of course a function which requires the Chain Rule to differentiate produces a derivative function which is more complicated than any of those. So we need an approach for "undoing" the Chain Rule. That rule is called the *Substitution Rule.*

Recall that $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$. For instance, $\frac{d}{dx}\left(e^{x^2+3x}\right) = e^{x^2+3x}(2x+3)$. But then e^{x^2+3x} is an antiderivative of $e^{x^2+3x}(2x+3)$, so we can see that

$$\int e^{x^2+3x}(2x+3)dx = e^{x^2+3x} + C$$

In fact, since we know that, for any function f, we have $\frac{d}{dx}(e^{f(x)}) = e^{f(x)}f'(x)$, then we recognize that $\int e^{f(x)}f'(x)dx = e^{f(x)} + C$. That is, letting $u = f(x)$, we have $\frac{du}{dx} = f'(x)$. And treating $\frac{du}{dx}$ as *a ratio of* **differentials**, we can rearrange $\frac{du}{dx} = f'(x)$ to $du = f'(x)dx$, to get

$$\int \underbrace{e^{f(x)}}_{e^u}\underbrace{f'(x)dx}_{du} = \int e^u du = e^u + C = e^{f(x)} + C$$

That is, when we let $u = f(x)$, we recognize $e^{f(x)}$ as e^u, and we realize (from above) that $f'(x)dx = du$, so we make both those substitutions to get an easy integral. After that, we do the integration, and then substitute back in the answer. (But the du has gone away when we integrated, so it's only u that we need to substitute back for.) i.e. we replace u by $f(x)$, because $u = f(x)$ was where we started.

In general, if the integrand function is a derivative obtained using the Chain Rule, then the integral has the form

$$\int f(g(x))g'(x)dx$$

To convert this into an integral we easily recognize how to deal with, we perform a **substitution**. We observe that $g(x)$ is the "inner function" of some composite function, so we let $u = g(x)$. This gives $\frac{du}{dx} = g'(x)$, so that $du = g'(x)dx$. Then we replace $g(x)$ by u, and also $g'(x)dx$ by du, to get

$$\int f(g(x))g'(x)dx = \int f(u)du$$

and hopefully this is an integral we know.

Example 4.11. Find the following indefinite integrals.

(a) $\displaystyle\int 2xe^{(x^2)}dx$ (b) $\displaystyle\int (x^4-2x)^{10}(4x^3-2x)dx$ (c) $\displaystyle\int \frac{2x+5}{(x^2+5x-9)^2}dx$

(d) $\displaystyle\int \frac{2x+5}{x^2+5-9}dx$ (e) $\displaystyle\int 3x^2\sin x^3\,dx$

Solution:
(a) For $\int 2xe^{(x^2)}dx$ we see $e^{(x^2)}$, which can be considered as e^u where $u=x^2$, so we think this substitution may work. For $u=x^2$ we have $\frac{du}{dx}=2x$ so that $du=2xdx$. Since $2x$ and dx *do* appear, it seems this substitution is working. We have

$$\int 2xe^{(x^2)}dx=\int \underbrace{e^{(x^2)}}_{e^u}\,\underbrace{2xdx}_{du}=\int e^u du=e^u+C=e^{(x^2)}+C$$

(Don't forget to substitute back. The answer should have x's in it, not u's.)

(b) In $\displaystyle\int (x^4-2x)^{10}(4x^3-2)dx$, we see the form u^n, where $u=x^4-2x$. (That is, we have (*something*)^n for some value of n, and we recognize this as u^n where u is *the "something" in the brackets.*) Maybe this substitution (i.e. $u=x^4-2x$) will transform the integral into something easy. If $u=x^4-2x$, then $\frac{du}{dx}=4x^3-2$. But that *is* the second term in the integrand – great! From this relationship we get $du=(4x^3-2)dx$, and we carry out the substitution(s).

$$\int \underbrace{(\underbrace{x^4-2x}_{u})^{10}}_{u^{10}}\underbrace{(4x^3-2)dx}_{du}=\int u^{10}du=\frac{u^{11}}{11}+C=\frac{(x^4-2x)^{11}}{11}+C$$

(c) For $\int \frac{2x+5}{(x^2+5x-9)^2}dx$, we once again see the form u^n, this time for $u=x^2+5x-9$. Perhaps this substitution would work. We get $\frac{du}{dx}=2x+5$, which we *do* see in the integrand as well, so we'll try substituting $u=x^2+5x-9$ and $du=(2x+5)dx$.

$$\begin{aligned}\int \frac{2x+5}{(x^2+5x-9)^2}dx &= \int \frac{1}{(\underbrace{x^2+5x-9}_{u})^2}\underbrace{(2x+5)dx}_{du}=\int \frac{1}{u^2}du=\int u^{-2}du\\ &= \frac{u^{-2+1}}{-2+1}+C=\frac{u^{-1}}{-1}+C=-u^{-1}+C\\ &= -\frac{1}{u}+C=-\frac{1}{x^2+5x-9}+C=C-\frac{1}{x^2+5x-9}\end{aligned}$$

(d) For $\displaystyle\int \frac{2x+5}{x^2+5x-9}dx$, which looks much the same, it's a bit different. We don't see (*something*)^n ... but we do see $\frac{1}{u}$, for $u=x^2+5x-9$ (i.e. we have (*something*)$^{-1}$, but it doesn't look like that). Perhaps what we have is something of the form $\frac{u'}{u}$, which we know is $\frac{d}{dx}(\ln|u|)$. If $u=x^2+5x-9$, then $du=(2x+5)dx$ (as in part (c)), and $2x+5$ *is* what we see in the numerator! We get

$$\int \frac{2x+5}{x^2+5x-9}dx=\int \underbrace{\left(\frac{1}{x^2+5x-9}\right)}_{\frac{1}{u}}\underbrace{(2x+5)dx}_{du}=\int \frac{1}{u}du=C+\ln|u|=C+\ln|x^2+5x-9|$$

(e) For $\int 3x^2 \sin x^3 \, dx$ we see $\sin u$ in the integrand, where $u = x^3$. This gives $du = 3x^2 \, dx$, which is also in the integrand, so we get

$$\int 3x^2 \sin x^3 \, dx = \int \underbrace{(\sin x^3)}_{\sin u}\underbrace{(3x^2)dx}_{du} = \int \sin u \, du = C - \cos u = C - \cos x^3$$

Next, consider something like $\int xe^{x^2+3}dx$. Of course we can write this as $\int e^{x^2+3}(x)dx$. We see e^{x^2+3}, so we think that the substitution $u = x^2 + 3$ will work. That gives $\frac{du}{dx} = 2x$, so that $du = 2xdx$. But we don't have $2xdx$ anywhere after the integral symbol. Just xdx. But wait! If $du = 2xdx$, then $xdx = \frac{du}{2} = \left(\frac{1}{2}\right) du$. Aha! We can substitute e^u for e^{x^2+3} and $\left(\frac{1}{2}\right) du$ for xdx, and maybe that will give us something easy.

$$\int xe^{x^2+3}dx = \int \underbrace{e^{x^2+3}}_{e^u}\underbrace{xdx}_{\frac{1}{2}du} = \int e^u \left(\frac{1}{2}\right) du = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{e^{x^2+3}}{2} + C$$

In general, instead of having $g'(x)$ appearing in the integrand we may have (only) some constant multiple of $g'(x)$, i.e. $kg'(x)$. And often the k multiplier is invisible, so that the problem is that there's a constant *missing* from the integrand, as with x instead of $2x$. But for $\int f(g(x))(kg'(x))dx$, we can substitute $u = g(x)$, which gives $du = g'(x)dx$ so that $kg'(x)dx = kdu$. We get

$$\int f(g(x))(kg'(x))dx = \int (f(u))(k)dx = k\int f(u)du$$

Note: It must be only a **constant** that's missing, or we can't do this.

The Substitution Rule: To find $\int f(g(x))(kg'(x))dx$, substitute $u = g(x)$, so that $du = g'(x)dx$ gives $kg'(x)dx = kdu$ to get

$$\int f(g(x))(kg'(x))dx = \int kf(u)du = k\int f(u)du$$

Example 4.12. Find $\int e^{kx}dx$.

Solution:
We have the form e^u where $u = kx$ so that $du = kdx$ and $dx = \frac{1}{k}du$. So we get

$$\int e^{kx}dx = \int e^u \left(\frac{1}{k}\right) du = \frac{1}{k}\int e^u du = \frac{1}{k}(e^u) + C = \frac{e^u}{k} + C = \frac{e^{kx}}{k} + C$$

Notice: That's a general result that's worth remembering.

Useful Rule: $\frac{e^{kx}}{k}$ is an antiderivative of e^{kx}.

Whenever you need to use, or think you need to use, the Substitution Rule, you should think carefully about what you're doing and take a very methodical approach. We can describe a procedure that breaks it down into steps that you should follow.

Procedure for using the Substitution Rule to find $\int h(x)dx$:

Step 1: Identify the product in the integrand. i.e. express $h(x)$ as $a(x)b(x)$ for some functions a and b. You may need to re-write the integrand in order to do this.
That is, often if you're thinking the substitution rule will work, it's because you have a product in the integrand. But sometimes the product may be hidden – disguised as a quotient, or with an invisible 1 multiplier, or hidden even more subtly (see for instance Examples 4.17 and 4.19).

Step 2: Identify one term of the product as a composite function. That is, try to find $g(x)$ such that either $a(x)$ or $b(x)$ is equal to $f\left(g(x)\right)$ for some functions f and g for which f is easy to integrate. Call the term that's a composite function $a(x)$. (That is, if that's the term you were going to call $b(x)$, switch the names and call the *other* one $b(x)$.)

Step 3: Let $u = g(x)$ (the $g(x)$ found in Step 2) and find $\frac{du}{dx} = g'(x)$. Does $b(x) = kg'(x)$, for some constant k? If so, let $b(x)dx = kdu$. (In the simpler cases, k may just be 1.)

Step 4: At this point we have

$$\int h(x)dx = \int a(x)b(x)dx = \int f(g(x))b(x)dx = \int (f(u))(k)du = k\int f(u)du$$

Step 5: Integrate $k\int f(u)du$ and re-express in terms of x. (That is, substitute back in $g(x)$ for u.)

Example 4.13. Find $\displaystyle\int (x^5-3)^4x^4dx$.

Solution: First, we look at the integral and see that we don't recognize it as something straightforward. And it can't easily be restated as something straightforward. This suggests that we probably need to use the substitution rule.

Step 1: The product is obvious here. $(x^5-3)^4x^4 = a(x)b(x)$ where $a(x) = (x^5-3)^4$ and $b(x) = x^4$.

Step 2: The first term of the product, $a(x) = (x^5-3)^4$, is a function of (x^5-3). That is, we have $a(x) = f(g(x))$ where $g(x) = x^5 - 3$ and the f function takes the input and raises it to the fourth power. So $a(x) = (g(x))^4$, where $g(x) = x^5 - 3$.

Step 3: We let $u = x^5 - 3$. This gives $\frac{du}{dx} = 5x^4$. The second term of the product, $b(x) = x^4$, is not exactly equal to $\frac{du}{dx}$. However, it *is* a constant multiple of $\frac{du}{dx}$. That is, $x^4 = \frac{1}{5}(5x^4) = \left(\frac{1}{5}\right)\frac{du}{dx}$. The missing constant was 5, so the constant multiplier we need is $k = \frac{1}{5}$.

We let $\left(\frac{1}{5}\right)du = x^4dx$. That is, we have $\frac{du}{dx} = 5x^4$, so $du = 5(x^4dx)$ and $b(x)dx = x^4dx = \left(\frac{1}{5}\right)du$.

Step 4: So far we have

$$\int (x^5-3)^4x^4dx = \int (\underbrace{x^5-x}_{u})^4\underbrace{x^4dx}_{\frac{1}{5}du} = \int (u^4)\left(\frac{1}{5}\right)du = \frac{1}{5}\int u^4du$$

Step 5: We carry on and do the integration, then substitute back in $x^5 - 3$ for u:

$$\frac{1}{5}\int u^4du = \frac{1}{5}\left(\frac{u^5}{5}\right) + C = \frac{u^5}{25} + C = \frac{(x^5-3)^5}{25} + C$$

If you get stuck while using the procedure, there are 3 possibilities:

1. Something a bit more complicated is going on. We'll look at some examples of that after we've done a few more examples that work as fairly direct substitutions. (See Examples 4.20 through 4.22.)

2. You've chosen the wrong substitution. Try going back to Step 2 and look for a different choice of $g(x)$ to substitute for.

3. The substitution rule is the wrong approach.
   If $h(x)$ (i.e. the integrand) can be expanded or otherwise restated, maybe the integral can be found by more direct means. For instance, for $\int \frac{(x+3)^2}{x}dx$, expand the numerator, then divide each term of the numerator by x.
   If not, maybe an approach we haven't learnt yet is needed. (But not if it's a question in the homework for this unit.)

Also, there are certain forms of integrand that you should be able to easily identify as situations in which the substitution rule is needed, and for which you should also be able to easily identify the substitution you need. These are summarized in the following table:

Basic Form	Substitution	Resulting Form
$\int\ [g(x)]^n g'(x)dx$	Let $u = g(x)$	$\int u^n du$
$\int \frac{g'(x)}{(g(x))^n}dx$ with $n \neq 1$	Let $u = g(x)$	$\int u^{-n} du$
$\int\ \frac{g'(x)}{g(x)}dx$	Let $u = g(x)$	$\int \frac{1}{u}du$
$\int\ g'(x)e^{g(x)}dx$	Let $u = g(x)$	$\int e^u du$
$\int\ g'(x)\sin(g(x))\,dx$	Let $u = g(x)$	$\int \sin u\ du$
(and similarly for other trig functions)		

We have actually already seen at least one example of each of these. For instance, in Example 4.13 we had the first form shown in the table. Let's look at some more examples.

Example 4.14. Find $\displaystyle\int \frac{3x^2+2x}{2x^3+2x^2+7}dx$.

Solution:
Clearly, this is not an integral we can find without the substitution rule. That is, we don't immediately recognize the integrand as the derivative of some function, or as fitting any of our straightforward rules of integration. Nor do we see any way to simplify the integrand to an easier form.

Step 1: This time, the product is disguised as a quotient. However, we can rewrite the quotient as

$$\frac{3x^2+2x}{2x^3+2x^2+7} = \left(\frac{1}{2x^3+2x^2+7}\right)(3x^2+2x)$$

That is, we have $a(x) = \frac{1}{2x^3+2x^2+7}$ and $b(x) = 3x^2 + 2x$.

Step 2: $a(x) = \dfrac{1}{2x^3+2x^2+7}$ is a composite function, with the form $\dfrac{1}{g(x)}$. That is, we have $a(x) = \dfrac{1}{u}$, where $u = 2x^3+2x^2+7$.

Note: From the table, the form we have is the third one shown, so we let u equal the denominator and try to get $\frac{u'}{u}$ as the integrand.

Step 3: We try this substitution. For $u = 2x^3+2x^2+7$, we have $\frac{du}{dx} = 6x^2+4x$. We don't have $b(x) = \frac{du}{dx}$. However, $6x^2+4x = 2(3x^2+2x)$, so we do have $2b(x) = \frac{du}{dx}$, i.e., $b(x) = \left(\frac{1}{2}\right)\frac{du}{dx}$. Therefore $\left(\frac{1}{2}\right)du = (3x^2+2x)dx$.

Step 4: We have

$$\int \frac{3x^2+2x}{2x^3+2x^2+7}dx = \int \left(\frac{1}{u}\right)\left(\frac{1}{2}\right)du = \frac{1}{2}\int \frac{1}{u}du$$

Step 5: And now we can easily find the answer.

$$\frac{1}{2}\int \frac{1}{u}du = C + \frac{1}{2}\ln|u| = C + \frac{1}{2}\ln|2x^3+2x^2+7| = C + \ln\sqrt{|2x^3+2x^2+7|}$$

Example 4.15. Find $\displaystyle\int 3x^2e^{(x^3)}dx$.

Solution:
Again, this is an integral we don't recognize, so we need the substitution rule.

Step 1: We have an obvious product here, $b(x) = 3x^2$ and $a(x) = e^{(x^3)}$.

Step 2: $a(x) = e^{(x^3)}$ is a function of x^3. That is, $a(x) = e^u$, where $u = x^3$.

Step 3: Using $u = x^3$, we get $\frac{du}{dx} = 3x^2$. This time $b(x) = \frac{du}{dx}$, (i.e. we don't have to worry about a constant multiplier) so we use $du = 3x^2dx$.

Steps 4 & 5: We have:

$$\int 3x^2e^{(x^3)}dx = \int e^u du = e^u + C = e^{(x^3)} + C$$

Example 4.16. Find $\displaystyle\int \frac{4-6\sqrt{x}}{\left(x-\sqrt{x^3}\right)^{10}}dx$.

Solution:
Clearly we will need the substitution rule. Now that we have been through it a few times, we don't need to explicitly write out all the steps. We have

$$\int \frac{4-6\sqrt{x}}{\left(x-\sqrt{x^3}\right)^{10}}dx = \int \left(\frac{1}{\left(x-\sqrt{x^3}\right)^{10}}\right)(4-6\sqrt{x})dx$$

Looking at the first term in the product we see that the problem is similar to the second form shown in the table, although perhaps not exactly. We try letting $u = x-\sqrt{x^3} = x - x^{\frac{3}{2}}$. This gives $\frac{du}{dx} = 1-\frac{3}{2}x^{\frac{1}{2}} = 1-\frac{3}{2}\sqrt{x}$. But the numerator is $4-6\sqrt{x}$. Is this a constant multiple of $1-\frac{3}{2}\sqrt{x}$?

From the first term, the multiplier would have to be 4. And $4 \times \frac{3}{2} = 2 \times 3 = 6$, so we see that $4 - 6\sqrt{x} = 4(1 - \frac{3}{2}\sqrt{x}) = 4\frac{du}{dx}$. Therefore $(4 - 6\sqrt{x})dx = 4du$.

We get:

$$\begin{aligned}\int \frac{4 - 6\sqrt{x}}{\left(x - \sqrt{x^3}\right)^{10}} dx &= \int \frac{1}{u^{10}}(4)du = 4\int u^{-10}du \\ &= 4\left(\frac{u^{-9}}{-9}\right) + C = C - \frac{4}{9}\left(x - \sqrt{x^3}\right)^{-9} \\ &= C - \frac{4}{9\left(x - \sqrt{x^3}\right)^9}\end{aligned}$$

Example 4.17. Find $\int \tan x \, dx$.

Solution:
Notice that we don't know a trig function whose derivative is $\tan x$, so we don't already know an antiderivative of $\tan x$. And we don't see a product – or do we? In fact, $\tan x$ disguises a quotient and we have already seen that a quotient is a product in disguise. That is, we have $\tan x = \frac{\sin x}{\cos x}$, which can be written as a product as $\left(\frac{1}{\cos x}\right)(\sin x)$. So it's quite possible that a substitution will work.

In fact, we have the form $\frac{u'}{u}$ (with a missing constant). That is, with $\cos x$ in the denominator, our guess at a substitution would be $u = \cos x$, which gives $du = -\sin x \, dx$, so that $\sin x \, dx = -du = (-1)du$. Therefore we get

$$\begin{aligned}\int \tan x &= \int \frac{\sin x}{\cos x} \, dx = \int \left(\frac{1}{u}\right)(-1) \, du = -\int \frac{1}{u} \, du \\ &= C - \ln|u| = C - \ln|\cos x|\end{aligned}$$

However, we can "simplify" this to get rid of the negative:

$$-\ln|\cos x| = \ln\left(|\cos x|^{-1}\right) = \ln\left|(\cos x)^{-1}\right| = \ln\left|\frac{1}{\cos x}\right| = \ln|\sec x|$$

Therefore we can express the answer as

$$\int \tan x \, dx = C + \ln|\sec x|$$

Notice: The fact that $\ln|\sec x|$ is an antiderivative of $\tan x$ (and is how the antiderivative of $\tan x$ is usually expressed) is another one that's probably worth remembering.

Example 4.18. Find $\int te^{(t^2)} \cos e^{(t^2)} \sin\left(\sin e^{(t^2)}\right) dt$.

Solution:
This is pretty ugly, but maybe a substitution will work. The integrand function is a product of several terms, of which the most complicated is $\sin\left(\sin e^{(t^2)}\right)$, which is a trig function of a trig function of a function of t. Let's try $u = \sin e^{(t^2)}$, so that this most complicated term is simply $\sin u$.

We get:

$$\frac{du}{dt} = \frac{d}{dt}\left[\sin e^{(t^2)}\right] = \left[\cos e^{(t^2)}\right]\left[\frac{d}{dt}\left(e^{(t^2)}\right)\right] = \left[\cos e^{(t^2)}\right]\left(e^{(t^2)}\right)\left[\frac{d}{dt}\left(t^2\right)\right] = \left[\cos e^{(t^2)}\right]\left(e^{(t^2)}\right)(2t)$$

We have $du = 2te^{(t^2)} \cos e^{(t^2)} dt$, so that $te^{(t^2)} \cos e^{(t^2)} dt = \frac{1}{2} du$.
Thus we see that:

$$\int te^{(t^2)} \cos e^{(t^2)} \sin\left(\sin e^{(t^2)}\right) dt = \frac{1}{2}\int \sin u \, du = \frac{1}{2}(-\cos u) + C = C - \frac{\cos\left(\sin e^{(t^2)}\right)}{2}$$

Notice: You may find a question like this easier using multiple substitutions. For instance, your first instinct might be to let $u = e^{(t^2)}$, since that occurs in various places in the integrand. That's fine. But then you'll also need the substitution $v = \sin u$. Or perhaps you want to start with $u = t^2$. Again that works, but then you need to also substitute $v = e^u$ and then $w = \sin v$.

Sometimes we think we should use a substitution, but we just can't see a product in the integrand. Is one term in the product an invisible 1 multiplier? Or trickier yet, is the product disguised because the terms look too much alike? Consider the next example.

Example 4.19. Find $\displaystyle\int (e^{2x})^3 dx$.

Solution:
We see the form u^n for $u = e^{2x}$, so it seems like a substitution – specifically, that substitution – should be useful. However, we don't see a product. Let's see what we should have.

For $u = e^{2x}$, we need the Chain Rule to get $\frac{du}{dx} = e^{2x}\left[\frac{d}{dx}(2x)\right] = e^{2x}(2) = 2e^{2x}$. (Or, we remember that $\frac{d}{dx}(e^{kx}) = ke^{kx}$.) This gives $du = 2e^{2x}dx$, and (since we certainly don't see a 2 multiplier) $e^{2x}dx = \frac{1}{2}du$. Now, where's that e^{2x} that's supposed to be a term of the product hiding?

Oh! The integrand is $(e^{2x})^3$. We've got lots of e^{2x}'s. Three of them multiplied together. We just need to factor one out! That is, $(e^{2x})^3 = (e^{2x})^2(e^{2x})$. If we write the integrand that way, we've got the product we need. We get:

$$\begin{aligned}\int (e^{2x})^3 dx &= \int \underbrace{(e^{2x})^2}_{u^2} \underbrace{(e^{2x}) dx}_{\frac{1}{2}du} = \int u^2 \left(\frac{1}{2}\right) du \\ &= \frac{1}{2}\int u^2 du = \left(\frac{1}{2}\right)\left(\frac{u^3}{3}\right) + C = \frac{u^3}{6} + C = \frac{\left(e^{2x}\right)^3}{6} + C\end{aligned}$$

Actually, for this particular problem, there's an easier approach we could have taken, for which we wouldn't even need the substitution rule at all, if we remember the Useful Rule we found earlier (from doing Example 4.12). We start by using properties of exponents to simplify the integrand. We have a base raised to a power, and then raised to another power, so we multiply the powers. That is, $(e^{2x})^3 = e^{(2x)(3)} = e^{6x}$. And the useful rule says that $\int e^{kx} dx = \frac{e^{kx}}{k} + C$. We get

$$\int (e^{2x})^3 dx = \int e^{6x} dx = \frac{e^{6x}}{6} + C$$

And of course since $(e^{2x})^3 = e^{6x}$, this is the same answer we got before.

Sometimes it is not so obvious how to proceed with a problem. It may appear that we have made an incorrect substitution when we have not. Consider the next example.

Example 4.20. Find $\int (x+2)^{50} x dx$.

Solution: We have $(x+2)^{50}$ appearing in the integrand, which suggests $u = x+2$ as an appropriate substitution. But then $\frac{du}{dx} = 1$, and letting $du = 1dx$, we have an extra x term left over. It would appear that this substitution doesn't work.
However, consider ... Since $u = x+2$, then $x = u-2$. Making this substitution for x, as well as the substitutions for $x+2$ and dx, allows us to perform the integration.
That is,

$$\int (x+2)^{50} x dx = \int (x)(x+2)^{50} dx = \int (u-2)u^{50} du = \int (u^{51} - 2u^{50}) du$$

$$= \frac{u^{52}}{52} - 2\left(\frac{u^{51}}{51}\right) + C = \frac{(x+2)^{52}}{52} - \frac{2(x+2)^{51}}{51} + C$$

That is, we had an *extra* term in the product, and that term was a function of x which could be expressed directly in terms of u, so we were able to express the entire integrand function in terms of u and perform the integration.

Example 4.21. Find $\int \frac{x^2+3x-2}{x+1} dx$.

Solution: This integral has none of the forms that we recognize, and also is not in any of our standard forms for using the substitution rule. We need to try to restate the integrand function in some sort of form we know how to tackle.

Using long division to divide $x+1$ into x^2+3x-2, we find that it goes in $x+2$ times with a remainder of -4. That is, we find that $x^2+3x-2 = (x+1)(x+2)+(-4)$. This allows us to restate the integrand function and then carry through the integration, as follows:

$$\int \frac{x^2+3x-2}{x+1} dx = \int \frac{(x+1)(x+2)-4}{x+1} dx = \int \left(\frac{(x+1)(x+2)}{x+1} - \frac{4}{x+1}\right) dx$$

So then

$$\int \frac{x^2+3x-2}{x+1} dx = \int \left(x+2-\frac{4}{x+1}\right) dx = \int (x+2) dx - 4\int \frac{1}{x+1} dx = \left(\frac{x^2}{2}+2x\right) - 4\int \frac{1}{u} du$$

where $u = x+1$ so that $du = dx$, and we get

$$\int \frac{x^2+3x-2}{x+1} dx = \frac{x^2}{2} + 2x - 4\ln|x+1| + C$$

Notice: For any constant k, to find $\int \frac{1}{x+k}\, dx$ we let $u = x+k$, so that $du = dx$. This gives

$$\int \frac{1}{x+k}\, dx = \int \frac{1}{u}\, du = C + \ln|u| = C + \ln|x+k|$$

It's worth remembering that $\ln|x+k|$ is an antiderivative of $\frac{1}{x+k}$ so that we don't need to carry out the substitution every time. (We'll use this antiderivative a lot for one of the integration techniques that we'll learn in Unit 8.)

Let's look at one last example.

Example 4.22. Find $\displaystyle\int \frac{x+3}{(x-2)^3}dx$.

Solution: We see $(x-2)^3$ in the denominator of the integrand, which suggests $u = x-2$ as a substitution. This gives $du = dx$. However, we once again have an extra term in the integrand. That is, we would expect the other term in the product to be just $\frac{du}{dx} = 1$, and instead we have $x+3$. But we can express $x+3$ in terms of u. Since $u = x-2$, then $x = u+2$, so $x+3 = (u+2)+3 = u+5$. So substituting $u = x-2$, $du = dx$ and $x+3 = u+5$ we get:

$$\begin{aligned}
\int \frac{x+3}{(x-2)^3}dx &= \int \frac{u+5}{u^3}du = \int\left(\frac{u}{u^3}+\frac{5}{u^3}\right)du = \int(u^{-2}+5u^{-3})du \\
&= \frac{u^{-1}}{-1}+\frac{5u^{-2}}{-2}+C = -\frac{1}{u}-\frac{5}{2u^2}+C = C-\frac{1}{x-2}-\frac{5}{2(x-2)^2}
\end{aligned}$$

Math 1225A/B

Unit 5:
The Definite Integral

(text reference: Sections 8.4 and 8.5

custom text pgs. 124 - 142)

5 The Definite Integral

Next, we're going to learn about some new notation. Recall that $\int f(x)dx$ is the general antiderivative of f, and that we said this is often referred to as an indefinite integral. That's to distinguish it from, but at the same time recognize the relationship to, a different idea which is called a *definite integral.* But before we introduce that concept, we need to review another concept that you learnt about in your Introductory Calculus course, but that we haven't talked about yet in this course. The idea of a function being *continuous.*

Recall the definition of continuity of a function at a point:

Definition 5.1. A function $f(x)$ is said to be **continuous** at a value c in the domain of f if and only if

$$\lim_{x\to c} f(x) = f(c)$$

That seemingly simple statement packs a lot of punch. In order for the statement $\lim\limits_{x\to c} f(x) = f(c)$ to be true, **all** of the following things must be true:

1. $f(c)$ must be defined,
2. $\lim\limits_{x\to c} f(x)$ must exist,
   (i.e., the function approaches the same finite limiting value from both sides at c.)
3. and these two numbers must be equal.

Also recall how we extend this idea of continuity to intervals:

Definition 5.2. A function $f(x)$ is said to be **continuous on a closed interval** $[a, b]$ if f is continuous at c for every value $c \in [a, b]$.

In layman's terms, a function is continous on an interval if you can draw the function without having to lift your pencil off the page.

And remember that discontinuities in a function occur at two kinds of places:

1. places where the function is not defined
   (for instance a place where the function has $denominator = 0$,)
2. places where the function suddenly jumps from one value to another.

When neither of those things happens, at least not in the interval $[a, b]$, then f is defined everywhere in $[a, b]$, and is continuous on this interval, and the antiderivative of f is also defined everywhere on $[a, b]$. And under those circumstances, the *definite integral* is defined. First, we define the notation.

Definition 5.3. If a function f is continuous on the closed interval $[a, b]$, then:

$\int_a^b f(x)dx$ is called the **definite integral of f from a to b**
where a and b are called the **limits of integration**
(the **lower limit** and the **upper limit** of integration, respectively)

Well, okay, so we have a piece of notation, which looks a lot like an indefinite integral except it's got these little numbers hanging off it. But *what is it*? Well, we're not going to talk about that right now. But we *are* going to learn how to calculate it. There's a really important-sounding theorem that tells us that. And the reason that it's got such an important-sounding name is because it really is the basis of integral calculus.

The Fundamental Theorem of Calculus

Definition 5.4. If f is continuous on $[a, b]$, and F is any antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Note: An indefinite integral, $\int f(x)dx$, is a function. But $F(b)$ and $F(a)$ are numbers, so a definite integral, $\int_a^b f(x)dx$, **is a number**. This is similar to (but not quite the same as) the idea of the *derivative function* versus the *derivative at a particular x-value*, i.e. $f'(x)$ versus $f'(a)$.

The theorem says that to find the value of $\int_a^b f(x)dx$ we need to:

1. Find any antiderivative, F, of the integrand function f.
2. Evaluate $F(a)$ and $F(b)$.
3. Calculate the difference, $F(b) - F(a)$.

Example 5.1. Evaluate $\int_1^2 x^3 dx$.

Solution:
Step 1: We know that $F(x) = \frac{x^4}{4}$ is an antiderivative of $f(x) = x^3$.
Step 2: We get $F(1) = \frac{1^4}{4} = \frac{1}{4}$ and $F(2) = \frac{2^4}{4} = 2^2 = 4$.
Step 3: Therefore we get

$$\int_1^2 x^3 dx = F(2) - F(1) = 4 - \frac{1}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$$

There's a piece of notation that you probably saw in your introductory calculus course which we haven't yet used in this course. It means, basically, "evaluated at". It's a vertical line (|), or sometimes a right (square) bracket ( ]) (used when brackets are wanted anyway), or like a right (square) bracket but without the bottom horizontal bit (⌉), and then it's got something saying what value to evaluate at. For instance if we write

$$\left.\frac{x}{3}\right|_{x=1} \quad \text{or} \quad \left[\frac{x}{3}\right]_{x=1} \quad \text{or} \quad \left.\frac{x}{3}\right]_{x=1}$$

what we're saying is "$\frac{x}{3}$ evaluated at $x = 1$", so each of these is just equal to $\frac{1}{3}$. We use this in something like the following:

> For the function $f(x) = \frac{x^2}{6}$, the slope of the tangent line to the graph of $y = f(x)$ at $x = 1$ is
>
> $$f'(1) = \left.\frac{dy}{dx}\right]_{x=1} = \left.\frac{2x}{6}\right]_{x=1} = \left.\frac{x}{3}\right]_{x=1} = \frac{1}{3}$$

Our reason for reviewing this is not so much because we're going to use this notation, but rather because we're going to use something related, which we think of as meaning "evaluated from" one value "to" another, but interpreted as the *difference* between the two values.

Definition 5.5. We use $F(x)|_a^b$ or $[F(x)]_a^b$ or $F(x)\rceil_a^b$ to denote $F(b)-F(a)$, where $F(x)$ may be shown either by name or in functional form. This is pronounced as "$F(x)$ evaluated from a to b".

For instance, for $F(x)=\dfrac{x^2}{3}$ we have

$$F(x)\rceil_1^3 = \frac{x^2}{3}\Bigg\rceil_1^3 = \frac{x^2}{3}\Bigg\rceil_{x=3} - \frac{x^2}{3}\Bigg\rceil_{x=1} = \frac{(3)^2}{3}-\frac{(1)^2}{3} = \frac{9}{3}-\frac{1}{3}=\frac{8}{3}$$

Similarly, we can write

$$\begin{aligned} \left[x^3+x^2-3x\right]_1^2 &= [x^3+x^2-3x]_{x=2}-[x^3+x^2-3x]_{x=1} \\ &= (2^3+2^2-3(2))-(1^3+1^2-3(1)) \\ &= (8+4-6)-(1+1-3)=6-(-1)=7 \end{aligned}$$

Using this notation, we can express the definite integral as $\int_a^b f(x)dx = F(x)\rceil_a^b$ where F is any antiderivative of f. That is, this notation expresses exactly what we need to do with the antiderivative function to calculate a definite integral. So for Example 5.1 we would express the calculations as follows.

Example 5.1. (*Revisited*) Evaluate $\int_1^2 x^3dx$.

Solution:

$$\int_1^2 x^3dx = \frac{x^4}{4}\Bigg\rceil_1^2 = \left[\frac{2^4}{4}-\frac{1^4}{4}\right] = \frac{16}{4}-\frac{1}{4}=\frac{15}{4}$$

Let's look at more examples of evaluating definite integrals.

Example 5.2. Evaluate $\int_{-1}^2 (x^2-x)dx$.

Solution:
Since $\frac{x^3}{3}-\frac{x^2}{2}$ is an antiderivative of x^2-x, we have:

$$\begin{aligned} \int_{-1}^2 (x^2-x)dx &= \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^2 = \left(\frac{2^3}{3}-\frac{2^2}{2}\right)-\left(\frac{(-1)^3}{3}-\frac{(-1)^2}{2}\right) \\ &= \left(\frac{8}{3}-2\right)-\left(-\frac{1}{3}-\frac{1}{2}\right) = \frac{8}{3}-2+\frac{1}{3}+\frac{1}{2} \\ &= \frac{9}{3}-2+\frac{1}{2} = 3-2+\frac{1}{2} = 1+\frac{1}{2}=\frac{3}{2} \end{aligned}$$

Notice: If $F(x)$ is any antiderivative of $f(x)$, then so is $F(x)+1$, and $F(x)-1$ and ... as we know, there are many antiderivatives of f (differing in the value of the constant term). Does it matter *which* antiderivative function we use to evaluate a definite integral?

No, because of the subtraction. For instance, in Example 5.2 we could have used the antiderivative function $F(x)=\frac{x^3}{3}-\frac{x^2}{2}+5$. Using that antiderivative function we get:

$$\begin{aligned} \int_{-1}^2 (x^2-x)dx &= \left[\frac{x^3}{3}-\frac{x^2}{2}+5\right]_{-1}^2 = \left(\frac{2^3}{3}-\frac{2^2}{2}+5\right)-\left(\frac{(-1)^3}{3}-\frac{(-1)^2}{2}+5\right) \\ &= \left(\frac{8}{3}-\frac{4}{2}\right)+5-\left(-\frac{1}{3}-\frac{1}{2}\right)-5 = \frac{8}{3}-2+\frac{1}{3}+\frac{1}{2}=\frac{3}{2} \end{aligned}$$

Whenever we calculate $\int_a^b F'(x)dx = F(b) - F(a)$, if there is a non-zero constant term in $F(x)$, that constant is the same in both $F(b)$ and $F(a)$, so the constant is added and then subtracted, so that the constants cancel out and the net effect is that the constant didn't need to be there. So just as in finding the general antiderivative of a function (i.e. finding an indefinite integral), for calculating a definite integral we normally use the antiderivative whose constant term is 0.

Example 5.3. Evaluate $\int_1^2 \left(e^x - \frac{1}{x}\right) dx$.

Solution:
(Notice that the integrand function, although not defined at $x = 0$, is continuous throughout $[1, 2]$.) We know that $F(x) = e^x - \ln|x|$ is an antiderivative of $f(x) = e^x - \frac{1}{x}$,so we get:

$$\begin{aligned}\int_1^2 \left(e^x - \frac{1}{x}\right) dx &= \left[e^x - \ln|x|\right]_1^2 \\ &= \big(e^2 - \ln 2\big) - \big(e^1 - \ln 1\big) \\ &= e^2 - (\ln 2) - e + 0 = e^2 - e - \ln 2\end{aligned}$$

Many of the properties of indefinite integrals carry over to definite integrals. For instance, let F and G be any antiderivatives of f and g, respectively. Then we see that

$$\begin{aligned}\int_a^b (f(x) + g(x))\, dx &= [F(x) + G(x)]_a^b = (F(b) + G(b)) - (F(a) + G(a)) \\ &= F(b) + G(b) - F(a) - G(a) = (F(b) - F(a)) + (G(b) - G(a)) \\ &= \int_a^b f(x)dx + \int_a^b g(x)dx\end{aligned}$$

That is, because of the sum rule for indefinite integrals, there is a corresponding sum rule for definite integrals. Likewise it's easy to show that the same is true for differences, i.e. for $f(x) - g(x)$ in the integrand. That is, it's easy to also show that

$$\int_a^b (f(x) - g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

Similarly, because of the constant multipler rule for indefinite integrals, again there is a corresponding constant multiplier rule for definite integrals. We get:

$$\int_a^b c[f(x)]dx = [cF(x)]_a^b = cF(b) - cF(a) = c\,[F(b) - F(a)] = c\int_a^b f(x)dx$$

Example 5.4. Evaluate $\int_1^2 \left(3e^x - \frac{1}{2x}\right) dx$.

Solution:
We use the difference rule and then the constant multiplier rule:

$$\begin{aligned}\int_1^4 \left(3e^x - \frac{1}{2x}\right) dx &= \int_1^4 3e^x dx - \int_1^4 \frac{1}{2x} dx = 3\int_1^4 e^x dx - \frac{1}{2}\int_1^4 \frac{1}{x} dx \\ &= 3\left[e^x\right]_1^4 - \left(\frac{1}{2}\right)\left[\ln|x|\right]_1^4 = 3(e^4 - e^1) - \left(\frac{1}{2}\right)\left[(\ln 4) - (\ln 1)\right] \\ &= 3(e^4 - e) - \left(\frac{1}{2}\right)\ln 4 = 3e(e^3 - 1) - \ln 4^{1/2} \\ &= 3e(e^3 - 1) - \ln\sqrt{4} = 3e(e^3 - 1) - \ln 2\end{aligned}$$

Example 5.5. Evaluate $\displaystyle\int_{\pi/6}^{\pi/4} \sec^2 t \tan t \; dt$.

Solution:
The integrand here is complicated — it's a product — but it's a product in which one term is the derivative of the other. That is, we recognize that $\sec^2 t = \frac{d}{dt}(\tan t)$. Letting $u = \tan t$ we have $du = \sec^2 t\; dt$ and so

$$\int \sec^2 t \tan t \, dt = \int u \; du = \frac{u^2}{2} + C = \frac{\tan^2 t}{2} + C$$

That is, we see that $\frac{\tan^2 t}{2}$ is an antiderivative of $\sec^2 t \tan t$, so we get

$$\int_{\pi/6}^{\pi/4} \sec^2 t \tan t \; dt = \left[\frac{\tan^2 t}{2}\right]_{\pi/6}^{\pi/4} = \frac{1}{2}\left[\left(\tan\frac{\pi}{4}\right)^2 - \left(\tan\frac{\pi}{6}\right)^2\right]$$

And we know that

$$\tan\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = \frac{1/\sqrt{2}}{1/\sqrt{2}} = 1 \qquad \text{and} \qquad \tan\frac{\pi}{6} = \frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

Therefore we get:

$$\int_{\pi/6}^{\pi/4} \sec^2 t \tan t \; dt = \frac{1}{2}\left[\left(\tan\frac{\pi}{4}\right)^2 - \left(\tan\frac{\pi}{6}\right)^2\right] = \frac{1}{2}\left[1^2 - \left(\frac{1}{\sqrt{3}}\right)^2\right] = \frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}$$

Recall: So far, we have only defined $\displaystyle\int_a^b f(x)dx$ where $[a, b]$ *is an interval.* That is, where $b > a$. We can extend the defintion to allow b to be the same as, or smaller than, a by defining 2 special cases.

Definition 5.6. Let f be any function which is continuous on some interval $[a, b]$. We define that:

1. $\displaystyle\int_a^a f(x)dx = 0$.
2. $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx$

Example 5.6. Evaluate $\displaystyle\int_a^{a^2} 2xdx$, where $a = 1$.

Solution:

$$\int_a^{a^2} 2xdx = \int_1^{1^2} 2xdx = \int_1^1 2xdx = 0$$

(using the first part of the definition above at the last step).

Example 5.7. Evaluate $\displaystyle\int_2^0 e^x dx$.

Solution:
This time we use the second part of the definition. We get:

$$\int_2^0 e^x dx = -\int_0^2 e^x dx = -\left[e^x\right]_0^2 = -(e^2 - e^0) = 1 - e^2$$

Notice: In that calculation, we explicitly used the definition. However, we could instead have just used it implicitly, to not worry about whether the upper limit of integration is larger or smaller than the lower limit of integration. That is, since $f(x) = e^x$ is continuous on $[0, 2]$, then by the above definition we know that the integral $\int_2^0 e^x dx$ is defined, so we just need to do:

$$\int_2^0 e^x dx = e^x\rceil_2^0 = e^0 - e^2 = 1 - e^2$$

There's another result which allows us to extend our use of definite integrals even farther. This result is presented in the following theorem.

Theorem 5.1. *Let f be any function and $[a, b]$ be any interval such that $f(x)$ is continuous on $[a, b]$ and let t be any value inside $[a, b]$. Then*

$$\int_a^b f(x)dx = \int_a^t f(x)dx + \int_t^b f(x)dx$$

That is, if t is some value between a and b so that $a < t < b$, we can "break up" the evaluation of the definite integral from a to b at the value t, going from a to t, and then going from t to b.

Proof:
If f is continuous on $[a, b]$, then for any t in $[a, b]$ it is also true that f is continuous on both $[a, t]$ and $[t, b]$, so both $\int_a^t f(x)\,dx$ and $\int_t^b f(x)\,dx$ are defined. Letting F be any antiderivative of f we see that

$$\int_a^t f(x)dx + \int_t^b f(x)dx = (F(t)-F(a)) + (F(b)-F(t)) = F(t)-F(t)+F(b)-F(a) = F(b)-F(a) = \int_a^b f(x)dx$$

Example 5.8. If it is known that $\displaystyle\int_1^{10} f(x)dx = 23$ and that $\displaystyle\int_2^{10} f(x)dx = 15$, find $\displaystyle\int_1^2 f(x)dx$.

Solution:
We are told that $\int_1^{10} f(x)dx = 23$ and $\int_2^{10} f(x)dx = 15$. Looking at the theorem and letting $a = 1$, $b = 10$ and $t = 2$, we get

$$\int_1^{10} f(x)dx = \int_1^2 f(x)dx + \int_2^{10} f(x)dx$$

which gives:

$$\int_1^2 f(x)dx = \int_1^{10} f(x)dx - \int_2^{10} f(x)dx = 23 - 15 = 8$$

As with indefinite integrals, it pays to look out for the "trick questions". If you're asked for the derivative of a definite integral, the answer is always 0, because the value of a definite integral is just a constant. That is, we could do perhaps a lot of work finding an antiderivative function F for the integrand function f to then calculate

$$\frac{d}{dx}\left(\int_a^b f(x)dx\right) = \frac{d}{dx}\left(F(b) - F(a)\right) = 0$$

(because $F(b) - F(a)$ is a constant). But why bother finding *which* constant we want the derivative of? We already know that $\int_a^b f(x)dx$ is just a constant, so its derivative is 0.

Likewise, if you're asked to evaluate a definite integral in which the integrand function is expressed as the derivative of something, then (just as we saw with indefinite integrals) there's no need to differentiate, and then integrate. We already know that the "something" whose derivative is the integrand is an antiderivative of that integrand, and we just need to evaluate the definite integral using that antiderivative. For instance, consider the next example.

Example 5.9. Evaluate $\int_{\pi/6}^{\pi/4} F'(x)dx$ where $F(x) = \sin^4 x$.

Solution:
We could use the chain rule to find $F'(x)$, and then the substitution rule to find an antiderivative of that function. But then again, why bother? That sounds like a lot of work, and we *already know an antiderivative of* $F'(x)$, since we know $F(x)$. That is, we simply recognize that F is an antiderivative of F', so with $F(x) = \sin^4 x = (\sin x)^4$ we have

$$\int_{\pi/6}^{\pi/4} F'(x)\,dx = F\left(\frac{\pi}{4}\right) - F\left(\frac{\pi}{6}\right) = \left(\sin\frac{\pi}{4}\right)^4 - \left(\sin\frac{\pi}{6}\right)^4 = \left(\frac{1}{\sqrt{2}}\right)^4 - \left(\frac{1}{2}\right)^4 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

Substitution and Definite Integrals

Now that we know how to evaluate straightforward definite integrals, we need to think about how to deal with definite integrals in which the integrand is more complicated, so that a more complicated integration technique is needed. Or do we? We already did one of these, in Example 5.5. There, we used an approach that will *always* work, any time the integrand function is something we know how to find an antiderivative of, by any means, no matter how complicated it may be:

Given any definite integral $\int_a^b f(x)dx$,

1. Find $F(x) = \int f(x)dx$.
2. Evaluate $\int_a^b f(x)dx = F(b) - F(a)$.

That is, you can *always* evaluate a definite integral by considering it as 2 problems, with the first being an indefinite integral. So first find any (or the general) antiderivative function, and then once you've got that, evaluate the definite integral using the Fundamental Theorem of Calculus.

Example 5.10. Evaluate $\int_1^2 e^{x^2+3x}(2x+3)dx$.

Solution:
We recognize that the integrand function is complicated – it looks like it probably needs the substitution rule. First, we consider the problem "Find $\int e^{x^2+3x}(2x+3)dx$". Looking at the integrand function, we suspect that the form is $e^u du$, with $u = x^2 + 3x$. We try that. It gives $du = (2x+3)dx$. Since we do see that in the integral, we go ahead with this substitution:

$$\int e^{x^2+3x}(2x+3)dx = \int e^u du = e^u + C = e^{x^2+3x} + C$$

so we see that $F(x) = e^{x^2+3x}$ is an antiderivative of $e^{x^2+3x}(2x+3)$, and we use this to evaluate the definite integral:

$$\int_1^2 e^{x^2+3x}(2x+3)dx = \left[e^{x^2+3x}\right]_1^2 = e^{2^2+3(2)} - e^{1^2+3(1)} = e^{4+6} - e^{1+3} = e^{10} - e^4$$

Well, okay, that works. So what do we need to talk about? Well, there's a shortcut. Let's think about what we did in this example. We found a substitution, i.e. a way to express the problem in terms of u instead of x, so that the problem when stated in terms of u is straightforward. And then we solved that straightforward problem, to get an answer expressed in terms of u. Next, we translated the answer in terms of u into an answer stated in terms of x. Finally, we translated the answer in terms of x into a number. That is, we evaluated the answer.

Remember that when we solve an indefinite integral, we must translate the answer to being stated in terms of x, because that's what the question asked for. If you're asked "what's the general antiderivative of $f(x)$" and your answer is stated in terms of u, your boss or whoever asked you for the antiderivative is just going to stare blankly at you and say "What's u?". (Only maybe they won't say it so politely.) It's just the same as if an American tourist asks you "How many miles is it from here to Toronto?" and you reply "about 200 kilometres". You need to convert the answer into the units the question was stated in, or you haven't answered the question.

On the other hand, when the answer to the question is a number which is independent of any units, then if you convert from one unit system to another along the way, it doesn't matter whether or not you convert back, so you really don't need to. Suppose an American tourist comes along and says "Back home, the town I live in is about 60 miles from the big city. How does the distance from here to Toronto compare to that?". You think to yourself "Let's see... 60 miles is about 100 kilometres, and Toronto's about 200 kilometres from here" so you tell the tourist "It's about twice as far". Would the answer have been any more correct if you'd added an extra step, thinking "... and that's about 120 miles" before saying it's twice as far? Would the tourist have been any more or less happy with the answer? No. The answer is that it's about twice as far whether you measure in miles or in kilometres. So there's no need to convert back in order to get the answer.

The answer to "evaluate this definite integral" is always a number. It's not expressed in terms of x, and so if you're using a substitution to find an antiderivative, it doesn't matter whether or not you substitute back, stating the antiderivative in terms of x, before calculating that answer. You can calculate it in terms of u ... as long as you do it properly.

Consider the following. Suppose we have the function $f(x) = (x-1)^2$ and we want to evaluate $f(2)$. We could expand $(x-1)^2$, to get $f(x) = x^2-2x+1$ and then plug in $x = 2$. But wouldn't it be easier to say "well, $(x-1)^2 = u^2$ where $u = x-1$, so I just need to calculate u when $x = 2$ and then square that"? That is, we can get the answer more quickly (and with less chance of error) as follows: when $x = 2$ then $x-1 = 1$ and so $f(2) = 1^2 = 1$. We can think of what we did as making the substitution $u = x-1$, and replacing the function $f(x) = (x-1)^2$ with the function $h(u) = u^2$ to calculate $f(2) = h(1) = 1$. Notice that in that approach, *we didn't square 2*. We weren't applying the h function to an x-value, we applied it to a u-value. Because that function was expressed in terms of u.

When we're evaluating a definite integral and we need the substitution rule, we can do something similar. Since the final answer is a number, it doesn't matter at all whether we calculate that number using an x-value in an antiderivative function that's stated in terms of x, or using a u-value in an antiderivative function that's stated in terms of u. So rather than translating the antiderivative that we found in terms of u, into an antiderivative expressed in terms of x, and then plugging in x-values, we could just use the antiderivative that we found (stated in terms of u), as long as we find the right u-values to plug in.

Suppose the limits of integration are some numbers a and b. Also, suppose the substitution we use is $u = g(x)$. When we find an antiderivative function, expressed in terms of u, the values we need to evaluate that antiderivative function at are not a and b, they're *the u-values that correspond to a and b*. And those u-values are $u = g(a)$ and $u = g(b)$.

Let's see how we would do Example 5.10 without substituting the antiderivative back in terms of x before calculating the answer.

Example 5.10. (*Revisited*) Evaluate $\displaystyle\int_1^2 e^{x^2+3x}(2x+3)dx$.

Solution:
We know that we want to use the substitution $u = x^2 + 3x$, which gives $du = (2x + 3)dx$. So we integrate $e^u du$ instead of $e^{x^2+3x}(2x+3)dx$. That is, we use the fact that $\int e^{x^2+3x}(2x+3)dx = \int e^u du$. But

$$\int_1^2 e^u du \neq \int_1^2 e^{x^2+3x}(2x+3)dx$$

because 1 and 2 are not u-values, they're x-values. We need to calculate the corresponding u-values. Since $u = x^2 + 3x$, then when $x = 1$ we have $u = 1^2 + 3(1) = 1 + 3 = 4$. And when $x = 2$ we have $u = 2^2 + 3(2) = 4 + 6 = 10$. So we can evaluate the given definite integral using the substitution $u = x^2 + 3x$ and $du = (2x + 3)dx$ where $x = 1 \Rightarrow u = 4$ and $x = 2 \Rightarrow u = 10$. We get

$$\int_1^2 e^{x^2+3x}(2x+3)dx = \int_4^{10} e^u du = e^u\big]_4^{10} = e^{10} - e^4$$

Because we converted our inputs, the limits of integration, from x-values to u-values, we get the same answer as we got the first time we did this problem.

Whenever we need to use the substitution rule while evaluating a definite integral, it's generally quicker to convert the limits of integration to values of the substituted variable instead of re-expressing the antiderivative function in terms of the original variable before doing the final calculations. We can express a rule for doing this. If it seems a bit complicated, remember it just describes what we did in this example. (It looks complicated because of course if we need to use a substitution, we're dealing with a composite function.)

Rule: Definite Integrals with Substitution
To evaluate $\displaystyle\int_a^b f(g(x))g'(x)dx$ using the substitution $u = g(x)$ (so that $du = g'(x)\,dx$),
calculate $\displaystyle\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

That is, when you substitute u for $g(x)$, and du for $g'(x)dx$, you **must** also substitute $g(a)$ for a and $g(b)$ for b (or else express the antiderivative function in terms of x before calculating the answer).

Example 5.11. Evaluate $\displaystyle\int_1^2 \frac{3x^2+2x+1}{x^3+x^2+x-2}dx$.

Solution:
We recognize that the integrand has the form $\dfrac{u'}{u}$ where $u = x^3+x^2+x-2$. And for this substitution, we get $du = (3x^2 + 2x + 1)dx$. Also, we see that

$$\begin{aligned} & x = 1 \Rightarrow u = 1^3 + 1^2 + 1 - 2 = 1 \\ \text{and} \quad & x = 2 \Rightarrow u = 2^3 + 2^2 + 2 - 2 = 12 \end{aligned}$$

That is, for $g(x) = x^3 + x^2 + x - 2$, we get $g(1) = 1$ and $g(2) = 12$, so our new integral is:

$$\int_1^2 \frac{3x^2+2x+1}{x^3+x^2+x-2}dx = \int_1^{12} \frac{1}{u}du = \ln|u|\big]_1^{12} = \ln 12 - \ln 1 = \ln 12$$

Example 5.12. Evaluate $\int_0^{3\pi} \sin\frac{x}{3}\,dx$.

Solution:
The angle in the trig function is not just x, so we need a substitution. Let $u = \dfrac{x}{3}$ so that $du = \frac{1}{3}\,dx$ and thus $dx = 3du$. We see that when $x = 0$, $u = \frac{0}{3} = 0$, and when $x = 3\pi$, $u = \frac{3\pi}{3} = \pi$. Therefore we have

$$\begin{aligned} \int_0^{3\pi} \sin\frac{x}{3}\,dx &= 3\int_0^{\pi} \sin u\,du &&= 3\left[-\cos u\right]_0^{\pi} \\ &= -3\cos\pi + 3\cos 0 &&= -3(-1) + 3(1) = 6 \end{aligned}$$

Example 5.13. Evaluate $\int_0^1 (2x - x^2)e^{(x^3-3x^2)}dx$.

Solution: We think we see the form $e^u k du$ with $u = x^3 - 3x^2$. We try that. We get $\frac{du}{dx} = 3x^2 - 6x = 3(x^2 - 2x) = -3(2x - x^2)$ and so we have $(2x - x^2)dx = -\frac{1}{3}du$.
We also need to find the limits of integration in terms of u. When $x = 0$ we have $u = 0^3 - 3(0^2) = 0$. Similarly, when $x = 1$ we have $u = 1^3 - 3(1^2) = -2$. Carrying out the substitution, we get:

$$\begin{aligned} \int_0^1 (2x - x^2)e^{(x^3-3x^2)}dx &= \int_0^{-2}\left(-\frac{1}{3}\right)e^u du = -\frac{1}{3}\int_0^{-2} e^u du \\ &= \frac{1}{3}\int_{-2}^{0} e^u du = \frac{1}{3}\left[e^u\right]_{-2}^{0} \\ &= \frac{1}{3}\left(e^0 - e^{-2}\right) = \frac{1}{3}\left(1 - \frac{1}{e^2}\right) = \frac{e^2-1}{3e^2} \end{aligned}$$

In the second line, we used the fact that $-\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$ to eliminate the negative. We didn't have to, but it made things less complicated so that we were less likely to make an arithmetic mistake.

Notice: Be careful with the new limits. As we saw here, we can sometimes have $g(b) < g(a)$ even though $b > a$. Be sure you set up the new integral going **from** $g(a)$ **to** $g(b)$, even if $g(a) > g(b)$. (In the example, since we also had a negative in the "missing constant", we re-expressed the integral as going *from* the smaller value *to* the larger value, instead of as the negative of an integral going *from* a larger value *to* a smaller value. But if we hadn't had the negative, we would have just left it as an integral going from a larger value to a smaller value, and evaluated it that way.)

Example 5.14. Evaluate $\int_{\pi/2}^{\pi} \dfrac{\cos x}{1+\sin x}\,dx$.

Solution:
We recognize the form $\frac{u'}{u}$, so we let $u = 1 + \sin x$ which gives $du = \cos x\,dx$. When $x = \frac{\pi}{2}$ we get $u = 1 + \sin\frac{\pi}{2} = 1 + 1 = 2$ and when $x = \pi$ we get $u = 1 + \sin\pi = 1 + 0 = 1$. Therefore we have

$$\int_{\pi/2}^{\pi} \frac{\cos x}{1+\sin x}\,dx = \int_2^1 \frac{1}{u}\,du = \left[\ln\mid u\mid\right]_2^1 = \ln 1 - \ln 2 = -\ln 2$$

The average value of f on an interval

Another use of the definite integral is to give us the *average value* of a function (f_{ave}) over a specific interval. We are familiar with the concept of the average value of two numbers. The average of a and b is given by $\frac{b+a}{2}$. For the function $f(x) = x$, the value of $f(x)$ on the interval $[a, b]$ increases in a straight line from $f(a) = a$ to $f(b) = b$, so it should be true that the average value of the function $f(x) = x$ on the interval $[a, b]$ is also $\frac{b+a}{2}$. For instance, we would want the average value of $f(x) = x$ on $[0, 2]$ to be $\frac{2+0}{2} = 1$. The definition of the average value of a function does satisfy this intuitive result, and allows us to extend the concept of average value to more complicated functions than $f(x) = x$, i.e. when $f(x)$ isn't going in a straight line over the interval $[a, b]$.

Definition 5.7. For any function f which is continuous on $[a, b]$, we define the average value of $f(x)$ on the interval [a,b] to be given by:

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)dx$$

We can check that for the function $f(x) = x$, when we apply this definition we do get the average value we wanted, for any interval $[a, b]$:

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{b-a}\int_a^b f(x)dx &&= \frac{1}{b-a}\int_a^b xdx &&= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b \\ &= \frac{1}{b-a}\left(\frac{b^2}{2}-\frac{a^2}{2}\right) &&= \frac{b^2-a^2}{2(b-a)} &&= \frac{(b+a)(b-a)}{2(b-a)} = \frac{b+a}{2} \end{aligned}$$

We can use the definition to find the average value of *any* function f over any interval $[a, b]$ on which it is continuous, provided we know an antiderivative of f. For instance, consider the next example.

Example 5.15. Find the average value of $f(x) = e^x + 3x^2$ on the interval [0,2].

Solution:
From the definition, with $a = 0$ and $b = 2$, we have:

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{b-a}\int_a^b f(x)dx &&= \frac{1}{2-0}\int_0^2 (e^x+3x^2)dx &&= \frac{1}{2}\left[e^x + x^3\right]_0^2 \\ &= \frac{1}{2}[(e^2+2^3)-(e^0+0^3)] &&= \frac{1}{2}(e^2+8-1) &&= \frac{e^2+7}{2} \end{aligned}$$

Notice that here we have

$$f(b) + f(a) = f(2) + f(0) = [e^2 + 3(2^2)] + [e^0 + 3(0^2)] = e^2 + 12 + 1 = e^2 + 13 \neq e^2 + 7$$

so $f_{\text{ave}} \neq \dfrac{f(b)+f(a)}{2}$. When $f(x)$ is a non-linear function, f_{ave} is **not** just the average of the function values at the endpoints of the interval $[a, b]$.

Example 5.16. Find the average value of $f(x) = \sin x$ on the interval $\left[\frac{\pi}{6}, \frac{\pi}{4}\right]$.

Solution:
We have $a = \frac{\pi}{6}$ and $b = \frac{\pi}{4}$ so we get:

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{\frac{\pi}{4} - \frac{\pi}{6}} \int_{\pi/6}^{\pi/4} \sin x \, dx = \frac{1}{\frac{3\pi}{12} - \frac{2\pi}{12}} \left[-\cos x\right]_{\pi/6}^{\pi/4} = \frac{1}{\pi/12} \left[\left(-\cos\frac{\pi}{4}\right) - \left(-\cos\frac{\pi}{6}\right)\right] \\ &= \frac{12}{\pi}\left(-\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\right) = \frac{12}{\pi}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right) = \frac{6(\sqrt{3} - \sqrt{2})}{\pi} \end{aligned}$$

Example 5.17. Find the average value of $f(x) = \dfrac{1 + \cos x}{x + \sin x}$ on the interval $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Solution:
The formula is

$$f_{\text{ave}} = \frac{1}{\frac{\pi}{2} - \frac{\pi}{6}} \int_{\pi/6}^{\pi/2} \frac{1 + \cos x}{x + \sin x}\, dx$$

and for this integral we need to use a substitution. We have the form $\frac{u'}{u}$ so we let $u = x + \sin x$. This gives $du = (1 + \cos x)\, dx$. When $x = \frac{\pi}{6}$ we see that $u = \frac{\pi}{6} + \sin\frac{\pi}{6} = \frac{\pi}{6} + \frac{1}{2} = \frac{\pi+3}{6}$, while $x = \frac{\pi}{2}$ gives $u = \frac{\pi}{2} + \sin\frac{\pi}{2} = \frac{\pi}{2} + 1 = \frac{\pi+2}{2}$. So we get:

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{\frac{\pi}{2} - \frac{\pi}{6}} \int_{\pi/6}^{\pi/2} \frac{1 + \cos x}{x + \sin x} dx \\ &= \frac{1}{\pi/3} \int_{(\pi+3)/6}^{(\pi+2)/2} \frac{1}{u}\, du \\ &= \frac{3}{\pi} \ln|u| \Big]_{(\pi+3)/6}^{(\pi+2)/2} \\ &= \frac{3}{\pi}\left[\left(\ln\frac{\pi+2}{2}\right) - \left(\ln\frac{\pi+3}{6}\right)\right] \\ &= \frac{3}{\pi}\left[(\ln(\pi+2) - \ln 2) - (\ln(\pi+3) - \ln 6)\right] \\ &= \frac{3}{\pi}\left[\ln(\pi+2) - \ln(\pi+3) + (\ln 6 - \ln 2)\right] \\ &= \frac{3}{\pi}\left[\ln\left(\frac{\pi+2}{\pi+3}\right) + \ln\left(\frac{6}{2}\right)\right] \\ &= \frac{3}{\pi}\left[\ln\left(\frac{\pi+2}{\pi+3} \times 3\right)\right] \\ &= \frac{3}{\pi}\left[\ln\left(\frac{3(\pi+2)}{\pi+3}\right)\right] \end{aligned}$$

(which a calculator says is approximately 0.8794)

Math 1225A/B

Unit 6:
The Area Between Two Curves

(text reference: Section 5.1

custom text pgs. 148 - 155)

6 The Area Between Two Curves

In this unit we will learn how to use definite integrals to calculate area. You know how to calculate the areas of certain shapes: length times height for a rectangle, πr^2 for a circle, and so forth. But how do we calculate the area of a less regularly-shaped region? Well, by using Calculus, and specifically the definite integral.

In fact, in a more rigorous Calculus course, the definite integral would be *defined* using the concept of area. The definition involves summing the areas of an infinite number of infinitessimally small "approximating rectangles" ... but you don't need to worry about that. We'll just state some theorems asserting that certain areas are given by certain integrals and accept them as true.

Definition 6.1. If $f(x) \geq 0$ on $[a, b]$, then the **area under the curve** $y = f(x)$ **from** $x = a$ **to** $x = b$ means the area of the region on the graph of $y = f(x)$ which lies *below* $y = f(x)$, and *above* the x-axis (i.e. the line $y = 0$), *between* the lines $x = a$ and $x = b$.

So "area under the curve" is shorthand for "area of the region below the curve and above the x-axis". For instance, we might want to know, or at least talk about, the region of the area shaded and marked R in Figure 1 below.

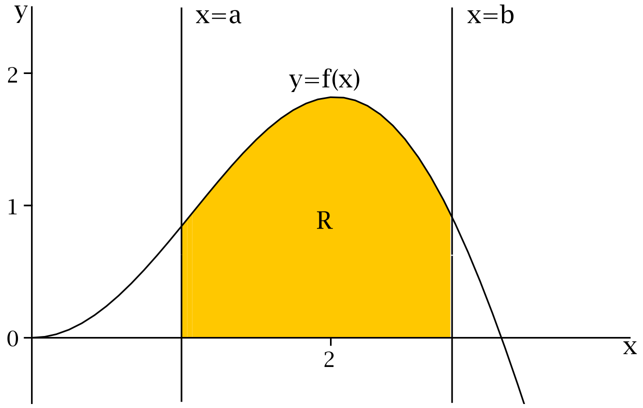


Figure 1: The region "under" the curve $y = f(x)$ between $x = a$ and $x = b$

Theorem 6.1. *If $f(x) \geq 0$ on $[a, b]$ then the area under $y = f(x)$ between $x = a$ and $x = b$ is:*

$$\text{Area} = \int_a^b f(x)dx$$

Example 6.1. Find the area under the curve $y = x^2$ between $x = 0$ and $x = 2$.

Solution:
We look at the graph, shown in Figure 2 (next page). Of course, $y = x^2$ is a parabola, "sitting on" the origin. We identify the region whose area we want to find. This region is shaded in Figure 2. Notice that the graph of $y = x^2$ lies (on or) above the x-axis everywhere, and in particular between $x = 0$ and $x = 2$. According to Theorem 6.1, the area of the region can be found by evaluating the integral of $f(x) = x^2$ from 0 to 2. We get:

$$\text{Area} = \int_0^2 x^2 dx = \left.\frac{x^3}{3}\right]_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$$

So the area is $\frac{8}{3}$ square units. (The units being whatever x and y are measured in.)

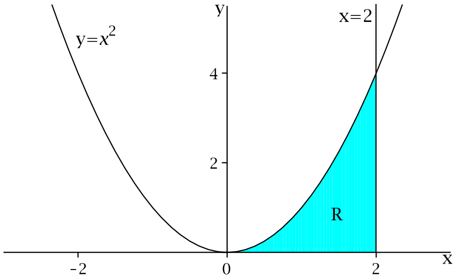


Figure 2: The region under $y = x^2$ between $x = 0$ and $x = 2$

Example 6.2. Find the area of the region under the curve $y = \dfrac{\sin x}{1 + \cos x}$ from $x = \dfrac{\pi}{3}$ to $x = \dfrac{\pi}{2}$.

Solution:
First of all, we need to be sure that the curve does lie above the x-axis on the interval $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$. Notice that $\sin x > 0$ on $(0, \pi)$, so on $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ the numerator is positive. We also know that $\cos x \geq 0$ on $\left[0, \frac{\pi}{2}\right]$, and so on this interval we have $1 + \cos x \geq 1 + 0$, i.e. $1 + \cos x \geq 1$ and thus on $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ the denominator is also positive. Therefore the function value is positive everywhere in $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ and so the curve does lie above the x-axis everywhere from $x = \dfrac{\pi}{3}$ to $x = \dfrac{\pi}{2}$. Therefore we see that

$$\text{Area } = \int_{\pi/3}^{\pi/2} \frac{\sin x}{1 + \cos x}\, dx$$

To evaluate this integral, we consider a substitution. Recognizing that $\sin x = -\frac{d}{dx}(\cos x)$, so that also $\sin x = -\frac{d}{dx}(1 + \cos x)$, we recognize (approximately) the form $\frac{u'}{u}$. That is, if we let $u = 1 + \cos x$, then $du = -\sin x\, dx$ and so $\sin x\, dx = -du$. Therefore we have $\displaystyle\int \frac{\sin x}{1 + \cos x}\, dx = -\int \frac{1}{u}\, du$. For the limits of integration, we see that when $x = \frac{\pi}{2}$ we have $u = 1 + \cos\frac{\pi}{2} = 1 + 0 = 1$, and when $x = \frac{\pi}{3}$ we have $u = 1 + \cos\frac{\pi}{3} = 1 + \frac{1}{2} = \frac{3}{2}$. Thus we get:

$$\text{Area } = \int_{\pi/3}^{\pi/2} \frac{\sin x}{1 + \cos x}\, dx = -\int_{3/2}^{1} \frac{1}{u}\, du = \int_{1}^{3/2} \frac{1}{u}\, du = [\ln|u|]_1^{3/2} = \ln\left(\frac{3}{2}\right) - \ln 1 = \ln 3 - \ln 2$$

Notice: Here, of course, we used the facts that $-\int_a^b f(t)\, dt = \int_b^a f(t)\, dt$ and that $\ln 1 = 0$.

Also Notice: We didn't draw the graph. We don't know what the graph looks like (or would have to do a fair amount of analysis to figure it out). But we didn't need to. We just needed to know that the curve lies above the x-axis on the relevant interval, and we were easily able to figure that out without seeing the graph.

As we have seen, the "area under" a curve $y = f(x)$ which lies above the x-axis really means the area between $y = f(x)$ and the x-axis (which is the line $y = 0$). What if instead we have a function which lies below the x-axis? If we have $f(x) \leq 0$ on the interval $[a, b]$, then $-f(x) \geq 0$ on $[a, b]$, and the area below $y = -f(x)$ between $x = a$ and $x = b$ is given by $\displaystyle\int_a^b (-f(x))dx = -\int_a^b f(x)dx$. So for a curve which lies below the x-axis, we can find the *area above* the curve by finding the area below the negative of the function. (And since this area must be non-negative, clearly the value of

$\int_a^b f(x)dx$ must be negative when $f(x) \leq 0$ on $[a, b]$.) Looking at this in a slightly different way, we can say that the area between $y = f(x)$ and $y = 0$, between $x = a$ and $x = b$, is given by $\int_a^b (f(x) - 0)dx$ when $f(x) \geq 0$ on $[a, b]$, and is given by $\int_a^b (0 - f(x))dx$ when $0 \geq f(x)$ on $[a, b]$. So we can say that the *area between* $y = f(x)$ *and the* x*-axis* on some interval $[a, b]$ in which the two functions (i.e. $y = f(x)$ and $y = 0$) do not cross is given by determining which of the two is the *"upper"* function, and which is the *"lower"* function, and calculating

$$\text{Area} = \int_a^b (\text{upper} - \text{lower})dx$$

Next, suppose we have two functions, $f(x)$ and $g(x)$. Suppose we have $f(x) \geq g(x)$ on $[a, b]$, and also suppose (for now) that $f(x) \geq 0$ on $[a, b]$. What is the area of the region that lies *between the curves* $f(x)$ *and* $g(x)$ from $x = a$ to $x = b$? For instance, consider the functions f and g shown in Figure 3. How can we find the area of the shaded region?

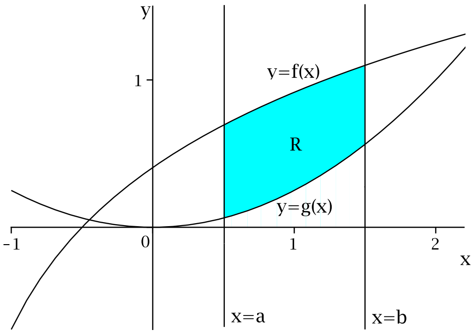


Figure 3: The region between $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$

If it is also true that $g(x) \geq 0$ on $[a, b]$, as in this diagram, then the area of the region is simply the area of the region between $y = f(x)$ and the x-axis minus the area of the region between $g(x)$ and the x-axis. And we know how to find each of these areas. We get:

$$\text{Area between } f \text{ and } g = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x) - g(x))dx$$

What if g lies below the x-axis? That is, what if we have $g(x) \leq 0$ (but still $f(x) \geq 0$) on $[a, b]$? Then the area of the region between $f(x)$ and $g(x)$ is the area under $f(x)$ plus the area above $g(x)$, which we have seen is the area under $-g(x)$, and so in this case we have

$$\text{Area between } f \text{ and } g = \int_a^b f(x)dx + \int_a^b (-g(x))dx = \int_a^b (f(x) - g(x))dx$$

and that's exactly the same as we had for the previous situation.

Now, what if $f(x)$ lies below the x-axis between $x = a$ and $x = b$? Then since $f(x) \geq g(x)$ on $[a, b]$, it must be true that $g(x)$ also lies below the x-axis, and lies further below than f. So in this case the area between the two curves is given by the area above $y = g(x)$ minus the area above $y = f(x)$. That is, we take the area under $y = -g(x)$ minus the area under $y = -f(x)$, so we get

$$\text{Area between } f \text{ and } g = \int_a^b (-g(x))dx - \int_a^b (-f(x))dx = \int_a^b [(-g(x)) - (-f(x))] = \int_a^b (f(x) - g(x))dx$$

Again, this is exactly the same result as before. So we don't need to worry about where f and g lie relative to the x-axis. All we need to know is that $f(x) \geq g(x)$ on the interval $[a, b]$. We get the following. First, we define what we mean by *the area between two curves*, and then we state the formula for calculating it.

Definition 6.2. If $f(x) \geq g(x)$ on $[a,b]$ then **the area (of the region) between** $y = f(x)$ **and** $y = g(x)$ **from** $x = a$ **to** $x = b$, or **on** $[a,b]$, means the area of the region which lies below $y = f(x)$ and above $y = g(x)$, between the lines $x = a$ and $x = b$.

Theorem 6.2. *If $f(x) \geq g(x)$ on $[a,b]$ then the area between $y = f(x)$ and $y = g(x)$ on $[a,b]$ is given by*

$$Area = \int_a^b (f(x) - g(x))dx$$

That is, given any two curves which don't cross between $x = a$ on the left and $x = b$ on the right, we can calculate the area between the two curves on the interval $[a,b]$ as

$$Area = \int_{left}^{right} (upper\ curve - lower\ curve)dx$$

Notice that this definition and formula also encompass the area under or above a curve, by considering the other "curve" to be the x-axis.

Example 6.3. Find the area of the region which lies between the curve $y = \sqrt{x}$ and the line $y = \frac{1}{2}$ on $[1,4]$.

Solution:
We can graph the two functions to see the region whose area we want to find. The graph is shown in Figure 4.

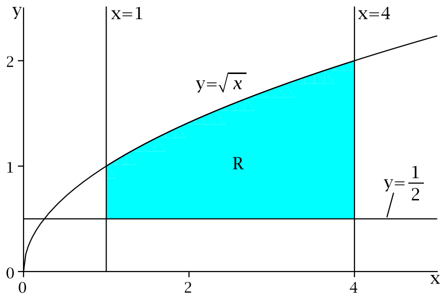


Figure 4: The region between $y = \sqrt{x}$ and $y = 1/2$ on the interval$[1,4]$

We see that the function $y = \sqrt{x}$ lies above the line $y = \frac{1}{2}$ everywhere on $[1,4]$, so the upper curve is $y = \sqrt{x}$ and the lower "curve" is $y = \frac{1}{2}$. We find the area using the formula from Theorem 6.2:

$$\begin{aligned}
\text{Area} &= \int_{\text{left}}^{\text{right}} (\text{upper curve} - \text{lower curve})dx &&= \int_1^4 \left(\sqrt{x} - \frac{1}{2}\right) dx \\
&= \int_1^4 \left(x^{1/2} - \frac{1}{2}\right) dx &&= \left[\frac{x^{3/2}}{3/2} - \left(\frac{1}{2}\right) x\right]_1^4 \\
&= \left[\left(\frac{2}{3}\right) x^{3/2} - \frac{x}{2}\right]_1^4 &&= \left[\left(\frac{2}{3}\right)(4^{3/2}) - \frac{4}{2}\right] - \left[\left(\frac{2}{3}\right)(1^{3/2}) - \frac{1}{2}\right]
\end{aligned}$$

$$\begin{aligned}
\text{So we have Area} &= \frac{2}{3}(\sqrt{4})^3 - 2 - \frac{2}{3}(1) + \frac{1}{2} &&= \frac{2}{3}(2^3 - 1) - 2 + \frac{1}{2} \\
&= \frac{2}{3}(8-1) - \left(2 - \frac{1}{2}\right) &&= \frac{14}{3} - \frac{3}{2} \\
&= \frac{28}{6} - \frac{9}{6} &&= \frac{19}{6}
\end{aligned}$$

Therefore the area of the region is $\frac{19}{6}$ square units.

When two curves intersect twice in an interval on which both are continuous, they *bound* a region between them. That is, there is a region whose boundary is everywhere composed of the two curves. We see a situation like this in Figure 5.

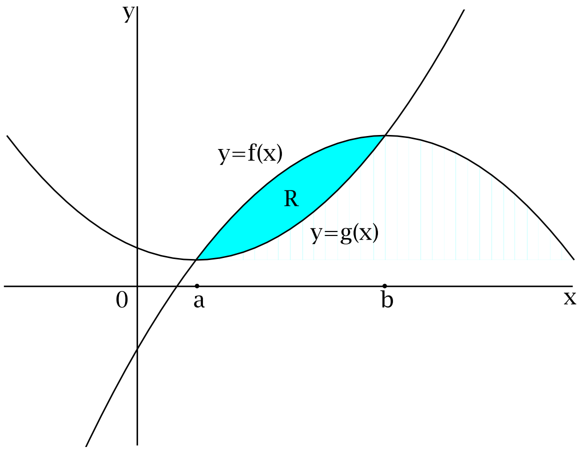


Figure 5: The region bounded by $y = f(x)$ and $y = g(x)$

Definition 6.3. If $y = f(x)$ and $y = g(x)$ intersect exactly twice, at $x = a$ and $x = b$ with $a < b$, and both f and g are continuous on $[a, b]$, then **the area (of the region) bounded by** $y = f(x)$ **and** $y = g(x)$ means the area between $y = f(x)$ and $y = g(x)$ on $[a, b]$.

Notice: We already know how to find this area, so we don't need a new formula. We simply use Area $= \int_a^b$ (upper curve - lower curve) dx.

Example 6.4. Find the area of the region bounded by the parabola $y = 5 - x^2$ and the line $y = 1$.

Solution:
To find the places at which two functions intersect, we equate them and solve for x. In this case we get:

$$5 - x^2 = 1 \quad \Rightarrow \quad -x^2 = 1 - 5 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$$

We see that the parabola and the line intersect in 2 places: once at $x = -2$ and again at $x = 2$. Notice that for any x-value between these values, i.e. any x such that $-2 < x < 2$, we have $x^2 < 4$ so that $-x^2 > -4$ and so $5 - x^2 > 5 - 4$. Therefore $5 - x^2 > 1$ throughout $[-2, 2]$, so $y = 5 - x^2$ is the upper curve and $y = 1$ is the lower curve, on $[-2, 2]$. The graph in Figure 6 (next page) shows the region bounded by these functions, which is the region which lies below the parabola and above the line, between $x = -2$ and $x = 2$.

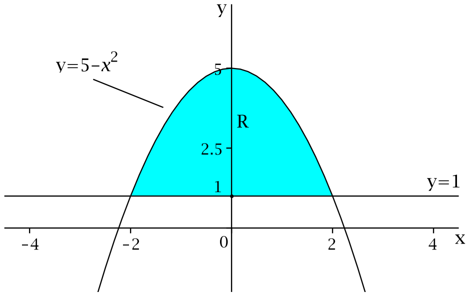


Figure 6: The region bounded by $y = 5 - x^2$ and $y = 1$

To find the area of this region, we simply evaluate $\int_{-2}^{2}(\text{upper curve} - \text{lower curve})dx$. We get:

$$\begin{aligned}
\text{Area} &= \int_{-2}^{2}[(5-x^2)-1]dx &&= \int_{-2}^{2}(4-x^2)dx \\
&= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} &&= \left(4(2) - \frac{(2)^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right) \\
&= 8 - \frac{8}{3} - \left[-8 - \left(\frac{-8}{3}\right)\right] &&= 8 - \frac{8}{3} + 8 - \frac{8}{3} \\
&= 16 - \frac{16}{3} &&= \frac{48-16}{3} = \frac{32}{3}
\end{aligned}$$

The area of the bounded region is $\frac{32}{3}$. Notice that we didn't actually need the graph for anything other than to display a picture of the region whose area we were finding. We found the relevant x-values and identified the upper curve before we looked at the graph.

Example 6.5. (a) Evaluate $\displaystyle\int_{-1}^{1} x^3 dx$.

Solution:

$$\int_{-1}^{1} x^3 dx = \left.\frac{x^4}{4}\right]_{-1}^{1} = \frac{(1)^4}{4} - \frac{(-1)^4}{4} = \frac{1}{4} - \frac{1}{4} = 0$$

Example 6.5. (b) Find the area between the curve $y = x^3$ and the x-axis on the interval $[-1, 1]$.

Solution:
We know that that the curve $y = x^3$ does not just run along the x-axis all the way from $x = -1$ to $x = 1$, and so there *is* some area. Therefore the area **is not** 0. What's going on here? Why doesn't $\displaystyle\int_{-1}^{1} x^3 dx$ give the area? Well, let's look at the graph. (See Figure 7, next page.)

Oh look! In fact, there are **two distinct subregions**, one below the x-axis and the other above, because the curve **crosses the axis** inside this interval. The area we want is actually the sum of the areas of these 2 distinct subregions. We must consider each subregion separately. Of course, it's at $x = 0$ that $y = x^3$ intersects $y = 0$, so the intervals defining the 2 regions are $[-1, 0]$ and $[0, 1]$.

On $[-1, 0]$ we have $x^3 \leq 0$, so the curve lies below the axis and it is the line $y = 0$ which is the upper curve. On the other hand, on $[0, 1]$ we have $x^3 \geq 0$, so the curve $y = x^3$ lies above the axis and is the upper curve. On each interval, we need to evaluate a definite integral in which the integrand

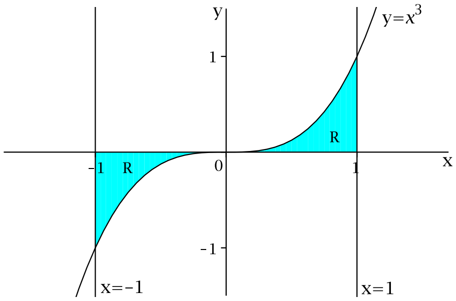


Figure 7: The graph of $y = x^3$ on $[-1, 1]$.

function has the form "upper curve − lower curve". We get:

$$\begin{aligned}\text{Total Area} &= \text{Area on } [-1,0] + \text{Area on } [0,1] &&= \int_{-1}^{0}(0-x^3)dx + \int_{0}^{1}(x^3-0)dx \\ &= -\int_{-1}^{0} x^3dx + \int_{0}^{1} x^3dx &&= -\left[\frac{x^4}{4}\right]_{-1}^{0} + \left[\frac{x^4}{4}\right]_{0}^{1} \\ &= -\left(\frac{0^4}{4} - \frac{(-1)^4}{4}\right) + \left(\frac{1^4}{4} - \frac{0^4}{4}\right) &&= -\left(0 - \frac{1}{4}\right) + \frac{1}{4} - 0 \\ &= \frac{1}{4} + \frac{1}{4} &&= \frac{1}{2}\end{aligned}$$

Notice that because one of the regions lies below the x-axis, to calculate the total area we needed to add $-\int_{-1}^{0} x^3dx$ and $\int_{0}^{1} x^3dx$. But in part (a) of this example, what we calculated was

$$\int_{-1}^{1} x^3dx = \int_{-1}^{0} x^3dx + \int_{0}^{1} x^3dx$$

and so we were adding $\int_{-1}^{0} x^3dx$ instead of $-\int_{-1}^{0} x^3dx$. That is, we were actually subtracting the area of the first region (i.e. area above $y = x^3$ on $[-1, 0]$) from the area of the second region (area below $y = x^3$ on $[0, 1]$). And because the 2 regions happen to have the same area, the answer was 0.

In general, $\int_{a}^{b}(f(x) - g(x))dx$ gives the **net area** below $y = f(x)$ and above $y = g(x)$ between $x = a$ and $x = b$. This is **not** the total area between $y = f(x)$ and $y = g(x)$ on $[a, b]$ **except in the specific circumstance** that $f(x) \geq g(x)$ throughout $[a, b]$. If $y = f(x)$ and $y = g(x)$ cross inside $[a, b]$ then there are 2 or more distinct subregions between $y = f(x)$ and $y = g(x)$ on $[a, b]$, and the area of each distinct subregion must be calculated separately.

Example 6.6. Find the area between the curve $y = x^2$ and the line $y = x$ on the interval $[0, 2]$.

Solution:
For the functions $f(x) = x^2$ and $g(x) = x$, setting $f(x) = g(x)$ gives

$$x^2 = x \quad\Rightarrow\quad x^2 - x = 0 \quad\Rightarrow\quad x(x-1) = 0 \quad\Rightarrow\quad x = 0 \text{ or } x = 1$$

The parabola and the line intersect once at $x = 0$ and again at $x = 1$. When 2 curves intersect outside, or at an endpoint of, the interval on which the area is being found, the intersection doesn't cause any problems, i.e. doesn't cause distinct regions which must be taken into account. So it doesn't matter that the parabola and the curve intersect at $x = 0$, which is an endpoint of $[0, 2]$. But the fact that they also intersect at $x = 1$, which lies inside $[0, 2]$, is important. This means that we need to consider two separate intervals, $[0, 1]$ and $[1, 2]$. We can see on the graph of these 2 functions (Figure 8) that there are, indeed, two distinct subregions whose areas must be calculated separately and added together.

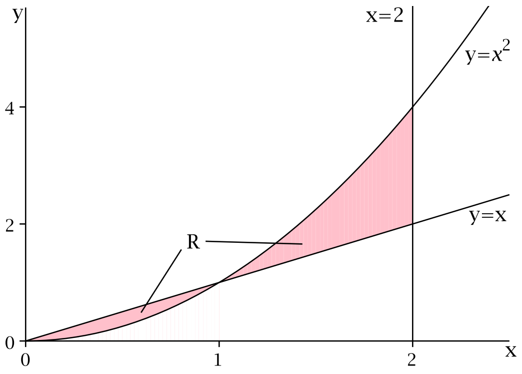


Figure 8: The region between $y = x^2$ and $y = x$ on $[0, 2]$.

Of course, to calculate the area of each subregion, we need to determine which is the upper curve in each region. Either by looking at the graph, or by realizing that for any x with $0 \leq x \leq 1$ we have $x^2 \leq x$ (e.g. for $x = \frac{1}{2}$, $x^2 = \frac{1}{4} < x$), while for any x such that $x \geq 1$ we have $x^2 \geq x$ (e.g. for $x = 2$, $x^2 = 4 > 2$), we can see that on $[0, 1]$ the upper curve is $y = x$ and the lower curve is $y = x^2$, but on $[1, 2]$ the upper curve is $y = x^2$ and the lower curve is $y = x$. We get:

$$\begin{aligned}
\text{Total Area} &= \text{Area on } [0,1] + \text{Area on } [1,2] \\
&= \int_0^1 (x - x^2)dx + \int_1^2 (x^2 - x)dx \\
&= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2 \\
&= \left[\left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right)\right] + \left[\left(\frac{2^3}{3} - \frac{2^2}{2}\right) - \left(\frac{1^3}{3} - \frac{1^2}{2}\right)\right] \\
&= \frac{1}{2} - \frac{1}{3} - 0 + \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2} \\
&= \frac{1}{2} + \frac{1}{2} + \frac{8}{3} - \frac{2}{3} - 2 = 1 + \frac{6}{3} - 2 = 1 + 2 - 2 = 1
\end{aligned}$$

Example 6.7. Find the area of the region(s) bounded by $y = x^3$ and $y = x$.

Solution:
We start by finding the places where $y = x^3$ and $y = x$ intersect:

$$x^3 = x \quad\Rightarrow\quad x^3 - x = 0 \quad\Rightarrow\quad x(x^2 - 1) = 0 \quad\Rightarrow\quad x(x+1)(x-1) = 0 \quad\Rightarrow\quad x = 0 \text{ or } x = -1 \text{ or } x = 1$$

We see that the functions intersect in 3 places: $x = -1$, $x = 0$ and $x = 1$. We cannot just consider the smallest and largest x-values at which they intersect, and consider the interval $[-1, 1]$, because they intersect, and in fact cross, *inside* this interval. We must consider two intervals, $[-1,0]$ and $[0,1]$, separately.

We need to determine the upper and lower curves on each of these intervals. To see which is the upper curve on $[-1,0]$, we can consider any x-value inside this interval and see which function has the larger value. For instance, at $x=-\frac{1}{2}$ we have $x^3=\left(-\frac{1}{2}\right)^3=-\frac{1}{8}>-\frac{1}{2}$, so $x^3>x$. (Remember, for negatives, bigger means closer to 0.) So on $[-1,0]$, we see that $y=x^3$ is the upper curve and $y=x$ is the lower curve. On the other hand, on $[0,1]$ we see that for $x=\frac{1}{2}$ we have $x^3=\frac{1}{8}<\frac{1}{2}$ and so $x>x^3$. Therefore on $[0,1]$, $y=x$ is the upper curve and $y=x^3$ is the lower curve. All of these findings are confirmed if we look at the graph.

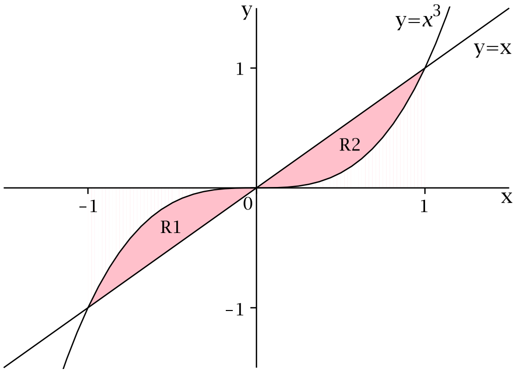


Figure 9: The region bounded by $y=x^3$ and $y=x$.

We find the area bounded by the curve and the line by adding the areas of the 2 subregions. We get:

$$\begin{aligned}
\text{Area} &= \text{Area on } [-1,0] + \text{Area on } [0,1] \\
&= \int_{-1}^{0}(x^3-x)dx+\int_{0}^{1}(x-x^3)dx \\
&= \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0}+\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{0}^{1} \\
&= \left[\left(\frac{0^4}{4}-\frac{0^2}{2}\right)-\left(\frac{(-1)^4}{4}-\frac{(-1)^2}{2}\right)\right]+\left[\left(\frac{1^2}{2}-\frac{1^4}{4}\right)-\left(\frac{0^2}{2}-\frac{0^4}{4}\right)\right] \\
&= \left[(0-0)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]+\left[\left(\frac{1}{2}-\frac{1}{4}\right)-(0-0)\right] \\
&= -\frac{1}{4}+\frac{1}{2}+\frac{1}{2}-\frac{1}{4}=\frac{1}{2}
\end{aligned}$$

The total area between $y=x^3$ and $y=x$ is $\frac{1}{2}$.

Notice: In this example, before looking at the graph, we determined which is the upper curve in each interval by realizing the following.

> **Fact:** If $y=f(x)$ and $y=g(x)$ do not intersect in (a,b), then whichever curve has the larger value *anywhere* inside (a,b) must have the larger value *everywhere* inside (a,b).

That is, if we know that the 2 curves do not intersect in (a,b), then we can deteremine which is the upper curve on $[a,b]$ by choosing any value $c\in(a,b)$, i.e. any c with $a<c<b$, and evaluating $f(c)$ and $g(c)$. Whichever function has the larger value at c is the upper curve on $[a,b]$.

Horizontal Slicing

At the beginning of our discussion of finding areas with definite integrals, it was mentioned in passing that in a more rigorous study of integration, a definite integral is *defined* as the sum of the areas of infinitely many infinitessimally small "approximating rectangles". That is, integrating is effectively the same as adding, except that instead of summing a finite number of identifiable pieces, it's taken to the extreme, with each piece being too small to quantify, and there being uncountably many of them. (This leads to some quite complicated work with limits, which is why we don't bother with that here.) This "explains" why finding areas involves evaluating definite integrals — effectively, we're finding and summing the areas of uncountably many invisibly small rectangles which fill the region. As stated previously, we don't need to concern ourselves with that definition of the definite integral. However, the idea of using approximating rectangles to find areas is useful in understanding what we need to learn about next. These approximating rectangles can be thought of as *slices* of the region.

In the area problems we've done so far, we were slicing the region vertically, even though we didn't realize it. Consider finding the area between two curves on some interval. Think about what a vertical slice of the region looks like. Figure 10 shows such a slice. Let the width of the slice be dx, some tiny "change in x" value so small that no matter how curvy the functions are, the slice has top and bottom edges which are approximately horizontal. If $f(x)$ is the function defining the upper boundary of the region, and $g(x)$ is the function defining the lower boundary of the region, then the slice of the region which goes from some value x on the left to the value $x + dx$ on the right is approximately a rectangle with width dx and height $f(x) - g(x)$. The area of this extremely skinny rectangle is therefore $(f(x) - g(x))dx$. Look familiar? The definite integral $\int_a^b (f(x) - g(x))dx$ can be thought of as saying "add up all the areas of slices like this, for x-values going from a up to b".

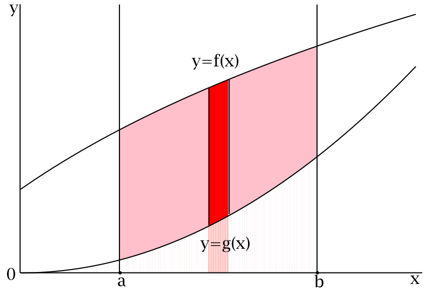


Figure 10: A vertical slice of the region between $y = f(x)$ and $y = g(x)$.

Another approach for finding the area of a region would be to add up a bunch of horizontal slices instead of a bunch of vertical slices. With the vertical slices, each slice runs from the bottom boundary to the top boundary, and the slices fill the whole region, from the left boundary all the way over to the right boundary. To slice horizontally, we would take a slice from the left boundary to the right boundary, and have slices that fill in the whole region, from the bottom boundary all the way up to the top boundary. That is, instead of considering two functions of x forming the top and bottom boundaries, and considering x-values going from the smallest x-value in the region (on the left) to the largest (on the right), we could consider two functions of y forming the left and right boundaries, and consider y-values going from the smallest y-value in the region (at the bottom) to the largest (at the top). Figure 11 (next page) shows two functions of y and a horizontal slice of the region between them.

So why would we want to do this? Well, in some situations, it's easier to find an area by expressing the curves in the form $x = f(y)$ and integrating with respect to y, instead of with respect to x. This is generally true when the region has more than one function of x defining it's upper boundary,

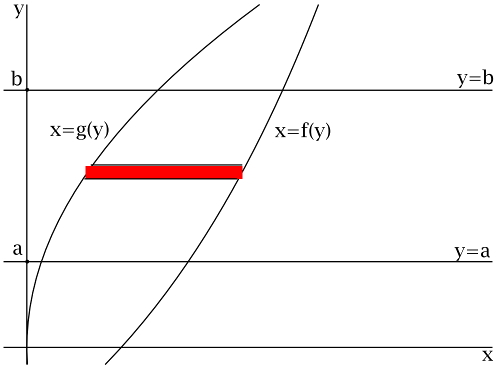


Figure 11: A horizontal slice of the region between $x = f(y)$ and $x = g(y)$.

and/or more than one function of x defining it's lower boundary, but has only one function of y defining each of the left and right boundaries. See for instance Example 6.9. But it's also true when the integration is simply easier when working with functions of y than when working with functions of x, such as for instance when expressing things in terms of x requires the substitution rule in order to evaluate an integral, but expressing things in terms of y doesn't. This is the situation in Example 6.8.

Before we look at examples, though, let's think about *how* we do it. Switching our focus from vertical to horizontal means switching which is the dependent variable. Instead of expressing y in terms of x, we express x in terms of y. That is, the roles of x and y are switched. With things expressed in terms of x, when the value of $y = f(x)$ gets larger it moves in the positive direction of y, so it is higher up. When we express things in terms of y, having a larger value for $x = f(y)$ means it moves in the positive direction of x, so it is farther to the right. So where we use "upper" and "lower" with vertical slicing, we need to use "rightmost" and "leftmost", respectively, for horizontal slicing. And whereas intervals of x-values run left-to-right on the graph, intervals of y-values run bottom-to-top. For actually expressing the formula we need to use, things don't look much different, except for the fact that we're expressing x as a function of y and we integrate with respect to y. But for looking at the graph and knowing how to set things up, it's important to remember these things. The formula for finding areas using horizontal slicing is given in a theorem that looks very much like Theorem 6.2.

Theorem 6.3. *If $f(y) \geq g(y)$ on $[a, b]$ then the area between $x = f(y)$ and $x = g(y)$ on $[a, b]$ is given by*

$$Area = \int_a^b (f(y) - g(y))dy$$

That is, given any two curves, expressed as functions of y which don't cross in the interval of y-values $[a, b]$ we can calculate the area between the two curves from $y = a$ at the bottom to $y = b$ at the top as

$$Area = \int_{bottom}^{top} (rightmost\ curve - leftmost\ curve)dx$$

Example 6.8. Find the area of the region bounded by $y^2 = 4 - x$ and $x = 0$.

Solution:
First, let's look at the graph of $y^2 = 4 - x$ and the region whose area we need to find. (Of course, the line $x = 0$ is just the y-axis.)

Notice: $y^2 = 4 - x$ is not a *function* of x. This *relation* has 2 different y-values for each x-value. (Well, for many of them, anyway.) A relation is only a function (of x) if each x-value has a unique

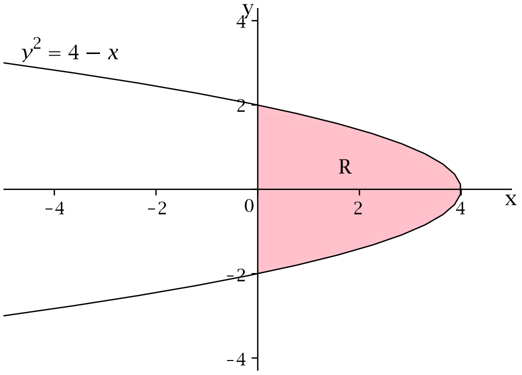


Figure 12: (a) The region bounded by $y^2 = 4 - x$ and $x = 0$.

y-value. Actually, the relation $y^2 = 4 - x$ is two different functions of x, the upper half-parabola and the lower half-parabola. That is, we have the two functions $y = \sqrt{4-x}$ and $y = -\sqrt{4-x}$.

Having identified these two functions, we could proceed in the usual way, integrating with respect to x, i.e. using vertical slicing. Let's see how that works.

Approach 1: Using vertical slicing
First, let's look at a vertical slice on the graph.

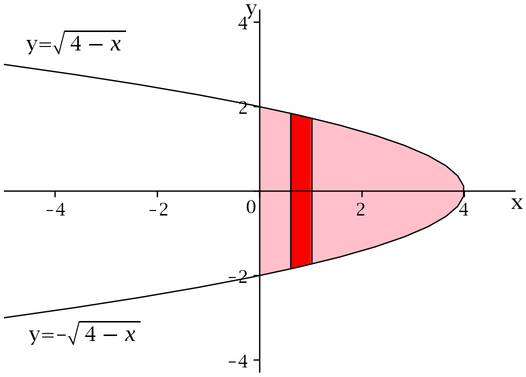


Figure 12: (b) A vertical slice of the region between $y = \sqrt{4-x}$ and $y = -\sqrt{4-x}$.

Considered vertically, the region lies between some upper curve and some lower curve. In this case, the upper curve (i.e. the upper boundary of the region) is the upper half of the parabola, $y = \sqrt{4-x}$, and the lower curve (the lower boundary of the region) is the lower half of the parabola, $y = -\sqrt{4-x}$. And the smallest x-value in the region is $x = 0$ on the left, while the largest is $x = 4$ on the right. That is, we consider the region to be the region between $y = \sqrt{4-x}$ and $y = -\sqrt{4-x}$ on the interval $[0, 4]$. Using our usual formula, we have:

$$\begin{aligned} \text{Area} &= \int_{\text{left}}^{\text{right}} (\text{upper curve} - \text{lower curve})dx = \int_0^4 [\sqrt{4-x} - (-\sqrt{4-x})]dx \\ &= \int_0^4 (\sqrt{4-x} + \sqrt{4-x})dx = \int_0^4 2\sqrt{4-x}\,dx = 2\int_0^4 \sqrt{4-x}\,dx \end{aligned}$$

Now to evaluate this integral, we need to recognize that the integrand is a composite function, so we need a substitution. Letting $u = 4 - x$ we get $du = -dx$, so $dx = -du$. When $x = 0$ we get

$u = 4 - 0 = 4$ and when $x = 4$ we have $u = 4 - 4 = 0$. Therefore we have

$$\begin{aligned}\text{Area} &= 2\int_0^4 \sqrt{4-x}\,dx = 2\int_4^0 \sqrt{u}(-1)du = -2\int_4^0 \sqrt{u}du = 2\int_0^4 u^{1/2}du \\ &= 2\left[\frac{u^{3/2}}{3/2}\right]_0^4 = 2\left[\left(\frac{2}{3}\right)u^{3/2}\right]_0^4 = \left(\frac{4}{3}\right)\left(4^{3/2} - 0^{3/2}\right) = \left(\frac{4}{3}\right)(\sqrt{4})^3 \\ &= \frac{4(2^3)}{3} = \frac{32}{3}\end{aligned}$$

So we see that the area of the region bounded by $y^2 = 4 - x$ and $x = 0$ is $\frac{32}{3}$ (square units).

Approach 2: Use horizontal slicing

We were asked to find the area of the region bounded by $y^2 = 4 - x$ and $x = 0$. Both of these are actually functions of y. That is, with some minor rearranging we see that $y^2 = 4 - x$ can be expressed as $x = 4 - y^2$, and the other boundary is already in "x equals" form. In fact, looking back at our original graph (see Figure 12(a)) we notice that throughout the region, the right boundary is the parabola $x = 4 - y^2$ (although there it was labelled $y^2 = 4 - x$) and the left boundary is the line $x = 0$. So finding the area of the region will be easier using horizontal slicing. Considering the region in this way, we get slices as shown in Figure 12(c). Of course, we need to know what interval of y-values we're dealing with. This is given by the y-values at which the parabola and the line intersect. We can easily determine that $4 - y^2 = 0$ when $y = \pm 2$, so the interval of y-values is $[-2, 2]$, i.e. the region runs between $y = -2$ at the bottom and $y = 2$ at the top.

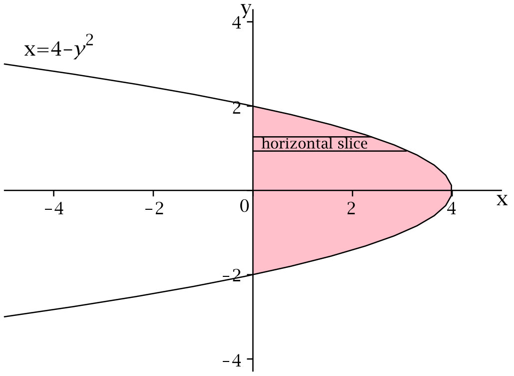


Figure 12: (c) A horizontal slice of the region between $x = 4 - y^2$ and $x = 0$.

Now, using Theorem 6.3 we get

$$\begin{aligned}\text{Area} &= \int_{\text{bottom}}^{\text{top}} (\text{rightmost} - \text{leftmost})dy = \int_{-2}^{2} [(4 - y^2) - 0]dy \\ &= \int_{-2}^{2} (4 - y^2)dy = \left[4y - \frac{y^3}{3}\right]_{-2}^{2} = \left(4(2) - \frac{2^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right) \\ &= \left(8 - \frac{8}{3}\right) - \left(-8 - \frac{-8}{3}\right) = 8 - \frac{8}{3} + 8 - \frac{8}{3} = 16 - \frac{16}{3} = \frac{32}{3}\end{aligned}$$

We got to the answer more easily this way, since we didn't need to use the substitution rule.

Example 6.9. Find the area of the region bounded by $y^2 = 4 - x$ and $y = 2 - x$.

Solution:
This is somewhat similar to the previous problem. It involves the same parabola, but a different line. However, using vertical slicing will be even more complicated in this situation than it was before. To see why, let's look at a graph of the region. (See Figure 13(a) next page.)

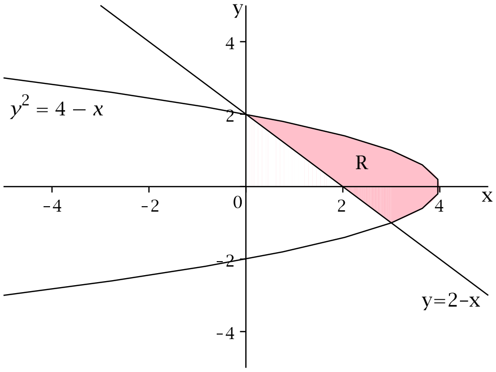


Figure 13: (a) The region bounded by $y^2 = 4 - x$ and $y = 2 - x$.

Notice the upper and lower boundaries of the region. The upper boundary is always the parabola, i.e. the upper half-parabola $y = \sqrt{4-x}$, as before. But the lower boundary ... that's different. In part of the region (at the right end), the lower boundary is the lower half-parabola $y = -\sqrt{4-x}$ like we had last time. But through most of the region, the lower boundary is the line $y = 2 - x$. If we consider a vertical slice of the region, the lower curve for calculating the height of the slice depends on what part of the region we're in. That is, there are actually 2 different subregions, R_1 and R_2, which we must consider separately (see Figure 13(b)).

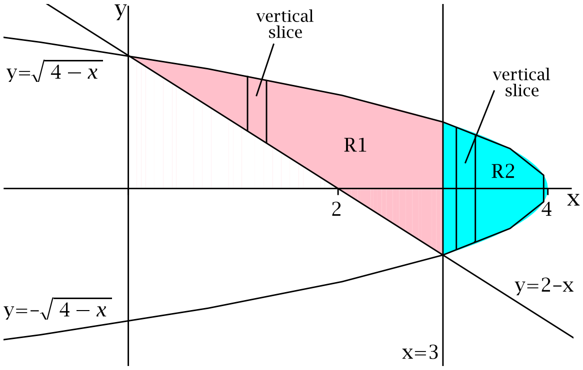


Figure 13: (b) For a vertical slice, the lower curve depends on where the slice is taken.

To find the area of the region bounded by $y^2 = 4 - x$ and $y = 2 - x$ using vertical slicing will be a lot of work. First, we need to determine the x-value which is the boundary between the two subregions. We need to find the x-values at which the parabola and the line intersect. When $y = 2 - x$ we have $y^2 = (2-x)^2 = 4 - 4x + x^2$. The curves $y^2 = 4 - x$ and $y^2 = 4 - 4x + x^2$ intersect when

$$4-x = 4-4x+x^2 \quad \Rightarrow \quad 4-x-4+4x-x^2 = 0 \quad \Rightarrow \quad 3x-x^2 = 0 \quad \Rightarrow \quad x(3-x) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = 3$$

We see that the rightmost x-value at which they intersect, which is the one at which the lower boundary changes from being the line to being the parabola, is $x = 3$. That is, R_1 runs from $x = 0$

to $x = 3$, and R_2 runs from $x = 3$ to $x = 4$. So the total area of the bounded region is the area between the upper half-parabola and the line, from $x = 0$ to $x = 3$, plus the area between the upper and lower half-parabolas, from $x = 3$ to $x = 4$. So we need to evaluate

$$\text{Area} = \text{Area of } R_1 + \text{Area of } R_2 = \int_0^3 [\sqrt{4-x} - (2-x)]dx + \int_3^4 [\sqrt{4-x} - (-\sqrt{4-x})]dx$$

We need to evaluate 2 definite integrals, and each will be quite a bit of work, requiring the substitution rule ... goodness!

On the other hand, look again at Figure 13(a), and think about the left and right boundaries of the region. The left boundary is always the line, and the right boundary is always the parabola. All we really need to do here is express each in "x equals" form and use horizontal slicing. As in the previous example, the parabola can be expressed as $x = 4 - y^2$, and the line $y = 2 - x$ could just as well be stated as the line $x = 2 - y$. So the region can be said to run from $x = 2 - y$ on the left to $x = 4 - y^2$ on the right, as shown in Figure 13(c). Of course, we need to find the interval of y-values defining the region. Equating $4 - y^2$ and $2 - y$ we get

$$4-y^2 = 2-y \quad\Rightarrow\quad 4-y^2-2+y = 0 \quad\Rightarrow\quad 2+y-y^2 = 0 \quad\Rightarrow\quad (2-y)(1+y) = 0 \quad\Rightarrow\quad y = 2 \text{ or } y = -1$$

We see that the line and the parabola intersect at $y = -1$ at the bottom and at $y = 2$ at the top, so the region runs over the interval of y-values $[-1, 2]$.

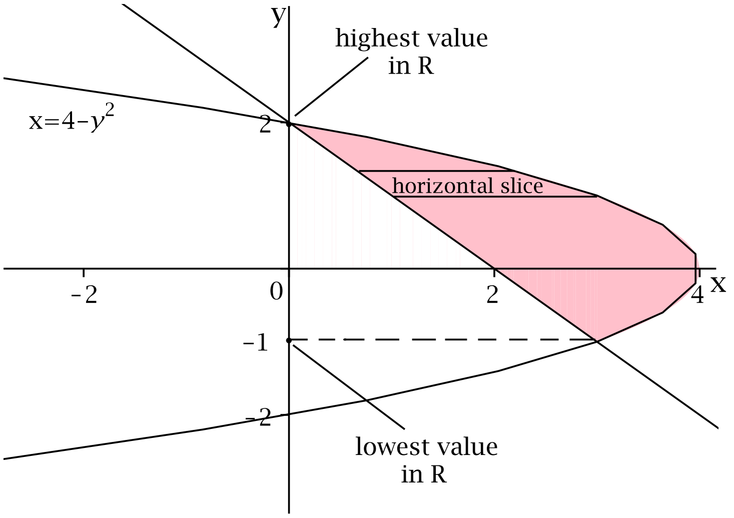


Figure 13: (c) A horizontal slice has the same left and right boundaries no matter where it is.

Notice that even if we weren't looking at the graph we could still easily determine that the parabola lies to the right of the line throughout this interval. Since we know that they don't intersect anywhere within the interval, we can just check any y-value inside the interval to see which function has the larger value. For instance, $y = 0$ is in the interval $[-1, 2]$. When $y = 0$, the curve $x = 4 - y^2$ has $x = 4 - y^2 = 4 - 0^2 = 4$, while the line $x = 2 - y$ has $x = 2 - y = 2 - 0 = 2$ so we see that the parabola has the larger x-value, i.e. is further to the right. We use the formula from

Theorem 6.3 again to get:

$$\begin{aligned}
\text{Area} &= \int_{\text{bottom}}^{\text{top}} (\text{rightmost} - \text{leftmost})dy && = \int_{-1}^{2} [(4-y^2)-(2-y)]dy \\
&= \int_{-1}^{2} (2+y-y^2)dy && = \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^{2} \\
&= \left(2(2) + \frac{2^2}{2} - \frac{2^3}{3}\right) - \left(2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3}\right) && = \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} - \frac{-1}{3}\right) \\
&= 6 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3} && = 8 - \frac{9}{3} - \frac{1}{2} \\
&= 5 - \frac{1}{2} && = \frac{9}{2}
\end{aligned}$$

This was much easier than evaluating the integrals necessary for finding the answer by vertical slicing. We see that the area of the region bounded by $y^2 = 4 - x$ and $y = 2 - x$ is 4.5 square units.

Math 1225A/B

Unit 7:
Volumes of Revolution

(text reference: Section 5.2

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7 Volumes of Revolution

Next, we want to think about a new kind of problem, which is another application of definite integrals. The phrase *volume of revolution* refers to the volume of a solid whose shape can be generated by revolving some planar region about some axis in three-space. That is, in this kind of problem, we have some region in "the plane", such as a particular region under a curve $y = f(x)$ or a particular region between two curves $y = f(x)$ and $y = g(x)$. But we are not just working in this plane, because we use a third dimension to revolve this region about a line. Therefore the region moves through a third dimension, going in a circle around the line, while keeping the distance between that line and the furthest edge of the region fixed.

There are 2 different situations which can arise. In the first kind of problem, the *axis of rotation*, i.e. the line about which the region is to be revolved, is a boundary of that region. So when the region revolves, that boundary actually stays in place, but turns. Like when a roast is turned on a spit. The spit stays where it is, but turns. And the roast is touching the spit everywhere (along the length of the roast), so there's no space between the roast and the spit. Now forget the roast. Imagine instead that you've got an irregularly shaped thin plank of wood, with one flat edge, and you glue the flat edge of the thin piece of wood to the spit. And imagine the spit being infinitely thin, like a line in space, which has no depth to it, and likewise the piece of wood is impossibly thin (but still stiff, so that it doesn't flop but maintains its shape). And now turn the spit, so that the wood revolves through space. As the wood revolves around the spit, it sweeps out a 3-dimensional region corresponding to some solid. For instance, if the wood isn't actually very irregular, but is simply a semi-circle, then as it goes around it sweeps out a region corresponding to a sphere. It's the volume of this region that's swept out by the wood as it revolves, i.e. of the solid it would generate if it could be in all those positions at once, that we want to find.

The other situation arises when the axis of rotation is not a boundary of the region being revolved. They may touch in one spot, or the region may everywhere be at some distance from the axis. So now we still have the spit, but the piece of wood isn't glued along the flat edge. It may not even have a flat edge. But you've found some way (don't ask me how) to "attach" it to the spit. Perhaps with one corner glued to the spit (securely enough to support the whole piece of wood). Or with several spots along a non-flat edge glued to the spit. Or even with the wood not even touching the spit, but somehow affixed in place, in a way that will cause it to turn when the spit turns. (That's the part where it's hard to imagine *how* it's attached. Maybe the spit is really the USS Enterprise and has a tractor beam directed at the wood, but the wood, which is really a Klingon ship, has engine burners turned on exerting just enough force to maintain its distance from the Enterprise, but the Enterprise spins around and as it does so, the tractor beam pulls the Klingon ship around with it ... ? ... But we don't care about the how. Just imagine that as the spit rotates, the piece of wood revolves around it.)

In this situation, when you turn the spit and the wood revolves, it again sweeps out a solid region. But this time that "solid" includes a hole, where the region *between* the wood and the spit went around. In the simplest case, where the wood is a rectangular strip which has been "attached" parallel to (but not touching) the spit, the "solid" that's been swept out is a hollow cylinder. The cylinder has substance, where the actual piece of wood moved through space. But then it's hollow inside, where the space between the wood and the spit was, as it went around.

So the two different situations can be described as follows: The region, which has the axis of rotation as a boundary, sweeps out a "solid solid" as it is revolved, i.e. a solid which doesn't have any hole(s) in it. Or the region, which does not have the axis of rotation as a boundary, sweeps out a "solid" which has a hole in it, i.e. a "solid" which is hollow, even if there's one (or more) place(s) where the hole goes away (like a cone, which is hollow, but *is* closed at one end, or a basketball, which is hollow but has no visible hole in it). We start by seeing how to find the volume in the "solid solid" situation. That is, first we'll look at the situation in which a region which has the axis of rotation as a boundary is revolved about that axis.

For instance, consider the region R shown in Figure 1. We want to learn how to determine the volume of the solid generated when this region is revolved about the x-axis.

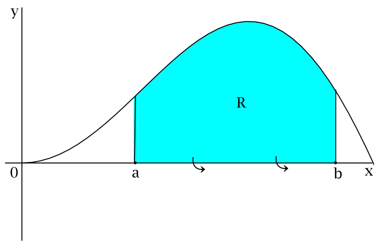


Figure 1: Region R will be revolved about the x-axis.

Before we do that, recall that in the previous unit about finding the area of a region, in order to understand that we had been "slicing" the region vertically and could instead slice horizontally, we talked (briefly) about the idea that evaluating a definite integral corresponds to adding up an infinite number of infinitely small objects which are "approximating rectangles". We use this concept again to understand how to use a definite integral to find the volume of a solid of the type described above. If we can set things up so that a solid of revolution is generated by revolving the region which lies between the graph of some continuous function $f(x)$ and some horizontal line, such as the x-axis, on an interval $[a, b]$, with the axis of rotation being that line, we can then calculate the volume quite easily. We do this by considering the volume of the solid obtained when a small slice of the region is rotated about the line. That is, we revolve an "approximating rectangle" about the axis to generate a small section of the solid which is being generated.

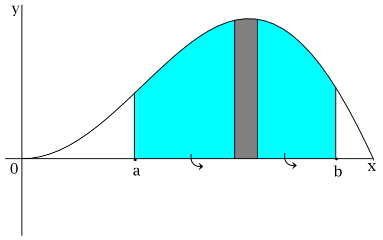
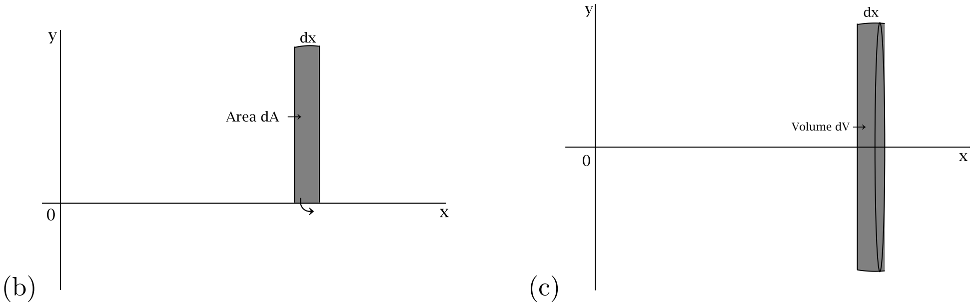


Figure 2: (a) Consider what happens when a slice of the region is revolved about the x-axis.

In calculating area with vertical slices, or strips, we called the width of the slice (strip) dx. It represents the change in the value of x corresponding to this particular slice. Likewise, we can call the *area* of the slice dA, representing the change in area resulting from including this particular slice in the region whose area is being calculated. Similarly, revolving the slice about the axis of rotation produces a *disk* with a particular volume, so including this particular slice causes a change in the volume of the solid, and we can refer to the volume of the disk generated by the slice as dV.



(b) (c)

Figure 2: When we revolve a slice with area dA, it generates a solid disk with volume dV.

But the slice was just one piece of the region which was to be revolved, so this disk is just one piece of the solid which is generated.

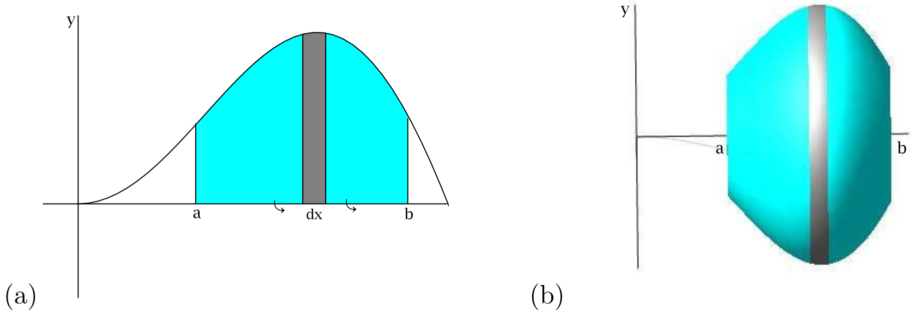


Figure 3: When we revolve the whole region, the solid generated includes the disk swept out by the slice.

We find the total volume of the entire solid of revolution by adding up the volumes of solids obtained by rotating infinitely many of these small slices, where each slice is infinitely small in width. That is, we find the total volume of the solid by evaluating a definite integral, $V = \int_a^b dV$.

We use the same general approach even when the solid of revolution is more complicated than the simple case described above. The procedure for finding the volume of *any* solid of revolution (whether the region has the axis of rotation as a boundary or not) can be expressed quite easily:

Theorem 7.1. Procedure for Finding the Volume of a Solid of Revolution:

1. *Sketch the region to be revolved and the axis of rotation.*
2. *Take a slice of the region perpendicular to the axis of rotation. Revolve the slice about the axis and calculate the volume of the solid that it generates, and call that volume dV.*
3. *Integrate dV, over the whole of the region (i.e. from one end of the region to the other), to find the volume of the entire solid.*

The Method of Disks

In the simplest case, which we have already described, the region being rotated lies between some function $f(x)$ and a horizontal line, such as the x-axis, and we are rotating this region about that line. Referring to Figure 2(b) again, we see that the small slice (which is vertical so that it is perpendicular to the axis of rotation) is approximately a rectangle with height $f(x)$ and width dx, and so the area of the slice is $dA = f(x)dx$. When this rectangle is rotated about the x-axis, it produces a disk-shaped solid, that is, a very small (sideways) cylinder (or disk), with radius $r = f(x)$ and (sideways) height $h = dx$ (i.e. the width of the slice). Of course, the volume of a cylinder of radius r and height h is $\pi r^2 h$. So we have the volume of the disk being

$$dV = \pi r^2 h = \pi\,(f(x))^2\,dx$$

Thus, we see that the volume of the solid obtained by rotating the entire region (which runs from $x = a$ to $x = b$) about the x-axis is given by

$$V = \int_a^b dV = \int_a^b \pi r^2 h = \int_a^b \pi\,(f(x))^2\,dx = \pi \int_a^b (f(x))^2\,dx$$

Since the solid generated by the small slice of the region has the shape of a disk, this approach is known as the **method of disks**. This method is used whenever the axis of rotation is one of the boundaries of the region being revolved.

Theorem 7.2. The Method of Disks
Consider the region lying between the curve $y = f(x)$ and the line $y = c$, from $x = a$ to $x = b$. When this region is revolved about the line $y = c$, the volume of the resulting solid of revolution is given by

$$V = \pi \int_a^b r^2 dx$$

where r is the distance of the boundary curve $y = f(x)$ from the axis of rotation, $y = c$, so that $r = |f(x) - c|$

Notice: When the axis of rotation is the x-axis, i.e. the line $y = 0$, and the curve $y = f(x)$ lies above it, we have $r = |f(x) - c| = |f(x) - 0| = f(x)$, so we get our previous result:

$$V = \int_a^b \pi r^2 dx = \int_a^b \pi \left(f(x) - 0\right)^2 dx = \int_a^b \pi \left(f(x)\right)^2 dx$$

Example 7.1. Consider the region R bounded by the line $y = 3 - 2x$, the x-axis and the y-axis. Find the volume of the solid generated when region R is revolved about the x-axis.

Solution:
We start by sketching the region R, and considering a vertical slice of the region, with width dx.

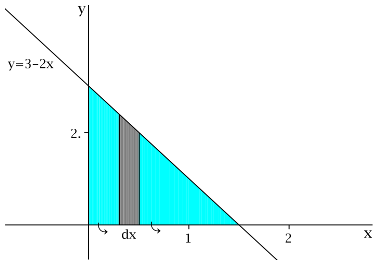


Figure 4: (a) The region R, and a slice of it.

When this triangular region is revolved about the x-axis, it will generate a cone-shaped solid, as shown here. Notice that the slice shown in Figure 4(a) has swept out a disk, which is part of the cone.

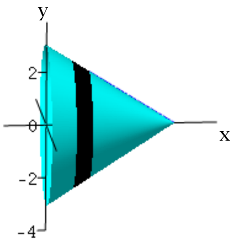


Figure 4: (b) The triangular region will generate a cone-shaped solid of revolution.

Let's think more carefully about that disk. The radius of the disk is the same as the height of the slice – which is just the distance between the top boundary of the region and the x-axis. And since the top boundary of the region is the line $y = 3 - 2x$, then that height is given by $f(x) = 3 - 2x$, so the radius of the disk is $r = 3 - 2x$.

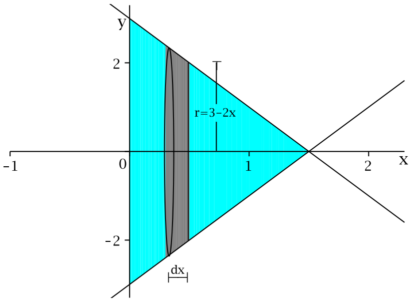


Figure 4: (c) The disk generated by the slice has radius $r = 3 - 2x$.

This disk with radius $r = 3-2x$ and "sideways height" dx has volume $dV = \pi r^2 dx = \pi(3-2x)^2 dx$. When we rotate the entire region R about the x-axis, we are revolving infinitely many similar infinitely thin slices to produce infinitely many thin disks. The slices start at the left end of the region, at $x = 0$, and run across the region (from left to right) to the right edge of the region, which is where the line $y = 3 - 2x$ intersects the x-axis, at height 0, which is at $x = \frac{3}{2}$. Therefore we have slices of the region running from $x = 0$ to $x = \frac{3}{2}$, so the volume of the solid is given by $V = \int_0^{3/2} dV = \pi \int_0^{3/2} r^2 dx = \pi \int_0^{3/2} (3 - 2x)^2 dx$. We get:

$$\begin{aligned} V &= \pi \int_0^{3/2} (3-2x)^2 dx = \pi \int_0^{3/2} (9 - 12x + 4x^2)\, dx = \pi \left[9x - 6x^2 + \frac{4x^3}{3} \right]_0^{3/2} \\ &= \pi \left[9\left(\frac{3}{2}\right) - 6\left(\frac{3}{2}\right)^2 + \frac{4}{3}\left(\frac{3}{2}\right)^2 \right] - \pi(0) = \pi \left[\frac{27}{2} - \frac{27}{2} + \frac{9}{2} \right] = \frac{9\pi}{2} \end{aligned}$$

We see that the volume of the cone generated by revolving region R about the x-axis is $\frac{9\pi}{2}$ cubic units.

Example 7.2. Find the volume of the solid generated when the region which lies below the curve $y = x^2$, from $x = 0$ to $x = 2$, is revolved about the x-axis.

Solution: As before, we can draw the region which is to be revolved, and think about what happens when a slice of the region is revolved about the x-axis. As we see below, a slice of the region has height $f(x) = x^2$, and sweeps out a disk of radius $r = f(x) = x^2$.

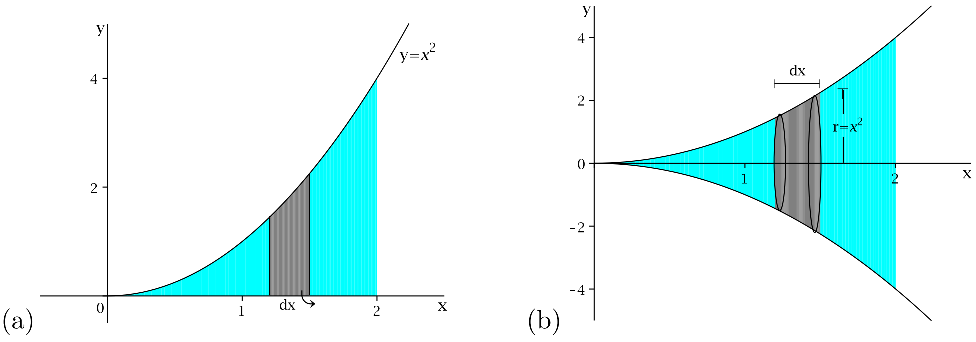


Figure 5: A vertical slice of the region has height x^2, which is also the radius of the disk.

Thus the volume of the disk is $dV = \pi r^2 dx = \pi(x^2)^2 dx = \pi x^4 dx$. Considering slices running from $x = 0$ to $x = 2$, we see that the volume of the whole solid generated is

$$V = \int_0^2 dV = \pi \int_0^2 x^4 dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \pi \left[\frac{2^5}{5} - 0\right] = \frac{32\pi}{5}$$

The solid generated is shown below. This solid has volume $\frac{32\pi}{5}$ cubic units.

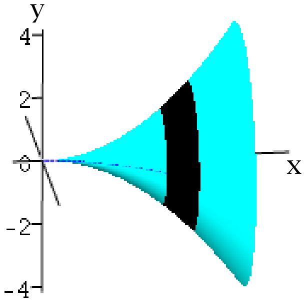


Figure 5: (c) The solid whose volume we have just found.

We don't actually need pictures to do this sort of thing. Of course, it can help, but we just need to know the procedure for using the Method of Disks, and carry it through. If we have a horizontal axis of rotation, we take a vertical slice of the region, so that the slice has height $|f(x) - c|$, where $y = f(x)$ is the boundary of the region (away from the axis of rotation) and $y = c$ is the axis of rotation. The disk generated by a slice of width dx has radius equal to the height of the slice and the volume of the disk is $dV = \pi r^2 dx$. We find the volume of the whole solid generated by integrating dV from the left end of the region to the right end of the region. So if the region lies between $x = a$ and $x = b$, we have $V = \pi \int_a^b r^2 dx$.

Example 7.3. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$ about the x-axis.

Solution:
The axis of rotation is the x-axis, i.e. the line $y = 0$, which is horizontal. The region lies below $y = \sqrt{x}$, so the boundary function is $f(x) = \sqrt{x}$. A slice of width dx has height $y = f(x) - 0 = \sqrt{x}$, which is the radius of the disk generated by revolving the slice, so we have $r = \sqrt{x}$.

The region runs from the intersection of $y = \sqrt{x}$ with $y = 0$, at $x = 0$, to $x = 4$. (That is, the region has upper boundary $y = \sqrt{x}$, lower boundary $y = 0$, left boundary $x = 0$ and right boundary $x = 4$.) When the region is revolved about the x-axis (which is the lower boundary), a slice of height $f(x) = \sqrt{x}$ and width dx produces a washer which has radius $r = f(x)$ and volume $dV = \pi r^2 dx$. So when the entire region from $x = 0$ to $x = 4$ is revolved, the solid generated has volume

$$V = \int_0^4 \pi \left(\sqrt{x}\right)^2 dx = \pi \int_0^4 x dx = \pi \left[\frac{x^2}{2}\right]_0^4 = \pi \left(\frac{16}{2} - \frac{0}{2}\right) = 8\pi$$

The Method of Washers

The method of disks is, as previously stated, used whenever the axis of rotation is a boundary of the region being rotated. In other situations, as we have already discussed, the region being revolved may be at some distance from the axis of rotation, in which case the solid of revolution has a "hole" in the centre. In this kind of situation, the "disk" generated by revolving a slice of the region about the axis of rotation has a hole in it, caused by the fact that there is space between the slice and the axis. So when the slice of the region is revolved, it isn't actually a disk that's generated, but rather a *washer*. And a washer has 2 different radii associated with it. There's the radius of the whole washer, which we call the **outer radius**, and also the radius of the hole, which we refer to as the **inner radius**. When dealing with this kind of situation, we always use R to denote the outer radius and r to denote the inner radius.

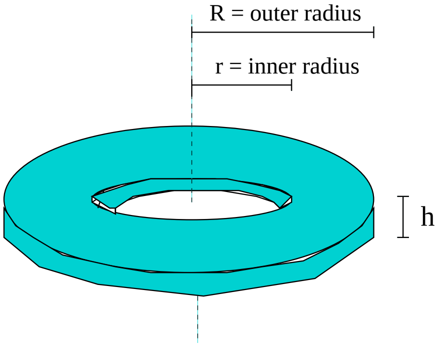


Figure 6: The washer generated when a slice of a region which doesn't touch the axis of roation is revolved about that axis.

Figure 6 shows a "washer", obtained by rotating a strip about an axis of rotation which it doesn't touch. The washer in the picture has inner radius r (the radius of the hole), outer radius R, and height h. *Note:* In this figure, either a *horizontal* slice was revolved about a *vertical* axis, or else somebody has knocked the washer over, so that the 'height' is actually height, rather than being 'sideways height'. A washer produced by revolving a vertical slice about a horizontal axis would be standing on edge, like the disk was.

In this kind of situation, we follow the same basic steps as before, as outlined in Theorem 7.1 on page 99. However in Step 2, the calculation of dV, the volume of the solid generated by the slice, is different than what we did in the method of disks, because of the hole. The hole doesn't count (i.e. isn't solid and doesn't contribute to the volume) and so we have to subtract the volume of the hole from the volume of what would have been a disk. And of course the hole is just a smaller disk. It's as if you had a disk, and someone came along and cut a smaller disk out of the centre of it. What you'd be left with is a washer. Of course, if we had a solid disk of radius R (the outer radius of the washer), its volume would be $\pi R^2 h$. And the smaller disk of radius r, which has been removed, leaving the hole in the washer, has volume $\pi r^2 h$. So the volume of the washer is

$$dV = \pi R^2 h - \pi r^2 h = \pi(R^2 - r^2)h$$

When the axis of rotation is horizontal, so that the slice is vertical with width dx (giving the 'height' of the washer as $h = dx$), and the region runs from $x = a$ to $x = b$, then the volume of the solid generated when the whole region is revolved about the axis is given by

$$V = \int_a^b dV = \int_a^b \pi(R^2 - r^2)dx = \pi \int_a^b (R^2 - r^2)dx$$

This is the formula that we use for the **method of washers**.

Theorem 7.3. The Method of Washers
Suppose that the region lying between the curves $y = f(x)$ and $y = g(x)$, from $x = a$ to $x = b$, is to be revolved about the line $y = c$, where this axis of rotation lies at some distance from the region (it may touch, but not cross, the boundary of the region in some places). Also, suppose that everywhere in the interval $[a, b]$ the distance of the curve $y = f(x)$ from the line $y = c$ is at least as far as the distance of the curve $y = g(x)$ from the line. The volume of the solid of revolution generated by revolving the region about the axis is given by:

$$V = \int_a^b \pi(R^2 - r^2)dx$$

where R is the distance of the farther boundary of the region from the axis of rotation, so that $R = |f(x) - c|$, and r is the distance of the nearer boundary of the region from the axis of rotation, so that $r = |g(x) - c|$.

Notice: In saying that $y = f(x)$ lies at least as far from $y = c$ as $y = g(x)$ does, we are requiring that the two curves may touch at one or more places in the interval, but they don't cross, and are saying that when one is farther away than the other, it is always $y = f(x)$ which is farther from the axis of rotation. In the simplest case, where the axis of rotation is the x-axis and both $f(x)$ and $g(x)$ are non-negative on the interval (i.e. lie above the x-axis), this just means that $y = f(x)$ is the "upper" curve and $y = g(x)$ is the "lower" curve. And in that case, we simply have $R = f(x)$ and $r = g(x)$. We can think of this, not only in the simplest case but in general, as "the curve that would make the bigger disk" being the one that gives R and "the curve that would make the smaller disk" being the one that gives r. That is, if we imagine revolving a slice of the region that lies between the curve and the axis about the axis, for each curve separately, where the two slices are taken at the same place in the interval $[a, b]$, two disks would be generated. Taking a slice of the region *between* the curves at that same place in the interval produces a washer, and thinking of the disks generated by considering each curve separately, the smaller of these 2 disks is the hole in the washer, and is generated by the curve that lies closer to the axis. As previously discussed, the washer is effectively the larger disk with the smaller disk removed from the middle of it.

Example 7.4. Find the volume generated by revolving the region bounded by $y = x^2 + 2$, $y = 1$, $x = 0$ and $x = 2$ about the x-axis.

Solution: We can sketch the region in question, and draw a slice perpendicular to the x-axis (with width dx). We look at the solid obtained by revolving this strip as shown in Figure 7 (next page).

We see that there is a space between the region which is going to be revolved and the axis about which it is to be revolved, so that when a slice of the region is revolved about this axis, a washer will be produced. Also, $y = x^2 + 2$ lies above $y = 1$ everywhere in the region (as it does everywhere), and both lie above the x-axis, which is the axis of rotation, so that the "upper" curve, $y = x^2 + 2$, is farther away from the x-axis than the "lower curve", the line $y = 1$, throughout the interval. Therefore when we take a slice of the region between $y = x^2 + 2$ and $y = 1$ anywhere in the interval $[0, 2]$, it is $y = x^2 + 2$ that determines the "top" of the slice and which, when the slice is revolved about the x-axis, determines the outside edge of the washer produced. And the "bottom" of the slice is given by $y = 1$, so that this curve (line) is what produces the hole in the washer and is determining the inner edge of the washer. The washer generated by revolving a slice of the region is shown in Figure 7 and is examined in more detail in Figure 8, on the next page. In the latter picture, we see that (as expected) the outer radius of the washer is given by the value of $f(x) = x^2 + 2$ and the inner radius is just $g(x) = 1$, i.e. is constant, because the line $y = 1$ is a constant distance from the x-axis. So we have $R = x^2 + 2$ and $r = 1$. And as always when the axis of rotation is horizontal, the "height" of the washer (i.e. the width, which would be the height if the washer was knocked over) is $h = dx$.

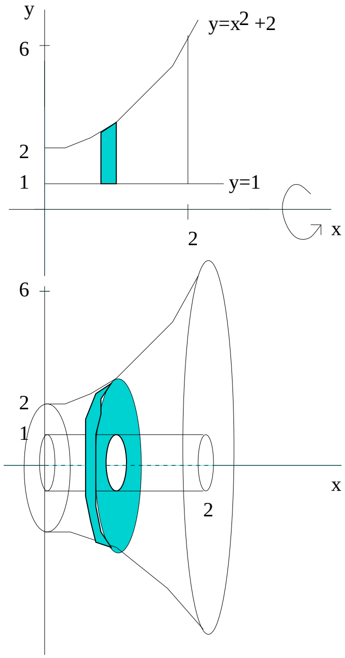


Figure 7: Revolving a slice of the region about the x-axis produces a washer.

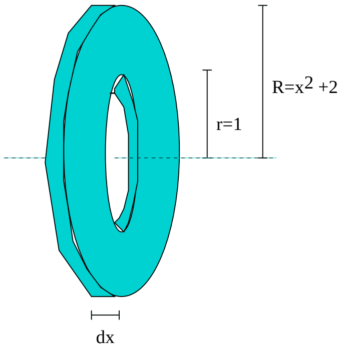


Figure 8: Finding the volume of the washer.

Therefore the volume of the washer generated by the slice is given by:

$$dV = \pi(R^2 - r^2)h = \pi[(x^2+2)^2 - (1^2)]dx = \pi[(x^4+4x^2+4)-1]dx = \pi(x^4+4x^2+3)dx$$

And since the region being revolved, and therefore also the resulting solid of revolution, runs from $x=0$ to $x=2$, these are the limits of integration when we integrate dV to get the volume of the entire solid. So we get:

$$V = \int_0^2 dV = \pi\int_0^2 (x^4+4x^2+3)dx = \pi\left[\frac{x^5}{5} + 4\left(\frac{x^3}{3}\right) + 3x\right]_0^2 = \pi\left[\left(\frac{32}{5}+\frac{32}{3}+6\right)-0\right] = \frac{346}{15}\pi$$

Example 7.5. Find the volume of the solid generated by revolving the region bounded by $y=x$, $y=2$ and $x=0$ about the x-axis.

Solution:
You should sketch the region and examine a vertical slice of it (since the axis of rotation is horizontal). That's easy enough to do. The region is triangular. You should see that the region is not bounded by the axis of rotation (although it does touch the axis at one end of the region). Therefore we need the method of washers. The upper boundary of the region is the line $y=2$ and the lower

boundary is the line $y = x$. That is, when the slice is revolved about the x-axis it will produce a washer with outer radius $R = 2$ and inner radius $r = x$. Therefore the volume of the washer is $dV = \pi(R^2 - r^2)\,dx = \pi(2^2 - x^2)\,dx = \pi(4 - x^2)\,dx$.

The left boundary of the region is the y-axis, i.e. the line $x = 0$. Since the line $y = x$ intersects the line $y = 2$ at $x = 2$, this is the x-value at the right edge of the region. That is, the region runs from $x = 0$ to $x = 2$, so these are the limits of integration for finding the volume of the whole solid. We get

$$V = \int_0^2 dV = \pi \int_0^2 (4 - x^2)\,dx = \pi \left[4x - \frac{x^3}{3}\right]_0^2 = \pi\left(8 - \frac{2^3}{3}\right) = \pi\left(8 - \frac{8}{3}\right) = \frac{16\pi}{3}$$

Example 7.6. Find the volume of the solid obtained when the region bounded by $y = x^2$ and $y = x$ is revolved about the x-axis.

Solution:
First, we need to identify the region bounded by $y = x^2$ and $y = x$. We start by finding the x-values at which the parabola $y = x^2$ and the line $y = x$ intersect:

$$x^2 = x \qquad \Rightarrow \qquad x^2 - x = 0 \qquad \Rightarrow \qquad x(x-1) = 0 \qquad \Rightarrow \qquad x = 0 \text{ or } x = 1$$

Therefore the region runs from $x = 0$ to $x = 1$. And on the interval $[a, b] = [0, 1]$, we have $x^2 \leq x$. For instance, at $x = \frac{1}{2}$ we have $x^2 = \frac{1}{4} < \frac{1}{2}$. Therefore the line $y = x$ lies above the parabola $y = x^2$ everywhere between the 2 places where they intersect. That is, the region being revolved lies above $y = x^2$ and below $y = x$ from $x = 0$ to $x = 1$. Notice that the lower boundary, $y = x^2$, lies above the x-axis everywhere in $[0, 1]$ except at the left endpoint, where the line intersects the x-axis. So there is space between the region being revolved and the axis of rotation, except that the region just touches the axis at one end. Therefore when this region is revolved about the axis there will be a hole in the resulting solid of revolution, but the left end will be closed, i.e. the hole doesn't quite go all the way through the solid.

When we take a slice of the region — vertically, so that the slice is perpendicular to the axis of rotation — the slice has the line $y = x$ defining its top edge and the curve $y = x^2$ defining its bottom edge. And since this slice, if taken anywhere except exactly at $x = 0$, lies strictly above the x-axis, then when the slice is revolved about the x-axis, a washer will be produced. This washer will have as its outer radius the value defining the top edge of the slice, so we have $R = x$, and will have as its inner radius the value defining the bottom edge of the slice, so that $r = x^2$. And recalling that the region runs from $x = 0$ to $x = 1$, using the formula from the Method of Washers (Theorem 7.3) we see that the volume of the solid generated by revolving the whole region is

$$\begin{aligned} V &= \pi \int_a^b (R^2 - r^2)dx = \pi \int_0^1 (x^2 - (x^2)^2)dx = \pi \int_0^1 (x^2 - x^4)dx \\ &= \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi \left[\left(\frac{1^3}{3} - \frac{1^5}{5}\right) - \left(\frac{0^3}{3} - \frac{0^5}{5}\right)\right] \\ &= \pi \left[\frac{1}{3} - \frac{1}{5}\right] = \frac{5-3}{15}\pi = \frac{2\pi}{15} \end{aligned}$$

Notice: The region in this problem is easy to sketch, and you should do so, identifying a vertical slice of the region and sketching the washer generated by revolving the slice. However we didn't really need to do that in order to solve the problem. Sketching the region, a slice of it and the washer generated can help you to recognize that it *is* the method of washers that we need here, and also to identify the outer and inner radius (i.e. which curve gives which). But we can solve a problem like this without a diagram as long as we're careful about (1) checking that there's only a single region being revolved, i.e. the curves don't cross inside the region, which tells us that we don't need to consider sub-regions; (2) realizing that both the “top” and the “bottom” (so in particular the bottom) of the region lie at a distance from the axis of rotation in part or all of the region, which tells us that the Method of Washers is required; and (3) properly identifying which curve is closer to the axis of rotation, which identifies which is giving the inner radius (and therefore also which is giving the outer radius) of the washer.

Horizontal Slicing

So far, all of the Volume of Revolution problems we have done have involved revolving some region about a horizontal axis of rotation. But of course that doesn’t have to be the case. In other situations we can have a region being revolved about a vertical axis. (And in fact we could also have an axis of rotation which is neither horizontal nor vertical, but we don’t consider that situation in this course.) Looking again at the general procedure for finding the volume of a solid generated by revolving a region about some axis of rotation, in Theorem 7.1, we see that there was nothing in that procedure which specified that the axis of rotation must be horizontal. It simply specifies that the slice of the region used to determine dV, the change in volume resulting from revolving a particular slice (i.e. the volume of the disk or washer), must be perpendicular to the axis of rotation. (It is this perpendicularity which gives the disk or washer its shape.) So with a horizontal axis of rotation, we use a vertical slice of the region, as we have done in all of the examples so far. But if we have a vertical axis of rotation, then we need to take a horizontal slice.

In the previous unit, where we were calculating the area of a region, we saw that using horizontal slices instead of vertical slices requires interchanging the roles of x and y, and the same is true here. The function(s) defining the boundary (boundaries) of the region need(s) to express x as a function of y instead of expressing y as a function of x, and the interval on which the region lies must be expressed with the endpoints being the smallest and largest y-values in the region instead of the smallest and largest x-values, and then we integrate with respect to y, rather than with respect to x. Everything else is the same. Except that of course this means that the nearer and farther boundaries of the region (to/from the axis of rotation) are determined by horizontal distance (i.e. farther left or right) rather than vertical distance (lower or higher).

So with a vertical axis of rotation, we have a region defined by $x = f(y)$, and perhaps also $x = g(y)$, running from $y = a$ to $y = b$, i.e. on the interval $[a, b]$ where these are values of y. The slice we consider is horizontal, so that when it is revolved about the axis the disk or washer produced is lying flat, rather than standing on edge, and so the radius of the disk, or the radii of the washer, is/are function(s) of y rather than of x. And the height (which actually is height this time, rather than “sideways height”, i.e. width) is dy rather than dx. Other than that, the procedure is exactly the same as before. The volume of the disk or washer gives dV, and the volume of the entire solid is given by $V = \pi \int_a^b dV$. However, when we expressed both the Method of Disks and the Method of Washers, we were specifically assuming a horizontal axis of rotation, so we should restate these methods for use with a vertical axis of rotation, i.e. with horizontal slicing.

Theorem 7.4. The Method of Disks for a vertical axis of rotation
Consider the region lying between the curve $x = f(y)$ and the line $x = c$, from $y = a$ to $y = b$. When this region is revolved about the line $x = c$, the volume of the resulting solid of revolution is given by

$$V = \pi \int_a^b r^2 dy$$

where r is the distance of the boundary curve $x = f(y)$ from the axis of rotation, $x = c$, so that $r = |f(y) - c|$

(Notice that the only thing which has changed here, from Theorem 7.2, is that all the x's have changed to y's, and all the y's have changed to x's.)

Example 7.7. Consider the region R bounded by the lines $y = x$, $y = 2$ and $x = 0$. Find the volume of the solid generated when R is revolved about the y-axis.

Solution:
This is the same region which you drew for Example 7.5, but there it was being revolved about a horizontal axis (the x-axis) which was not a boundary of the region, whereas now it is being revolved about the y-axis, a vertical axis, which *is* a boundary of the region (the line $x = 0$).

Because the axis of rotation is vertical, we need to use horizontal slicing. (Remember, we must always slice perpendicular to the axis of rotation.) And because the axis of rotation is a boundary of the region, we use the method of disks. The region, with a horizontal slice, is shown below. The (approximate) disk generated by revolving the slice is also shown.

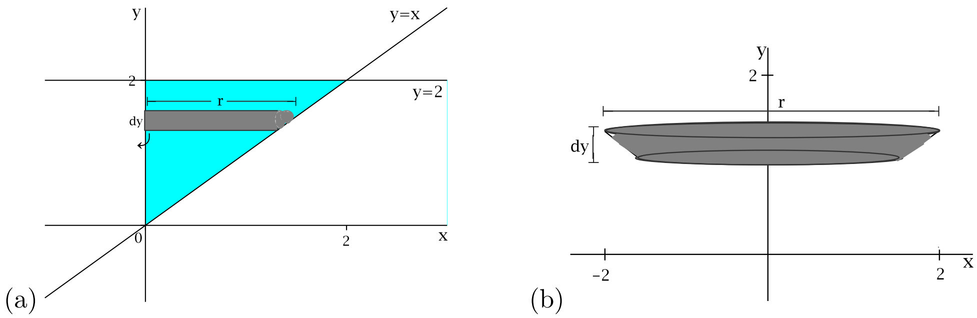


Figure 9: A horizontal slice of the region has length $x = y$, which is also the radius of the disk it sweeps out.

We see that the radius of the disk is given by the length of the slice, which is the distance of the line $y = x$ from the y-axis. That is, the distance from $x = 0$ to $x = y$, which is just y, the height at which the slice is taken. So $r = y$. And the height of the disk is the height of the slice, which is dy. Therefore the volume of the disk is $dV = \pi r^2 dy = \pi y^2 dy$.

To find the volume of the whole solid, we integrate (as usual), but this time we are integrating with respect to y, so the limits of integration are y-values, not x-values. That is, the region lies to the left of the line $y = x$ and to the right of the y-axis, from $y = 0$ at the bottom to $y = 2$ at the top. When we take infinitely many infinitely thin horizontal slices and revolve them, producing infinitely many infinitely thin disks, those disks are stacked one on top of the other, with the bottom-most at $y = 0$ and the top-most at $y = 2$. It is these bottom and top y-values which give the limits of integration when we have a vertical axis of rotation. Therefore the volume of the entire solid of revolution is given by $V = \int_0^2 dV$. And with $r = y$ so that $dV = \pi r^2 dy = \pi y^2 dy$ we get:

$$V = \pi \int_0^2 y^2 dy = \pi \left[\frac{y^3}{3}\right]_0^2 = \pi \left(\frac{2^3}{3} - 0\right) = \frac{8\pi}{3}$$

The solid is shown on the next page.

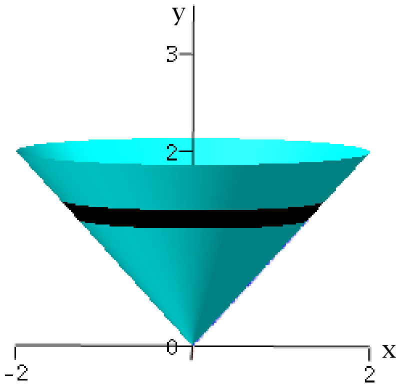


Figure 9: The solid generated in Example 7.7 is a cone. Its volume is $8\pi/3$.

Example 7.8. Find the volume of the solid generated when the region bounded by the parabola $y^2 = 4 - x$ and the line $x = 0$ is revolved about the y-axis.

Solution:
In Example 6.8, we found the area of this region using horizontal slicing. As we did there, we need to express the parabola as $x = 4 - y^2$ instead of as $y^2 = 4 - x$. We can look again at this region, and a horizontal slice of it, as shown in Figure 10.

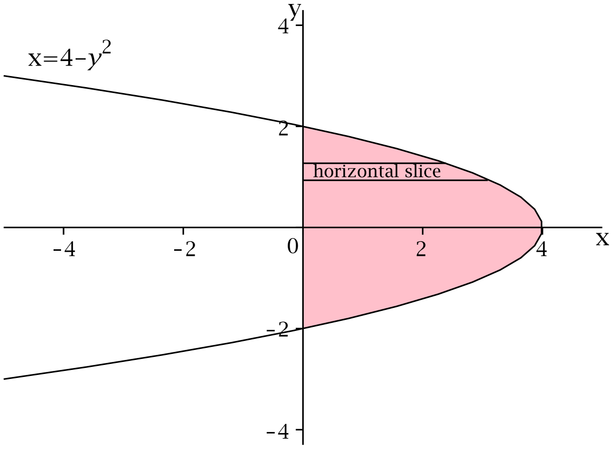


Figure 10: (a) A horizontal slice of the region between $x = 4 - y^2$ and $x = 0$.

Once again the boundary line $x = 0$ is just the y-axis, and so the axis of rotation is one of the boundaries of the region, so we use the Method of Disks. The radius of the disk is given by the distance of the right end of the slice from the axis of rotation, so we have $r = 4 - y^2$. As before, the height of the slice, and therefore also of the disk, is dy. And as we found in Example 6.8, the region extends from $y = -2$ to $y = 2$, so these are the y-values which give the limits of integration. Therefore the volume of the disk is

$$dV = \pi r^2 h = \pi r^2 dy = \pi(4 - y^2)^2 dy$$

and we find the volume of the entire solid of revolution by summing infinitely many of these disks, when each has infinitely small height, i.e. by integrating with respect to y, over the entire region. We get

$$\begin{aligned}
V &= \pi\int_{-2}^{2} r^2 dy = \pi\int_{-2}^{2}(4-y^2)^2 dy = \pi\int_{-2}^{2}(16-8y^2+y^4)dy \\
&= \pi\left[16y - 8\left(\frac{y^3}{3}\right)+\frac{y^5}{5}\right]_{-2}^{2} = \pi\left\{\left[16(2)-8\left(\frac{2^3}{3}\right)+\frac{2^5}{5}\right]-\left[16(-2)-8\left(\frac{(-2)^3}{3}\right)+\frac{(-2)^5}{5}\right]\right\} \\
&= \pi\left[\left(32-\frac{8(8)}{3}+\frac{32}{5}\right)-\left(-32-\frac{8(-8)}{3}+\frac{(-32)}{5}\right)\right] = \pi\left[\left(32-\frac{64}{3}+\frac{32}{5}\right)-\left(-32+\frac{64}{3}-\frac{32}{5}\right)\right] \\
&= \pi\left[\left(32-\frac{64}{3}+\frac{32}{5}\right)+\left(32-\frac{64}{3}+\frac{32}{5}\right)\right] = 2\pi\left[32\left(1-\frac{2}{3}+\frac{1}{5}\right)\right] \\
&= 64\pi\left(\frac{15-10+3}{15}\right) = 64\pi\left(\frac{8}{3}\right) = \frac{512}{3}\pi
\end{aligned}$$

(*Notice:* The bulk of the work there was the arithmetic. The calculus part was very straightforward.)

Here's a picture of the solid whose volume we have just found, with the disk generated by the slice from Figure 10(a) shown.

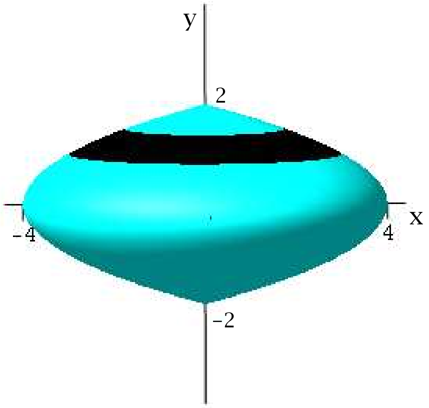


Figure 10: (b) The solid generated in Example 7.8

Example 7.9. Find the volume of the solid which is produced by revolving the region bounded by $y = 3 - 2x$, $x = 0$ and $y = 0$ about the y-axis.

Solution:
We have a vertical axis of rotation and so we need to express the region using functions of y, and as extending between 2 y-values, and take a horizontal slice of the region. Since the y-axis, i.e. the line $x = 0$, is a boundary of the region, we use the Method of Disks. The function providing the other boundary is $y = 3 - 2x \Rightarrow x = \frac{3-y}{2}$, and the region extends from $y = 0$ to the y-intercept of the line $x = \frac{3-y}{2}$ (i.e. the place at which the lines $x = \frac{3-y}{2}$ and $x = 0$ intersect). We see that

$$\frac{3-y}{2} = 0 \qquad \Rightarrow \qquad 3 - y = 0 \qquad \Rightarrow \qquad y = 3$$

and so the region extends from $y = 0$ to $y = 3$.

We can tell without looking at a sketch of the region that since $\frac{3-y}{2} \geq 0$ throughout $[0, 3]$, the horizontal slice will have length $\frac{3-y}{2}$ and height dy, which generates a disk with radius $r = \frac{3-y}{2}$

when the slice is revolved about the y-axis. (This is the same region as in Example 7.1. You may want to look again at Figure 4(a) and think about a horizontal slice instead of the vertical slice shown there.) The volume of the solid generated by revolving the region about the y-axis is

$$\begin{aligned} V &= \pi\int_0^3 r^2 dy = \pi\int_0^3 \left(\frac{3-y}{2}\right)^2 dy = \pi\int_0^3 \frac{(3-y)^2}{4} = \frac{\pi}{4}\int_0^3 (9-6y+y^2)dy \\ &= \frac{\pi}{4}\left[9y-3y^2+\frac{y^3}{3}\right]_0^3 = \frac{\pi}{4}\left[\left(9(3)-3(3)^2+\frac{3^3}{3}\right)-0\right] = \frac{\pi}{4}\left[(27-27+3^2)-0\right] = \frac{9\pi}{4} \end{aligned}$$

Of course, sometimes we need the Method of Washers with a vertical axis of rotation. We can state the method, by adapting Theorem 7.3, once again (as we did for the Method of Disks) simply interchanging the roles of x and y. That is, we change all the y's to x's and all the x's to y's in the statement of the method.

Theorem 7.5. The Method of Washers for a vertical axis of rotation
Suppose that the region lying between the curves $x = f(y)$ and $x = g(y)$, from $y = a$ to $y = b$, is to be revolved about the line $x = c$, where this axis of rotation lies at some distance from the region (it may touch, but not cross, the boundary of the region in some places). Also, suppose that everywhere in the interval $[a, b]$ the distance of the curve $x = f(y)$ from the line $x = c$ is at least as far as the distance of the curve $x = g(y)$ from the line. The volume of the solid of revolution generated by revolving the region about the axis is given by:

$$V = \int_a^b \pi(R^2 - r^2)dy$$

where R is the distance of the farther boundary of the region from the axis of rotation, so that $R = |f(y) - c|$ and r is the distance of the nearer boundary of the region from the axis of rotation, so that $r = |g(y) - c|$.

Notice that here the distances of the boundaries of the region from the vertical line that is the axis of rotation are horizontal distances, so we need to identify the leftmost and rightmost edges of the region in finding the nearer and farther boundaries. (Which is which depends on whether the axis of rotation lies to the left or to the right of the region.)

Example 7.10. Find the volume of the solid generated when the region bounded by $y^2 = 4 - x$ and $y = 2 - x$ is revolved about the y-axis.

Solution:
Since the axis of rotation is vertical, we need to use horizontal slicing. Therefore we need to think of the bounding functions as $x = 4 - y^2$ and $x = 2 - y$, instead of the way they were stated in the problem. The region, and a horizontal slice of it, are shown in Figure 11 (next page). We see that the furthest edge from the y-axis, the rightmost boundary, is the parabola $x = 4 - y^2$, while the nearer edge is the leftmost boundary, the line $x = 2 - y$. Because the y-axis is not a boundary of the region (except at one point), the solid generated will have a hole in it. That is, we can see that revolving the slice about the y-axis will produce a washer. The washer will have outer radius $R = 4 - y^2$ (i.e. the further edge of the slice) and inner radius $r = 2 - y$ (the nearer edge of the slice). And of course the height of the washer is dy, so the volume of the washer is

$$\begin{aligned} dV &= \pi(R^2 - r^2)\, dy = \pi\left[(4-y^2)^2 - (2-y)^2\right] dy \\ &= \pi\left[(16-8y^2+y^4) - (4-4y+y^2)\right] dy \\ &= \pi\left[12+4y-9y^2+y^4\right] dy \end{aligned}$$

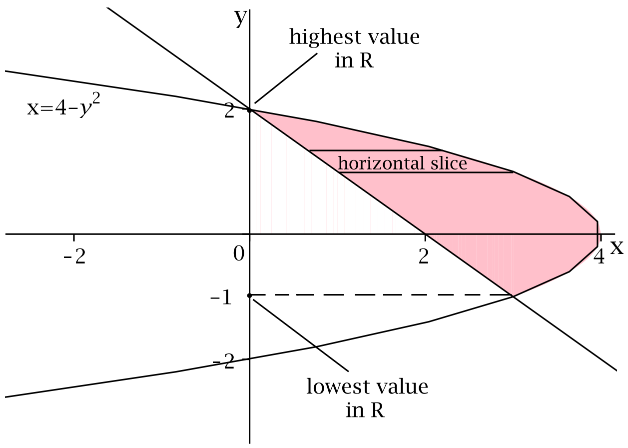


Figure 11: A horizontal slice of the region between $x = 4 - y^2$ and $x = 2 - y$.

As we found when we calculated the area of this region, in Example 6.9, the region runs from $y = -1$ to $y = 2$, so these are the limits of integration we use in calculating $V = \int dV$. We get:

$$\begin{aligned} V &= \pi \int_{-1}^{2} (12 + 4y - 9y^2 + y^4)\, dy = \pi \left[12y + 2y^2 - 3y^3 + \frac{y^5}{5} \right]_{-1}^{2} \\ &= \pi \left[\left(24 + 8 - 3(8) + \frac{32}{5} \right) - \left(-12 + 2 - 3(-1) + \frac{(-1)}{5} \right) \right] \\ &= \pi \left[8 + \frac{32}{5} - \left(-7 - \frac{1}{5} \right) \right] = \pi \left(15 + \frac{33}{5} \right) = \frac{108\pi}{5} \end{aligned}$$

Example 7.11. Find the volume of the solid produced by revolving the region bounded by $y = x^2$ and $y = x$ about the y-axis.

Solution:
The region is the same one which in Example 7.6 we revolved about the x-axis. But this time we have a vertical axis of rotation, so we will need to slice the region horizontally. We saw in that earlier example that the line and the parabola intersect at the points $(0, 0)$ and $(1, 1)$. (That is, we saw that they intersect when $x = 0$ or 1, and since $y = x$ everywhere on the line, the points of intersection are (0,0) and (1,1).) The region has the line $y = x$ forming the left/top boundary and the parabola $y = x^2$ forming the right/bottom boundary. That is, everywhere in the region, the line is above and to the left of the parabola, except of course at the endpoints of the region, where the two intersect. (Go ahead and sketch the region if you're having difficulty envisioning this.)

When we were slicing vertically, we considered the region to have the line as its top boundary and the parabola as its bottom boundary, with left edge occurring at $x = 0$ and right edge occurring at $x = 1$. Now, since we are slicing horizontally, we need to consider the line to be the left boundary and the parabola to be the right boundary, with the bottom edge occurring at $y = 0$ and the top edge occurring at $y = 1$. And of course we need to express the left and right boundaries as functions of y. So we express the line as $x = y$, and as in the previous example we have the positive half-parabola $x = \sqrt{y}$ (forming the right boundary this time, rather than the left boundary). And since the relevant x-values are all positive, we also know that the region lies to the right of the axis of rotation, so the farther boundary is the right boundary and the nearer boundary is the left boundary.

Therefore we have the region defined as from $g(y) = y$ on the left to $f(y) = \sqrt{y}$ on the right, running from $y = 0$ to $y = 1$, i.e. on the interval $[a, b] = [0, 1]$. And of course this region does not have the y-axis as a boundary, so there is space between the region and the axis of rotation (the line $x = c$ for $c = 0$). This means that revolving a horizontal slice of the region about the y-axis produces a washer whose outer radius is $R = |f(y) - c| = |\sqrt{y} - 0| = \sqrt{y}$ and whose inner radius is $r = |g(y) - c| = |y - 0| = y$. Thus the Method of Washers gives the volume of the solid as:

$$\begin{aligned} V &= \pi \int_a^b (R^2 - r^2) dy = \pi \int_0^1 \left[(\sqrt{y})^2 - (y)^2 \right] dy = \pi \int_0^1 (y - y^2) dy = \pi \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 \\ &= \pi \left[\left(\frac{1^2}{2} - \frac{1^3}{3} \right) - \left(\frac{0^2}{2} - \frac{0^3}{3} \right) \right] = \pi \left(\frac{1}{2} - \frac{1}{3} \right) = \pi \left(\frac{3}{6} - \frac{2}{6} \right) = \frac{\pi}{6} \end{aligned}$$

All of our examples so far have involved revolving a region about an axis of the xy-plane. But other horizontal or vertical lines can also be used as the axis of rotation. So let's finish up with an example that has a different vertical line as the axis of rotation.

Example 7.12. Find the volume of the solid generated by revolving the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the line $x = -1$.

Solution:
We have a vertical axis of rotation, so when we take a slice of the region perpendicular to this axis, we need a horizontal slice. Therefore the region has left and right boundaries which are functions of y and extends between two y-values. The y-axis (i.e. the line $x = 0$) is one of the boundaries of the region and the curve $y = x^3 \to x = \sqrt[3]{y}$ is the other. The region runs between $y = 8$ and the y-value at which the two boundaries intersect, which is of course at $\sqrt[3]{y} = 0 \Rightarrow y = 0$. Therefore the region corresponds to the interval of y-values $[a, b] = [0, 8]$. Notice that although the y-axis is a boundary of the region, this is *not* the axis of rotation this time. The axis of rotation is the line $x = -1$, which lies to the left of the line $x = 0$. And since $x = \sqrt[3]{y} \geq 0$ throughout the interval $[0, 8]$, then this curve lies to the right of the y-axis, so the y-axis is the boundary nearer to the axis of rotation, and there is a space between the region and the axis of rotation, which means that the Method of Washers is required. That is, when we take a horizontal slice of the region, there will be a space between the slice and the axis about which it is going to be revolved, which will result in a washer being generated by revolving the slice. Figure 12(a) (next page) shows the region, a horizontal slice of it, and the washer generated by the revolution.

The washer, which has height dy, has outer radius given by the farther edge of the slice. (See Figure 12(b) next page.) That is, for purposes of the Method of Washers the function $x = \sqrt[3]{y}$ is $f(y)$ and the function $x = 0$ is $g(y)$. And the axis of rotation is $x = -1$, so we have $c = -1$. This gives the outer radius as $R = |f(y) - (-1)| = \sqrt[3]{y} + 1$ and the inner radius as $r = |g(y) - (-1)| = 0 + 1 = 1$.

Notice that as we saw once before, since the nearer boundary of the region is a line parallel to the axis of rotation, the inner radius is constant. No matter. We simply need to plug the values of R and r into the formula for the volume of the solid of revolution, along with the boundary values $a = 0$ and $b = 8$.

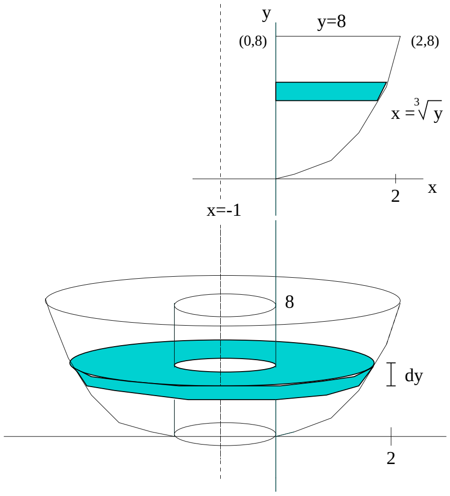


Figure 12: (a) There is space between the vertical axis of rotation and a horizontal slice of the region.

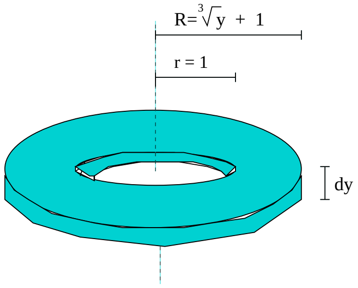


Figure 12: (b) The washer has height dy and the inner and outer radii are functions of y.

We get:

$$\begin{aligned}
V &= \pi\int_a^b (R^2-r^2)dy = \pi\int_0^8 \left[(\sqrt[3]{y}+1)^2-(1)^2\right]dy = \pi\int_0^8\left[\left(y^{\frac{1}{3}}+1\right)^2-1\right]dy \\
&= \pi\int_0^8\left[\left((y^{\frac{1}{3}})^2+2y^{\frac{1}{3}}+(1)^2\right)-1\right]dy = \pi\int_0^8\left(y^{\frac{2}{3}}+2y^{\frac{1}{3}}\right)dy = \pi\left[\frac{y^{\frac{5}{3}}}{\frac{5}{3}}+2\left(\frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right)\right]_0^8 \\
&= \pi\left[\frac{3}{5}y^{\frac{5}{3}}+\frac{3}{2}y^{\frac{4}{3}}\right]_0^8 = \pi\left\{\left[\frac{3}{5}\left(8^{\frac{5}{3}}\right)+\frac{3}{2}\left(8^{\frac{4}{3}}\right)\right]-\left[\frac{3}{5}\left(0^{\frac{5}{3}}\right)+\frac{3}{2}\left(0^{\frac{4}{3}}\right)\right]\right\} \\
&= \pi\left[\frac{3}{5}\left(\sqrt[3]{8}\right)^5+\frac{3}{2}\left(\sqrt[3]{8}\right)^4-0\right] = 3\pi\left[\frac{2^5}{5}+\frac{2^4}{2}\right] = 3\pi\left[\frac{32}{5}+8\right] = 3\pi\left(\frac{72}{5}\right) = \frac{216}{5}\pi
\end{aligned}$$

Note: In the assigned homework, the region is always revolved about either the x-axis or the y-axis. But as we see in this example (and in the more general statements of the Method of Disks and the Method of Washers given in these notes), revolving about some other vertical or horizontal line is really no different – the general approach is exactly the same.

Math 1225A/B

Unit 8:
Other Integration Methods

(text reference: Sections 9.1, 9.2 and 9.5

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8 Other Integration Methods

In this section we learn two new integration techniques. We know that if an integrand function is the product of a composite function and the derivative of the "inner function" of that composite function, then we can use the Substitution Rule to find an antiderivative function. But what if we have some other kind of product? Then the Substitution Rule doesn't work. For instance, how do we find $\int xe^x dx$? The Substitution Rule doesn't help. In this section we'll learn another rule that often works in those situations. Also, what if the integrand function could be written as the sum of several rational functions of the form $\frac{u'}{u}$, for different u's, but we can't recognize that because they've been brought to a common denominator and expressed as a single rational function? How can we break the integrand up into its constituent parts, which would each be easy to integrate? For instance, $\int \frac{3}{x} dx$ is easy and so is $\int \frac{4}{x+4} dx$, which means that $\int \left(\frac{3}{x} + \frac{4}{x+4}\right) dx$ is also easy. But what if it's presented as $\int \frac{7x+12}{x^2+4x} dx$ (which is what $\frac{3}{x} + \frac{4}{x+4}$ looks like when it's brought to a common denominator)? The Substitution Rule doesn't help with this integral. So how can we get (back) to the sum that would have been easy to integrate? The other method we learn in this section tells us how to do that. And after that, we'll learn about definite integrals which have one or both limits of integration being infinite. These are not "proper" definite integrals, because the definition requires finite limits of integration. But some integrals of this type do have a finite value – we'll learn how to determine whether that's the case.

Integration By Parts

Let's think about the product rule for differentiation. We know how to differentiate a product. The product rule says that $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$. Or we could express the same rule as

$$\frac{d}{dx}[uv] = v\frac{du}{dx} + u\frac{dv}{dx}$$

Now, suppose we integrate both sides of this equation:

$$\begin{aligned} \int \frac{d}{dx}[uv]dx &= \int \left(v\frac{du}{dx} + u\frac{dv}{dx}\right) dx \\ \text{so that } \int \frac{d}{dx}[uv]dx &= \int v\left(\frac{du}{dx}\right) dx + \int u\left(\frac{dv}{dx}\right) dx \end{aligned}$$

Let's think about the integral on the left hand side.

We know that $\int f'(x)dx = f(x) + C$. Or, expressed another way, that $\int \frac{d}{dx}[f(x)]dx = f(x) + C$. That is, $f(x)$ is an antiderivative of the derivative of $f(x)$, no matter how that derivative is expressed. And so whenever we have the form "the integral of the derivative of this function", that's simply equal to "this function", plus an arbitrary constant. And that's exactly what the left hand side of the equation above says. That is, "the integral of the derivative of uv" is simply equal to uv (plus an integration constant). And as we've seen before, as long as there's still an integral in there somewhere, we don't need to worry about adding the integration constant. We can just ignore that until we're doing the last integration, and the "$+C$" we write then represents all the arbitrary constants we might have added along the way. That is, we know that uv is an antiderivative of $\frac{d}{dx}[uv]$, so we can simply replace the integral on the left hand side of the equation with uv, and we don't need to worry about saying "$+C$" as long as there's still an integral on the right hand side of the equation. So we have

$$\int \frac{d}{dx}[uv]dx = \int v\left(\frac{du}{dx}\right) dx + \int u\left(\frac{dv}{dx}\right) dx \qquad \Rightarrow \qquad uv = \int v\left(\frac{du}{dx}\right) dx + \int u\left(\frac{dv}{dx}\right) dx$$

But of course $\left(\frac{du}{dx}\right) dx = du$ and $\left(\frac{dv}{dx}\right) dx = dv$. That is, if we consider $\frac{du}{dx}$ or $\frac{dv}{dx}$ as a ratio of differentials (as we did in the Substitution Rule), then we can "cancel" the dx's, and just be left with the differential in the numerator of the ratio. So now we have $uv = \int vdu + \int udv$, which we can rearrange to get $\int udv = uv - \int vdu$.

This insight allows us to find the general antiderivative of *some* products — the product of one term whose derivative would be less complicated than the term itself, and another term that's easy to integrate. That is, suppose we have something of the form

$$\int a(x)b(x)dx$$

where (1) $b(x)$ is something whose antiderivative can easily be found, i.e. we have $b(x) = \frac{dv}{dx}$ for some obvious function v; and (2) $a(x)$ is something whose derivative is less complicated, i.e. $a(x) = u$ where $\frac{du}{dx} = a'(x)$ is relatively uncomplicated. Then (hopefully) $\int vdu$ is an easier integral than $\int udv = \int a(x)b(x)dx$ and we can solve the integral by

$$\int a(x)b(x)dx = \int udv = uv - \int vdu$$

For instance, consider the integral $\int xe^x dx$. As mentioned previously, the Substitution Rule won't help for that. (Go ahead, try it. You'll just get stuck.) But e^x is certainly easy to integrate. If we consider that the e^x in the integrand is $\frac{dv}{dx}$ then we get $v = e^x$ as an antiderivative. (That is, from $dv = e^x dx$ we get $v = e^x$.) And x is certainly easy to differentiate, and gets less complicated when we do that. If we consider the x term in the product that is the integrand to be u then we get $\frac{du}{dx} = 1$, so that $du = dx$. So we can solve $\int xe^x dx$ by considering it to be $\int udv$ and using the fact that this is equal to $uv - \int vdu$. Using $u = x$ and $dv = e^x dx$, so that $du = dx$ and $v = e^x$ we get:

$$\int xe^x = \int udv = uv - \int vdu = (\underbrace{x}_{u})(\underbrace{e^x}_{v}) - \int \underbrace{e^x}_{v}\underbrace{dx}_{du} = xe^x - e^x + C = e^x(x-1) + C$$

Note: We started out talking about the product rule for differentiation. However what we're doing here *isn't* undoing the product rule. When we use the Substitution Rule, we really are *undoing* the Chain Rule. But what we've just done here doesn't correspond to undoing the Product Rule. Rather, it's an entirely new rule, which is derived from, but otherwise has nothing to do with, the Product Rule. This new rule allows us to find the general antiderivative (i.e. the indefinite integral) of *some* product functions, by considering the **parts**, i.e. the terms of the product in the integrand, separately. For this reason, the rule is called *Integration by Parts.*

Rule: Integration By Parts
Given $\int u(x)v'(x)dx$
Step 1: Let $u = u(x)$ and $dv = v'(x)dx$.
Step 2: Find $du = u'(x)dx$ and $v = v(x)$, where v is any antiderivative of $v'(x)$
Step 3: Find $\int u(x)v'(x)dx = \int udv$ using the fact that $\int udv = uv - \int vdu$.

Example 8.1. Find $\int x \ln x dx$.

Solution:
Looking at the integrand, we don't recognize $x \ln x$ as the derivative of any function. Also, we can quickly determine that the substitution rule won't work here. That is, we *don't* have some composite function times the derivative of the "inner function" of that composite function. (Have a look at it

and think about what substitution you might try, and then *if* you find one you think might work, try it and see that you get stuck.) Since what we have is a product, we try Integration By Parts. That is, we consider whether it will help to think of this integral as $\int udv$ for some functions u and v'.

The terms of the product in the integrand are x and $\ln x$. In deciding which should be u and which should be v', we need to realize that u must be something we can easily differentiate, and v' must be something we can easily integrate. It doesn't matter whether or not we can integrate u. Nor does it matter whether v' gets more or less complicated when it's differentiated. In this case, since we don't know how to integrate $\ln x$, the choice is fairly obvious. We *do* know how to differentiate it, so choose this as the function to be differentiated, u. And then we consider the other part of the product, in this case just x, to be the function to be integrated, v'. That is, we:

$$\text{Let } u = \ln x \text{ and } dv = xdx$$

This gives $\frac{du}{dx} = \frac{d}{dx}(\ln x) = \frac{1}{x}$ so that $du = \frac{1}{x}dx$, and also $v = \int xdx = \frac{x^2}{2}$ ($+C$, but ignore that). Using the integration by parts formula, we see that

$$\int udv = uv - \int vdu$$

$$\begin{aligned}\text{becomes } \int x\ln x\,dx = \int \underbrace{(\ln x)}_{u}\underbrace{x\,dx}_{dv} &= \underbrace{(\ln x)}_{u}\underbrace{\left(\frac{x^2}{2}\right)}_{v} - \int \underbrace{\left(\frac{x^2}{2}\right)}_{v}\underbrace{\left(\frac{1}{x}\right)dx}_{du} \\ &= \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx \\ &= \frac{x^2\ln x}{2} - \frac{1}{2}\int x\,dx \\ &= \frac{x^2\ln x}{2} - \left(\frac{1}{2}\right)\left(\frac{x^2}{2}\right) + C \\ &= \frac{x^2\ln x}{2} - \frac{x^2}{4} + C = \frac{x^2[2(\ln x) - 1]}{4} + C\end{aligned}$$

Example 8.2. Find $\displaystyle\int xe^{3x}dx$.

Solution:
We have a product, and the substitution rule doesn't really look promising, because although e^{3x} is a composite function of the form e^u, the other term in the product (i.e. x) *isn't* the derivative of, or a constant multiple of the derivative of, that u.

That is, we could try the substitution $u = 3x$, but that gives $du = 3dx$, so $xdx = \frac{x}{3}du$ and we get $\int e^u\left(\frac{x}{3}\right)du$. There's still an x in there. With the substitution rule, after substituting in u, there *should not be any x's left*! Of course, maybe it's like those tricky ones at the end of the notes for the Substitution Rule, where you have to express x in terms of u and do more substituting to get something that's only in terms of u. Let's try that. It's not hard here. $u = 3x$ gives $x = \frac{u}{3}$, and so $\frac{x}{3}du$ becomes $\frac{u/3}{3}du = \frac{u}{9}du$. So now we've got $\int xe^{3x}dx = \int e^u\left(\frac{u}{9}\right)du = \frac{1}{9}\int ue^u$. We did manage to get rid of all the x's. But there's just one problem. What's the integral of ue^u? We don't know. Or if we do know, it's only because we already did that (as $\int xe^x dx$) *using Integration By Parts.* So substitution didn't really get us much closer to the answer in this case. We could carry on from this point and use Integration By Parts, but then we might as well have just used that from the start. So that's what we'll do.

Let's see. We've got $\int xe^{3x}dx$ and we want to use Integration By Parts. The integrand is a product of x and e^{3x}. Both of these are easy to differentiate and also relatively easy to integrate. Consider the e^{3x} term. Should it be u or v'? If we choose this for u we get $du = 3e^{3x}dx$ (not forgetting to use the Chain Rule, of course). Or if we choose it for v', i.e. choose $e^{3x}dx$ as dv, we get $v = \frac{e^{3x}}{3}$ (using the Useful Rule that $\frac{e^{kx}}{k}$ is an antiderivative of e^{kx}). Either way, what we have is a constant times what we started with, so neither of these is particularly either more or less complicated than what we started with. That means that e^{3x} is a good candidate for *either* u or v'. So then let's think about the other term, x. If we choose this for u, we get $du = dx$. That's certainly not more complicated. (That is, the derivative of x is a less complicated function than x is.) So x would be a reasonable choice for u. But if we choose x as v' (i.e. if we choose xdx for dv) then we get $v = \frac{x^2}{2}$. Something with an x^2 in it is certainly more complicated than something with just an x in it. That is, $\frac{x^2}{2}$ is more complicated than x. And we don't want things to get more complicated, we need them to get *less* complicated (or at worst, stay at the same degree of complication). So we would prefer to choose x for the u term.

Therefore we let $u = x$ and $dv = e^{3x}dx$. This gives $du = dx$ and $v = \frac{e^{3x}}{3}$ (as noted above). And so we have

$$\begin{aligned}\int xe^{3x}dx &= \int udv = uv - \int vdu = (x)\left(\frac{e^{3x}}{3}\right) - \int \frac{e^{3x}}{3}du \\ &= \frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}dx = \frac{xe^{3x}}{3} - \frac{1}{3}\left(\frac{e^{3x}}{3}\right) + C = \frac{e^{3x}}{3}\left(x - \frac{1}{3}\right) + C\end{aligned}$$

Example 8.3. Find $\displaystyle\int \ln x dx$.

Solution:
Oh my! We observed, in Example 8.1 that we *don't know* how to integrate $\ln x$, and that it therefore wasn't a good choice for v' when doing Integration By Parts. So if we don't know an antiderivative of $\ln x$, how are we supposed to find $\int \ln x dx$?

Well, we do know that $\ln x$ makes a good choice for u when we do Integration By Parts. Can we do something with that? We don't have a product here, though. Hmm. Oh wait. What if we write the integrand as a product anyway. That is, what if what we have *is* a product, but the other term is invisible! An invisible 1 multiplier. Can we do something with that?

That is, what we want to do here is to consider $\int \ln x dx$ as $\int (\ln x)1dx$. If we choose $u = \ln x$, that leaves 1 to be v', so we choose $dv = 1dx$. Then we get

$$du = \frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \text{and} \qquad v = x$$

Now using the Integration By Parts formula we get:

$$\begin{aligned}\int \ln x dx &= \int udv = uv - \int vdu = (\ln x)(x) - \int x\left(\frac{1}{x}\right)dx \\ &= x\ln x - \int 1dx = x(\ln x) - x + C = C - x + x\ln x\end{aligned}$$

Example 8.4. Find $\displaystyle\int x\sin x\,dx$.

Solution:
We don't recognize the integrand function as the derivative of any function, and it doesn't look like

a substitution will help. (Go ahead, try to find one.) But we see a product in the integrand, so perhaps Integration By Parts will work.

We need to decide which term in the product $x \sin x$ should be u, and which should be dv. Since x gets simpler when we differentiate and more complicated when we integrate, whereas $\sin x$ gets neither more nor less complicated under either differentiation or integration, we choose the substitution which makes x get less complicated. That is, we choose $u = x$ and $dv = \sin x dx$. This gives $du = dx$ and $v = -\cos x$, and so substituting into the Integration By Parts formula we get:

$$\begin{aligned} \int x \sin x \, dx &= \int u \, dv = uv - \int v \, du = x(-\cos x) - \int (-\cos x) \, dx \\ &= -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C = \sin x - x \cos x + C \end{aligned}$$

Example 8.5. Find $\int 2x \sec^2 x dx$.

Solution:
Again, we don't recognize the integrand function and can't find a helpful substitution, but the integrand is a product, so we try Integration By Parts.

The terms in the product are $2x$ and $\sec^2 x$. Since $2x$ becomes less complicated when differentiated and we can easily integrate $\sec^2 x$ (which also then becomes less complicated), we try $u = 2x$ and $dv = \sec^2 x dx$. This gives $du = 2dx$ and $v = \tan x$, so we have:

$$\begin{aligned} \int 2x \sec^2 x dx &= \int udv = uv - \int vdu = (2x)(\tan x) - \int (\tan x) 2dx \\ &= 2x \tan x - 2 \int \tan x dx = 2x \tan x - 2 \ln|\sec x| + C = 2x \tan x - \ln(\sec^2 x) + C \end{aligned}$$

Sometimes, we have to do the whole Integration By Parts procedure more than once. That is, we have a situation in which $\int udv$ is complicated, and $\int vdu$ is less complicated (or at least, not more complicated), but still too complicated, so we need to do the same sort of thing again.

Example 8.6. Find $\int x^2 e^x dx$.

Solution:
We have a product, and substitution doesn't look promising (try it), so we try Integration By Parts. We see that x^2 gets less complicated when differentiated, but more complicated when integrated, so we'll probably want to use that for u rather than for v'. And e^x gets neither more nor less complicated, no matter whether it's differentiated or integrated, so it's a good choice for either. Therefore we choose to let $u = x^2$ and $dv = e^x dx$. This gives $du = 2xdx$ and $v = e^x$, so we have

$$\int x^2 e^x dx = \int udv = uv - \int vdu = (x^2)(e^x) - \int e^x (2x) dx = x^2 e^x - 2 \int x e^x dx$$

Well, that last integral is less complicated, but still ... to find $\int x e^x dx$ we need to use Integration By Parts (unless we happen to remember what it is from last time we did it, using IBP). So, we use Integration By Parts again.

Notice that we want to continue making things less complicated, rather than heading back in the direction we came from. That is, we don't want to integrate the x term, because that takes us back to an x^2 term, which is what we started with. We want to make choices this time which are similar to the choices we made last time. So we once again choose $dv = e^x dx$, which leaves us with $u = x$. (Notice that x is a suitable choice for u because it gets less complicated when differentiated.) Those choices give $du = dx$ and $v = e^x$, so we have (as we have seen before)

$$\int xe^x dx = \int udv = uv - \int vdu = (x)(e^x) - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C$$

Now, we use that in what we had after the first Integration By Parts (remembering that we don't need to worry about multiplying the arbitrary constant by a constant, because that still gives just an arbitrary constant):

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx = x^2 e^x - 2(e^x(x-1)) + C = e^x(x^2 - 2x + 2) + C$$

Example 8.7. Find $\int e^x \sin x \, dx$.

Solution:
We have the integrand function $e^x \sin x \; dx$. This is clearly a product, and not one in which the substitution rule helps, so we consider using Integration By Parts. Which part should be u and which should be dv? Neither one gets either simpler or more complicated when differentiated or when integrated. There doesn't appear to be any way to decide which should be which. Oh well, let's just try one. If we let $u = \sin x$ and $dv = e^x \, dx$ then we get $du = \cos x \; dx$ and $v = e^x$. Therefore we have:

$$\int e^x \sin x \, dx = \int u \, dv = uv - \int v \, du = e^x \sin x - \int e^x \cos x \, dx$$

Oh, dear! That integral doesn't seem to be any easier than the one we started with. Does that mean we're stuck? Well, no. We can try using Integration By Parts again. Notice that if we now switch the role of e^x (i.e. if we choose that as u, instead of dv, as last time), we'll just go back where we came from and we truly will be stuck. As mentioned above, whenever we need to use Integration By Parts twice, we always want to make a similar choice for u and dv the second time as we made the first time. So we let $u = \cos x$ and $dv = e^x dx$. This gives $du = -\sin x \, dx$ and $v = e^x$, so we have

$$\int e^x \cos x \, dx = \int u \, dv = uv - \int v \, du = e^x \cos x - \int e^x (-\sin x) \, dx = e^x \cos x + \int e^x \sin x \, dx$$

But ... that looks very similar to what we started with. Did we go around in circles, in spite of making the choice the second time in a similar way to the first? Are we stuck? Actually, no. Putting what we just found into what we had above, we have:

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx\right)$$

That is, we have

$$\int e^x \sin x \, dx = (e^x \sin x) - (e^x \cos x) - \int e^x \sin x \, dx$$

The fact that the sign on the integral is different on the two sides of the equation, even though the integrals are the same, means that we're not stuck. We simply rearrange this equation – and since we're not going to do any more integration, we need to add C:

$$\begin{aligned}
\int e^x \sin x\,dx &= (e^x \sin x) - (e^x \cos x) - \int e^x \sin x\,dx \\
\Rightarrow \int e^x \sin x\,dx + \int e^x \sin x\,dx &= (e^x \sin x) - (e^x \cos x) + C \\
\Rightarrow 2\int e^x \sin x\,dx &= e^x(\sin x - \cos x) + C \\
\Rightarrow \int e^x \sin x\,dx &= \frac{e^x(\sin x - \cos x)}{2} + C
\end{aligned}$$

Notice: Making the choice the other way in the first integration by parts, i.e. choosing $u = e^x$ and $dv = \sin x\,dx$ also works. It leads to a situation just like the one we had here. (You should try it, to see.) For this problem, both ways of choosing u and dv work, and neither is easier than the other. Either way, we have to use Integration By Parts twice, and we end up with "integral = something - same integral", so we rearrange this equation and divide through by 2.

Also Note: Whenever you use some complicated method to find an antiderivative, it's a good idea to check that what you found really is an antiderivative of the integrand function you started with. Here, if you differentiate the right hand side above, using the product rule, you'll see that $\frac{d}{dx}\left(\frac{e^x(\sin x - \cos x)}{2} + C\right) = e^x \sin x$, so we really did find an antiderivative of $e^x \sin x$.

The next example is pretty complicated, but there are 2 different ways we can do it. We just have to think carefully and work our way through logically.

Example 8.8. Find $\int 2x^3 e^{x^2+5} dx$.

Solution:

Approach 1:

We see that the integrand is a product. And one of the terms of the product is a composite function, so our first thought should be that a substitution might work. Considering the e^{x^2+5} term, we have the form e^t for $t = x^2 + 5$, so we consider trying that substitution. (We usually use u as the substitution, but we're going to end up needing to do Integration By Parts as well, so we'll save u for using in that. It doesn't matter what the name of the variable we sub in is, so t will do as well as anything.) We would get $dt = 2xdx$. But the other term in the product isn't $2x$, it's $2x^3$, which is quite a bit more complicated. There are too many x's happening there. If we substitute $t = x^2 + 5$ and $dt = 2xdx$ then there's an x^2 left over (because $2x^3 e^{x^2+5} = (e^{x^2+5})(2x)(x^2)$). Of course, since we do have $t = x^2 + 5$, we can easily express x^2 in terms of t and do the sort of thing we were doing in the last few examples in Unit 4. So let's try that. We get $x^2 = t - 5$, so we make that substitution to get rid of the extra x^2. Substituting $t = x^2 + 5$, $dt = 2x\,dx$ and $x^2 = t - 5$, we get:

$$\int 2x^3 e^{x^2+5} dx = \int (e^{x^2+5})(x^2)(2x) dx = \int e^t (t-5) dt = \int te^t dt - 5\int e^t dt = \int te^t dt - 5e^t = -5e^t + \int te^t\,dt$$

But for $\int te^t dt$, we need Integration By Parts. We let $u = t$ and $dv = e^t dt$, so that $du = dt$ and $v = e^t$. (We've done this one a couple of times before.) We get:

$$\int te^t dt = \int udv = uv - \int vdu = te^t - \int e^t dt = te^t - e^t + C = e^t(t-1) + C$$

Now, we use that in what we got from using the Substitution Rule:

$$\int 2x^3 e^{x^2+5} dx = -5e^t + \int te^t dt = e^t(-5 + t - 1) + C = e^t(t-6) + C$$

And then we need to substitute back in terms of x, using $t = x^2 + 5$:

$$\int 2x^3 e^{x^2+5} dx = e^t(t-6) + C = e^{x^2+5}(x^2+5-6) + C = e^{x^2+5}(x^2-1) + C$$

Approach 2:
Since we ended up using Integration By Parts anyway, what if we just use that right from the start? Does that work? Let's see ...

We have $\int 2x^3 e^{x^2+5} dx$, so the integrand is the product of $2x^3$ and e^{x^2+5}. Let's try letting $u = e^{x^2+5}$ and $dv = 2x^3$. Then we must use the Chain Rule to get du as $du = e^{x^2+5}(2x)dx = 2xe^{x^2+5}dx$. And we just need the Power Rule (for integration) to get v as $v = 2\left(\frac{x^4}{4}\right) = \frac{x^4}{2}$. So we get:

$$\int 2x^3 e^{x^2+5} dx = \int udv = uv - \int vdu = (e^{x^2+5})\left(\frac{x^4}{2}\right) - \int \left(\frac{x^4}{2}\right)(2xe^{x^2+5})dx = \frac{x^4 e^{x^2+5}}{2} - \int x^5 e^{x^2+5} dx = ???$$

Oh my goodness! We ended up with something even more complicated than we started with! That's because we chose the x^3 term as the term to be integrated, and powers of x generally get *more* complicated, not less complicated, on integration. (At least, *positive* powers of x do.) So clearly that's not the way to go. Maybe a different choice of u and dv is needed. Let's see. We tried $u = e^{x^2+5}$, so maybe this should have been v' instead. But if we set $dv = e^{x^2+5}dx$, then we can't find v because we don't know any antiderivatives of e^{x^2+5}. (It's a composite function. We'd need the substitution rule.) But wait a minute ... We saw (on our first attempt at using IBP for this problem) that for $u = e^{x^2+5}$ we get $du = e^{x^2+5}(2x) = 2xe^{x^2+5}$. But that means that we *do know* an antiderivative of $2xe^{x^2+5}$. That is, we know that for $dv = 2xe^{x^2+5}$ we get $v = e^{x^2+5}$. And if we include $2x$ with the e^{x^2+5} in the part that's in dv then that leaves only x^2 to be the part that's u. And for $u = x^2$ we have $du = 2xdx$, so we get:

$$\begin{aligned}
\int 2x^3 e^{x^2+5} dx &= \int (x^2)(2xe^{x^2+5})dx = \int udv \\
&= uv - \int vdu = (x^2)(e^{x^2+5}) - \int (e^{x^2+5})(2x)dx \\
&= x^2 e^{x^2+5} - \int (2xe^{x^2+5})dx \\
&= x^2 e^{x^2+5} - e^{x^2+5} + C = e^{x^2+5}(x^2-1) + C
\end{aligned}$$

Notice: In the second last step, to get rid of the last integral, we recognize that what we have is $\int dv$, which is just $v + C$. And in effect we used the Substitution Rule *while* we were doing IBP, in recognizing that we needed the $2x$ with the e^{x^2+5} for dv (twice) and that e^{x^2+5} is an antiderivative of that. (Notice also that we got the same answer as in Approach 1.)

Recall that when we use the substitution rule with a definite integral, since we change the variable with respect to which we're integrating, we need to change the limits of integration to be values of this new variable.

With Integration By Parts, we're identifying not just one but two "new variables". So if we have a definite integral do we need to change the limits of integration? Actually, no. If you think more carefully about what we do with Integration By Parts, you'll realize that we're not actually working with those new variables. We identify a u and a dv, and use them to find du and v, but these are all expressed in terms of x (or whatever variable we started with), and when we actually integrate, we're integrating with respect to the original variable. Since we aren't really performing a change of variables, there is no need to change the limits of integration.

But using the IBP formula, we express an integral, $\int u\,dv$, as the sum of a function, uv, and an integral, $-\int v\,du$. If what we have is a definite integral, $\int_a^b u\,dv$, and we want to evaluate it more efficiently than by using the approach " find an antiderivative, i.e. solve the indefinite integral, and then use this antiderivative to find the value of the definite integral", then we need to realize that both terms of the sum must be evaluated *"from a to b"*, i.e. we know that $\int_a^b f(x)\,dx = F(b)-F(a)$, where F is any antiderivative of f. When we have $\int_a^b u\,dv$, so that $F(x) = uv - \int v\,du$ is an antiderivative, we get $F(b) - F(a) = uv\big]_a^b - \int_a^b v\,du$.

That is, after we have done the integration of $\int v\,du$, to get something of the form $uv-g(x)$, where u and v are also functions of x, and $g(x)$ is an antiderivative of uv', we would have $F(x) = uv-g(x)$, so $F(b) = uv\big]_{x=b} - g(b)$ and $F(a) = uv\big]_{x=a} - g(a)$, which gives

$$F(b) - F(a) = uv\big]_{x=b} - uv\big]_{x=a} - (g(b) - g(a)) = uv\big]_a^b - g(x)\big]_a^b$$

But of course $g(b) - g(a) = g(x)\big]_a^b = \int_a^b v\,du$, since $g(x)$ is an antiderivative of uv', so *before* doing that last bit of integrating we could express what we have as $uv\big]_a^b - \int_a^b v\,du$. Therefore, for a definite integral, the Integration By Parts formula can be expressed as

Rule: Definite Integrals with Integration By Parts

$$\int_a^b udv = uv\big]_a^b - \int_a^b vdu$$

That is, we have the same formula (i.e. exactly the same approach), but we evaluate everything, including the uv term, "from a to b".

(*Notice:* Sometimes we express what we're doing this way, but don't actually do the evaluation until we're finished integrating.)

Example 8.9. Evaluate $\displaystyle\int_0^3 xe^{-x}dx$.

Solution:
Once we have determined that (1) we don't recognize the integrand as something we know an antiderivative function for without doing any work, and (2) substitution doesn't help in this case, we try Integration By Parts. Letting $u = x$ and $dv = e^{-x}dx$, and remembering that $\dfrac{e^{kx}}{k}$ is an antiderivative of e^{kx}, we get $du = dx$ and $v = \dfrac{e^{-x}}{-1} = -e^{-x}$. Using the formula from our new rule we get:

$$\begin{aligned}\int_0^3 xe^{-x}dx &= \left[x\left(-e^{-x}\right)\right]_0^3 - \int_0^3 \left(-e^{-x}\right)\,dx = \left[-xe^{-x}\right]_0^3 + \int_0^3 e^{-x}dx \\ &= \left[-xe^{-x}\right]_0^3 + \left[-e^{-x}\right]_0^3 = \left[-xe^{-x} - e^{-x}\right]_0^3 = \left[-e^{-x}(x+1)\right]_0^3 \\ &= \left[-e^{-3}(3+1)\right] - \left[-e^{0}(0+1)\right] = -\frac{1}{e^3}(4) + 1(1) = \frac{4}{e^3} + 1\end{aligned}$$

Example 8.10. Find the area under the curve $y = x \ln x^2$ on the interval $[1, 3]$.

Solution:
We may not be sure what the region looks like, but we know that $\ln x \geq 0$ when $x \geq 1$, and since $x^2 \geq 1$ when $x \geq 1$ then it's also true that $\ln x^2 \geq 0$ whenever $x \geq 1$. And it's also true that $x > 0$ whenever $x \geq 1$, so for $f(x) = x \ln x^2$, we have $f(x) \geq 0$ everywhere in $[1, 3]$. Therefore we know that the area under $y = x \ln x^2$ on $[1, 3]$ is given by

$$\text{Area} = \int_1^3 x \ln x^2 dx$$

so we can find the area of the region even though we don't know what the region looks like.

Approach 1:
Try a substitution. We see something being done to x^2, and we also have an x multiplier, so that seems promising. Letting $u = x^2$ gives $du = 2xdx$ so that $xdx = \frac{1}{2}du$. And then when $x = 1$ we have $u = 1^2 = 1$, and when $x = 3$ we have $u = 3^2 = 9$, so we get

$$\text{Area} = \int_1^3 x \ln x^2 dx = \frac{1}{2} \int_1^9 \ln u \, du$$

But we don't know an antiderivative of $\ln u$. Or if we do, it's only because we've done that problem before, using Integration By Parts. So we can either remember what we got last time, or repeat the Integration By Parts analysis of this integral. (Go ahead. Find $\int \ln x dx$ using integration by parts. And then call the variable u instead of x.) What we end up with is that $(u \ln u) - u$ is an antiderivative of $\ln u$. (If that's not what you got, compare what you did with Example 8.3 on page 118 in Unit 4.) Using this, we get:

$$\begin{aligned} \text{Area} &= \frac{1}{2} \int_1^9 \ln u \, du = \left[\frac{(u \ln u) - u}{2} \right]_1^9 \\ &= \left[\left(\frac{9 \ln 9}{2} \right) - \frac{9}{2} \right] - \left[\left(\frac{1 \ln 1}{2} \right) - \frac{1}{2} \right] = 9 \left[\left(\frac{1}{2} \right) \ln 9 \right] - \frac{9}{2} - \left(\frac{0}{2} \right) + \frac{1}{2} \\ &= 9(\ln 9^{1/2}) - \frac{8}{2} = (9 \ln \sqrt{9}) - 4 = (9 \ln 3) - 4 \end{aligned}$$

Approach 2:
Since we ended up needing Integration By Parts, we could have just done that from the start. Well, actually, not quite from the start. We might as well simplify the integrand function, using properties of logarithms. We have:

$$\int_1^3 x \ln x^2 dx = \int_1^3 x(2 \ln x) dx = \int_1^3 2x \ln x \, dx$$

Now, we use Integration By Parts. As always, we need the $\ln x$ term to be u so that we don't have to integrate it. So we let $u = \ln x$ and $dv = 2xdx$. This gives $v = x^2$ and $du = \frac{1}{x}dx$. We get:

$$\begin{aligned} \text{Area} &= \int_1^3 2x \ln x \, dx & &= x^2 \ln x \Big]_1^3 - \int_1^3 x^2 \left(\frac{1}{x} \right) dx \\ &= \left[(3^2 \ln 3) - (1^2 \ln 1) \right] - \int_1^3 x dx & &= (9 \ln 3) - 0 - \left[\frac{x^2}{2} \right]_1^3 \\ &= (9 \ln 3) - \left(\frac{3^2}{2} - \frac{1^2}{2} \right) & &= (9 \ln 3) - \left(\frac{9 - 1}{2} \right) \\ &= (9 \ln 3) - 4 \end{aligned}$$

Either way, we see that the area under $y = x \ln x^2$ on the interval $[1, 3]$ is $(9 \ln 3) - 4$ square units.

Partial Fractions

Now, let's think about a different kind of problem. We'll start by doing a fairly easy integral. All it requires is the sum rule, and then a very easy substitution for one of the integrals in the sum.

Example 8.11. (a) Find $\int \left(\frac{3}{x}+\frac{4}{x+4}\right)dx$.

Solution:
We start by breaking the integral of a sum into the sum of 2 integrals, and pulling out the constant multipliers:

$$\int \left(\frac{3}{x}+\frac{4}{x+4}\right)dx = \int \frac{3}{x}dx + \int \frac{4}{x+4}dx = 3\int \frac{1}{x}dx + 4\int \frac{1}{x+4}dx$$

Now $\int \frac{1}{x}dx$ we recognize. We know that $\ln|x|$ is an antiderivative of $\frac{1}{x}$. And for $\int \frac{1}{x+4}dx$ we recognize the form $\frac{u'}{u}$, where $u = x+4$ with $\frac{du}{dx} = 1$. That is, we substitute $u = x+4$ so that $du = dx$ to get $\int \frac{1}{x+4}dx = \int \frac{1}{u}du$. And now we see that $\ln|u| = \ln|x+4|$ is an antiderivative of $\frac{1}{u} = \frac{1}{x+4}$. So we have

$$\int \left(\frac{3}{x}+\frac{4}{x+4}\right)dx = 3\int \frac{1}{x}dx + 4\int \frac{1}{x+4}dx = 3\ln|x| + 4\ln|x+4| + C$$

Notice: To save having to do a substitution like that every time (and it will come up a lot in this section), it's useful to remember that:

$$\text{For any constant } k,\ \int \frac{1}{x+k}dx = \ln|x+k| + C$$

This result is obtained in just the same was as $\int \frac{1}{x+4}dx = \ln|x+4| + C$ was obtained. That is, we can substitute $u = x+k$ so that $du = dx$ to get $\int \frac{1}{x+k}dx = \int \frac{1}{u}du = \ln|u|+C = \ln|x+k|+C$.

In that example, the integrand function was $\frac{3}{x}+\frac{4}{x+4}$. But usually in mathematics, for a sum of fractions (or rational functions) the terms in the sum are brought to a common denominator so that they can be added and the quantity (or function) is instead expressed as a single fraction (or rational function). In this case, we have

$$\frac{3}{x}+\frac{4}{x+4} = \left(\frac{3}{x}\right)\left(\frac{x+4}{x+4}\right)+\left(\frac{4}{x+4}\right)\left(\frac{x}{x}\right) = \frac{3(x+4)}{x(x+4)}+\frac{4x}{(x+4)x} = \frac{3x+12}{x^2+4x}+\frac{4x}{x^2+4x} = \frac{7x+12}{x^2+4}$$

Example 8.11. (b) Find $\int \frac{7x+12}{x^2+4x}dx$.

Solution:
Since we know that $\frac{7x+12}{x^2+4x} = \frac{3}{x}+\frac{4}{x+4}$ and we have just found $\int \left(\frac{3}{x}+\frac{4}{x+4}\right)dx$, then we see that

$$\int \frac{7x+12}{x^2+4x}dx = \int \left(\frac{3}{x}+\frac{4}{x+4}\right)dx = 3\ln|x| + 4\ln|x+4| + C$$

But how would we find that sum if we hadn't just done it? That is, how can we undo the process of bringing the terms of a sum to a common denominator to express them as a single fraction (or rational function)? How can we break a rational function into the parts that make up the sum?

Let's think about what we would do with the example we've just been doing. We have $\dfrac{7x+12}{x^2+4x}$. We can easily factor the denominator as $x^2+4x = x(x+4)$. Therefore we think that we can break the function up into 2 parts: one with x in the denominator and the other with $x+4$ in the denominator, where the parts sum to the original function. But we don't know what the numerators should be. Well, okay, let's just call them A and B for now. That is, we suppose that there are some values A and B for which

$$\frac{7x+12}{x^2+4x} = \frac{A}{x} + \frac{B}{x+4}$$

Now, we just have to solve for the values of A and B. We do that by bringing the right hand side of the above equation to a common denominator and making it look like the left hand side. We get:

$$\frac{A}{x} + \frac{B}{x+4} = \left(\frac{A}{x}\right)\left(\frac{x+4}{x+4}\right) + \left(\frac{B}{x+4}\right)\left(\frac{x}{x}\right) = \frac{Ax+4A+Bx}{x(x+4)} = \frac{(A+B)x+4A}{x^2+4x}$$

But this must be equal to what we started with. That is, we now know that

$$\frac{(A+B)x+4A}{x^2+4x} = \frac{7x+12}{x^2+4x}$$

And that means that we must have $(A+B)x+4A = 7x+12$. For that to be true, the coefficients of x must be the same on both sides of the equation, and the constant terms must also be the same on both sides. So it must be true that $4A = 12$, which tells us that $A = 3$, and also that $A+B = 7$, which gives $B = 7-A = 7-3 = 4$. So we have just found that

$$\frac{7x+12}{x^2+4x} = \frac{A}{x} + \frac{B}{x+4} = \frac{3}{x} + \frac{4}{x+4}$$

And we found that without using anything more than the assumption that there must be some value A and some value B that would give us a sum of that form.

What we have just done is to use to break the function up into **Partial Fractions**. The sum $\frac{3}{x}+\frac{4}{x+4}$ is called the **Partial Fraction Decomposition** of $\frac{7x+12}{x^2+4x}$. When can we do something like this? Well, whenever we have a rational function (a function which is a ratio of polynomials) where the highest power of x in the denominator is larger than the highest power of x in the numerator (we say that the denominator is a *higher degree polynomial*), and the denominator can be factored into distinct linear terms of the form $x+c$.

Finding the Partial Fraction Decomposition of $\frac{f(x)}{g(x)}$

Consider the rational function $h(x) = \frac{f(x)}{g(x)}$ for some polynomials $f(x)$ and $g(x)$, where f is a lower degree polynomial than g, and where g can be factored into *distinct* linear terms as $g(x) = (x+a)(x+b)(x+c)....$ We find the **Partial Fraction Decomposition** of h as follows:

Step 1: Factor the denominator as $g(x) = (x+a)(x+b)(x+c)...$

Step 2: Separate the fraction into parts, placing new unknowns in the numerators, to get

$$\frac{f(x)}{g(x)} = \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c} + ...$$

Step 3: Bring the parts to a common denominator and add them together.

Step 4: Equate the numerator sum to $f(x)$.

Step 5: Solve for the unknowns A, B, C, ...

In solving for the values of A, B, etc. in Step 5, it may be useful to use the zeroes of the denominator polynomial. To bring the parts to a common denominator, we multiply both the numerator and the denominator of each part by all of the factors of g *except* the one that's already in the denominator of that part. So if one of the factors of g is $x+a$, then setting $x=-a$ gives $x+a=0$, and for that value of x all of the numerators that have been multiplied by the linear factor $(x+a)$ will disappear, leaving just one unknown whose value can be easily found. You'll understand better from seeing an actual example.

Example 8.12. (a) Find the partial fraction decomposition of $\dfrac{1}{x^2-4}$.

Solution:
We have $h(x)=\frac{f(x)}{g(x)}$ where $f(x)=1$ and $g(x)=x^2-4$. We see that f and g are both polynomials, and that the highest power of x in the denominator (2, from x^2) is bigger than the highest power of x in the numerator (0, from x^0, i.e. the constant term).

Step 1:
We can factor the denominator as $x^2-4=(x+2)(x-2)$. We see that there are only linear terms, and each is distinct (i.e. there are no repeated factors), so we can proceed with the partial fraction decomposition.

Note: Because we are assuming that the numerator of each part is just a constant, the method we are using **only** works with linear factors, and **only** when there are no repeated factors. For other factorizations, different decomposition methods are needed.

Step 2:
We set up one part for each factor of the denominator function. Each part has that factor as the denominator and has a different unknown for the numerator.

$$\frac{1}{x^2-4}=\frac{A}{x+2}+\frac{B}{x-2}$$

Step 3:
We bring the parts in the right hand side of the equation to a common denominator and add them all together.

$$\begin{aligned}\frac{A}{x+2}+\frac{B}{x-2} &= \frac{A(x-2)}{(x+2)(x-2)}+\frac{B(x+2)}{(x+2)(x-2)}\\ &= \frac{A(x-2)+B(x+2)}{(x+2)(x-2)}\end{aligned}$$

Step 4:
At this point, we have

$$\frac{1}{x^2-4}=\frac{A(x-2)+B(x+2)}{(x+2)(x-2)}$$

The denominator on the right side is equal to the denominator on the left side, so it must also be true that the numerator on the left side is equal to the numerator on the right side. That is, we must have

$$A(x-2)+B(x+2)=1$$

Step 5:
To solve $A(x-2)+B(x+2)=1$ for the values of A and B, we can use the x-values which make $x-2$ and $x+2$ be 0. First, for $x=2$ we have $x-2=0$ and $x+2=2+2=4$ and so when $x=2$ we have

$$A(x-2)+B(x+2)=1 \quad\Rightarrow\quad A(0)+B(4)=1 \quad\Rightarrow\quad 4B=1 \quad\Rightarrow\quad B=\frac{1}{4}$$

And when $x = -2$, so that $x + 2 = 0$, we have $x - 2 = -2 - 2 = -4$ and so we get

$$A(x-2) + B(x+2) = 1 \quad \Rightarrow \quad A(-4) + B(0) = 1 \quad \Rightarrow \quad -4A = 1 \quad \Rightarrow \quad A = -\frac{1}{4}$$

Using each value in its corresponding part of the partial fraction decomposition, we get

$$\frac{1}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2} = \frac{-1/4}{x+2} + \frac{1/4}{x-2} = \frac{1/4}{x-2} - \frac{1/4}{x+2}$$

Example 8.12. (b) Find $\int \frac{1}{x^2-4}dx$.

Solution:
Having found the partial fraction decomposition, we can use it to find the integral. We have

$$\begin{aligned}
\int \frac{1}{x^2-4}dx &= \int \left(\frac{1/4}{x-2} - \frac{1/4}{x+2}\right) dx \\
&= \int \frac{1/4}{x-2}dx - \int \frac{1/4}{x+2}dx \\
&= \frac{1}{4}\int \frac{1}{x-2}dx - \frac{1}{4}\int \frac{1}{x+2}dx \\
&= \frac{1}{4}(\ln|x-2|) - \frac{1}{4}(\ln|x+2|) + C \\
&= \frac{1}{4}(\ln|x-2| - \ln|x+2|) + C \\
&= \frac{1}{4}\left(\ln\left|\frac{x-2}{x+2}\right|\right) + C \\
&= \ln\left|\frac{x-2}{x+2}\right|^{1/4} + C \\
&= \ln\sqrt[4]{\left|\frac{x-2}{x+2}\right|} + C
\end{aligned}$$

Example 8.13. Find $\int \frac{7x+5}{(x+1)(x-1)(x+2)}dx$.

Solution:
We need to decompose the integrand function into 3 parts, corresponding to the 3 linear factors in the denominator of this rational function:

$$\frac{7x+5}{(x+1)(x-1)(x+2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x+2}$$

We bring the right hand side to a common denominator:

$$\frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x+2} = \frac{A}{x+1} \times \frac{(x-1)(x+2)}{(x-1)(x+2)} + \frac{B}{x-1} \times \frac{(x+1)(x+2)}{(x+1)(x+2)} + \frac{C}{x+2} \times \frac{(x-1)(x+1)}{(x-1)(x+1)}$$

so we have

$$\frac{7x+5}{(x+1)(x-1)(x+2)} = \frac{A(x-1)(x+2) + B(x+1)(x+2) + C(x+1)(x-1)}{(x+1)(x-1)(x+2)}$$

Now equating numerators, we see that:

$$7x+5 = A(x-1)(x+2) + B(x+1)(x+2) + C(x+1)(x-1)$$

Again, the "zeroes" of the denominator polynomial help us to find the unknowns. When $x = -1$, we have $x+1 = 0$ and so both the B and C terms on the right hand side are 0. We get:

$$\begin{aligned} x=-1 \quad \Rightarrow& \quad 7(-1)+5 = A(-1-1)(-1+2) + B(-1+1)(-1+2) + C(-1-1)(-1+1) \\ \Rightarrow& \quad -7+5 = A(-2)(1) + B(0)(1) + C(-2)(0) \\ \Rightarrow& \quad -2 = A(-2) + 0 + 0 \\ \Rightarrow& \quad -2 = -2A \\ \Rightarrow& \quad A = 1 \end{aligned}$$

Similarly, when $x = 1$ we have $x-1 = 0$ and so both the A and C terms are 0, giving:

$$7(1)+5 = A(0) + B(1+1)(1+2) + C(0) \quad \Rightarrow \quad 12 = B(2)(3) \quad \Rightarrow \quad B = \frac{12}{6} = 2$$

Finally, when $x = -2$ we have $x+2 = 0$ so that both the A term and the B term are 0 and we get:

$$7(-2)+5 = 0+0+C(-2+1)(-2-1) \quad \Rightarrow \quad -14+5 = C(-1)(-3) \quad \Rightarrow \quad 3C = -9 \quad \Rightarrow \quad C = -3$$

We see that the partial fraction decomposition is:

$$\frac{7x+5}{(x+1)(x-1)(x+2)} = \frac{1}{x+1} + \frac{2}{x-1} - \frac{3}{x+2}$$

(We can check these calculations by bringing the right hand side of the decomposition to a common denominator and working out the numerator, to make sure we get the numerator that's shown on the left hand side.)

We can now use this decomposition to evaluate the integral:

$$\begin{aligned} \int \frac{7x+5}{(x+1)(x-1)(x+2)} dx &= \int \left(\frac{1}{x+1} + \frac{2}{x-1} - \frac{3}{x+2} \right) dx \\ &= \int \frac{1}{x+1} dx + 2 \int \frac{1}{x-1} dx - 3 \int \frac{1}{x+2} dx \\ &= \ln|x+1| + 2\ln|x-1| - 3\ln|x+2| + C \\ &= \ln|x+1| + \ln|x-1|^2 - \ln|x+2|^3 + C \\ &= \ln\left(\frac{|x+1|(x-1)^2}{|x+2|^3} \right) + C \\ &= \ln\left| \frac{(x+1)(x-1)^2}{(x+2)^3} \right| + C \end{aligned}$$

Improper Integrals

Recall how we defined the notation for definite integrals, in Definition 5.3:

> If a function f is continuous on the closed interval $[a, b]$, then:
> $\int_a^b f(x)dx$ is called the **definite integral of f from a to b**

This definition requires that $[a, b]$ be a closed interval. And that means that both a and b must be real, *finite*, numbers. Therefore $\int_a^b f(x)dx$ is only defined (so far) when both limits of integration are finite. We now extend that definition, to allow one or both of the limits of integration to be infinite. It is considered that *properly*, a definite integral should have finite limits of integration. So if it doesn't, it's considered to be *improper*.

Definition 8.1. An **improper integral** is a definite integral in which one or both of the limits of integration are infinite.

For instance, $\int_a^\infty f(t)dt$, $\int_{-\infty}^b f(t)dt$, and $\int_{-\infty}^\infty f(t)dt$ are all examples of improper integrals.

We evaluate an improper integral by determining what happens in the limit as the corresponding limit of integration approaches infinity. That is, we define an improper integral as a limit of a proper integral.

Definition 8.2.
(1) Let $f(t)$ be any function which is continuous on $[a, \infty)$, where a is any (finite) real number. We define:

$$\int_a^\infty f(t)dt = \lim_{b\to\infty} \int_a^b f(t)dt$$

(2) Let $f(t)$ be any function which is continuous on $(-\infty, b]$, where b is any (finite) real number. We define:

$$\int_{-\infty}^b f(t)dt = \lim_{a\to-\infty} \int_a^b f(t)dt$$

(3) Let $f(t)$ be any function which is continuous everywhere. We define:

$$\int_{-\infty}^\infty f(t)dt = \int_{-\infty}^a f(t)dt + \int_a^\infty f(t)dt$$

where a may be any arbitrarily chosen (finite) real number.

Note: We often use $a = 0$ in (3), but any convenient value can be used.

Definition 8.3. If a limit corresponding to an improper integral exists (i.e. has a finite value) then we say that the improper integral **converges** to the value of the limit; otherwise (i.e., if the limit is infinite or for some other reason does not exist) we say that the improper integral **diverges**.

That is, to evaluate an improper integral with one infinite limit of integration, we simply apply the definition, by evaluating a finite, i.e. proper, integral and evaluating a limit at ∞ or $-\infty$. If the limit exists and has some finite value L, then the improper integral converges to (i.e. has the value) L. Otherwise, we conclude that the improper integral diverges (i.e. has no defined value).

To evaluate an improper integral in which both limits of integration are infinite, we have to evaluate 2 improper integrals, each with only 1 infinite limit of integration. If either of these improper integrals diverges, then the improper integral with both limits of integration being infinite also diverges. That is, we have $\int_{-\infty}^{\infty} f(t)dt = \int_{-\infty}^{a} f(t)dt + \int_{a}^{\infty} f(t)dt$, so if either $\int_{-\infty}^{a} f(t)dt$ or $\int_{a}^{\infty} f(t)dt$ diverges (has no defined value) then the sum, i.e. $\int_{-\infty}^{\infty} f(t)dt$, also diverges. If both $\int_{-\infty}^{a} f(t)dt$ and $\int_{a}^{\infty} f(t)dt$ converge, say to L_1 and L_2 respectively, then $\int_{-\infty}^{\infty} f(t)dt$ converges to the sum of the limits, i.e. to $L_1 + L_2$. We can look at some examples of evaluating improper integrals.

Example 8.14. Does $\displaystyle\int_0^{\infty} x^3 dx$ converge or diverge?

Solution:
From the definition, we have:

$$\int_0^{\infty} x^3 dx = \lim_{b\to\infty} \int_0^b x^3 dx = \lim_{b\to\infty} \left.\frac{x^4}{4}\right]_0^b = \lim_{b\to\infty} \left[\frac{b^4 - 0^4}{4}\right] = \lim_{b\to\infty} \frac{b^4}{4} = \frac{1}{4} \lim_{b\to\infty} b^4$$

Since $b^4 \to \infty$ as $b \to \infty$, this limit does not have a finite value. Therefore $\int_0^{\infty} x^3 dx$ diverges.

Example 8.15. Determine whether $\displaystyle\int_{-\infty}^{1} e^{.01t} dt$ converges or diverges. If it converges, find its value.

Solution:
Again, we use the definition:

$$\int_{-\infty}^{1} e^{.01t} dt = \lim_{a\to-\infty} \int_a^1 e^{.01t} dt = \lim_{a\to-\infty} \left.\frac{e^{.01t}}{.01}\right]_a^1$$

And since $\frac{1}{.01} = 100$, we have

$$\int_{-\infty}^{1} e^{.01t} dt = \lim_{a\to-\infty} 100e^{.01t}\Big]_a^1 = \lim_{a\to-\infty} (100e^{.01} - 100e^{.01a}) = 100e^{.01} - \lim_{a\to-\infty} (e^{.01a})$$

Since $.01a \to -\infty$ as $a \to -\infty$, then we have e raised to a very large negative power. That is, $e^{.01a} = \frac{1}{e^{.01|a|}}$ since a is negative. So $e^{.01a} \to 0$ as $a \to -\infty$, i.e. $\lim_{a\to-\infty} e^{.01a} = 0$. Thus, $100e^{.01} - \lim_{a\to-\infty} e^{.01a} = 100e^{.01} - 0 = 100e^{.01}$. We see that the improper integral converges to $100e^{.01}$. That is, we see that $\displaystyle\int_{-\infty}^{1} e^{.01t} dt = 100e^{.01}$

Example 8.16. Evaluate $\displaystyle\int_{-\infty}^{\infty} xe^{9-x^2} dx$

Solution:
By definition, and choosing $a = 0$, we have:

$$\int_{-\infty}^{\infty} xe^{9-x^2} dx = \int_{-\infty}^{0} xe^{9-x^2} dx + \int_0^{\infty} xe^{9-x^2} dx$$

We need to evaluate both of these improper integrals. For each, we will need an antiderivative of xe^{9-x^2}.

To find $\int xe^{9-x^2}dx$, we use the substitution rule. Letting $u = 9 - x^2$ we have $du = -2xdx$ so that $xdx = -\frac{1}{2}du$. So we have:

$$\int xe^{9-x^2}dx = -\frac{1}{2}\int e^u du = -\frac{e^u}{2} + C = -\frac{e^{9-x^2}}{2} + C$$

That is, we see that $-\frac{e^{9-x^2}}{2}$ is an antiderivative of xe^{9-x^2}. So we have:

$$\int_{-\infty}^{0} xe^{9-x^2}dx = \lim_{a\to-\infty}\int_a^0 xe^{9-x^2}dx = \lim_{a\to-\infty}\left[-\frac{e^{9-x^2}}{2}\right]_a^0 = -\frac{1}{2}\left[\lim_{a\to-\infty}(e^{9-0} - e^{9-a^2})\right] = -\frac{e^9}{2}+\frac{1}{2}\left(\lim_{a\to-\infty} e^{9-a^2}\right)$$

As $a \to -\infty$, $a^2 \to \infty$, so $9 - a^2 \to -\infty$ and hence $e^{9-a^2} \to 0$. That is, $\lim\limits_{a\to-\infty} e^{9-a^2} = 0$, so we have

$$\int_{-\infty}^{0} xe^{9-x^2}dx = -\frac{e^9}{2} + \frac{1}{2}(0) = -\frac{e^9}{2}$$

Similarly, we have:

$$\int_0^\infty e^{9-x^2}dx = \lim_{b\to\infty}\int_0^b xe^{9-x^2}dx = \lim_{b\to\infty}\left[-\frac{e^{9-x^2}}{2}\right]_0^b = -\frac{1}{2}\left[\lim_{b\to\infty}(e^{9-b^2} - e^{9-0})\right] = \frac{e^9}{2}-\frac{1}{2}\left(\lim_{b\to\infty} e^{9-b^2}\right)$$

But $\lim\limits_{b\to\infty} e^{9-b^2} = 0$ (since as $b \to \infty$, $b^2 \to \infty$ and $9 - b^2 \to -\infty$) and so $\displaystyle\int_0^\infty xe^{9-x^2}dx = \frac{e^9}{2}$.

Since both $\int_{-\infty}^0 xe^{9-x^2}dx$ and $\int_0^\infty xe^{9-x^2}dx$ converge, then $\int_{-\infty}^\infty xe^{9-x^2}dx$ also converges. We see that

$$\int_{-\infty}^{\infty} xe^{9-x^2}dx = \int_{-\infty}^{0} xe^{9-x^2}dx + \int_0^\infty xe^{9-x^2}dx = -\frac{e^9}{2} + \frac{e^9}{2} = 0$$

Math 1225A/B

Unit 9:
Functions of Two or More Variables

(text reference: Sections 10.1 and 10.2

custom text pgs. 207 - 226)

9 Functions of Two or More Variables

When we talk about a "function of a certain number of variables" we mean *independent* variables. A function of the form $y = f(x)$ expresses the dependent variable y as a function of one independent variable, x. So x can have any value in it's domain, i.e. doesn't depend on anything, but the value of y is determined by, i.e. depends on, the value of x. So f is a function of only a *single* variable, x.

When we have a function of 2 or more independent variables, each of these variables is free to take on any value within the domain, no matter what the values of the other independent variables are (although there may be some domain restrictions on which values can occur together), and the function value is entirely determined by, i.e. depends on, the values of these independent variables. So we have something like $z = f(x, y)$ expressing that the value of the *dependent* variable z is determined by the values of the independent variables x and y, in the manner described by the function f, i.e. using the rule stated by the function f. And the domain of f is the set containing all pairs (x, y) for which f is defined.

For instance, the function $z = f(x, y)$ with $f(x, y) = x - y$ says that the value of z depends on the values of both x and y, being determined by the difference $x - y$. In this function, each of x and y may have any real value, and there are no restrictions on which values may occur together. No matter what real value x has, $x - y$ is defined for all real values of y, and vice versa. On the other hand, in the function $g(x, y) = \dfrac{1}{x - y}$, again each of x and y may have any real value, but there is a restriction on which values may occur together, because we cannot have $x - y = 0$. Therefore x can have any real value, and y can have any real value, but they cannot both have the same value at the same time. So $g(x, y)$ is defined for all pairs (x, y) with $x \in \Re$ and $y \in \Re$ and $x \neq y$. So *when* we have $x = 1$, we cannot also have $y = 1$, but we *can* have $y = 1$ any time that $x \neq 1$.

But as with a function of only a single variable, there may be some values of an independent variable which cannot ever occur, i.e. for which the multivariate function is never defined. For instance, the function $f(x, y) = \dfrac{y^2 \ln y}{x}$, is never defined when $x = 0$ and is also never defined when $y \leq 0$ (since natural logarithm is only defined for positive numbers). So f is defined for any $x \neq 0$ and $y > 0$. But in this case there are, within those values, no further restrictions, i.e. no restrictions on which values of x and y may occur together. For *any* value of x with $x \neq 0$ and *any* value of y with $y > 0$, the value of $z = f(x, y)$ is defined. However, there are also functions which have both kinds of restrictions, i.e. values which an independent variable may never have, and also values of the independent variables which cannot occur together. Consider, for instance, the function $g(x, y) = x^2 + e^y - \ln xy$. In the domain of this function, neither x nor y can ever be 0 (since if either is 0 then xy is 0 and $\ln xy$ is not defined). But x can have any non-zero real value, and so can y ... but only if x and y have the same sign, because as long as they do, then xy is positive, so $\ln xy$ is defined, but whenever x and y have different signs, xy is negative and so $\ln xy$ is not defined. So $z = g(x, y)$ is defined for any $x \neq 0$ and any $y \neq 0$ for which $xy > 0$, i.e. for all pairs (x, y) for which $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$.

Of course, we could have more than 2 independent variables. We can have something like $w = f(x, y, z)$, with 3 independent variables, or $y = g(x_1, x_2, x_3, x_4, x_5)$, which defines y as a function of 5 different independent variables. And there can be values of any one of the independent variables for which the function is never defined, and also restrictions about the interactions of the variables so that there are some values which, although allowed in general, cannot occur together.

As you surely already realize, we evaluate a function of 2 or more variables at a particular combination of values of the independent variables simply by "plugging in" the particular value of each independent variable and calculating what the rule which is the function says is the function value. For instance, if we have some function $f(x, y, z)$, we find $f(a, b, c)$ by setting $x = a$, $y = b$ and $z = c$ and calculating whatever the rule for f says should be done with those values.

Notice: The independent variables are always listed in the same order inside the brackets. So if the function f has been defined as $f(x, y, z, w)$ equals some expression, then $f(a, b, c, d)$ means set $x = a$, $y = b$, $z = c$ and $w = d$. But if the statement of the function said $f(w, x, y, z)$ equals that same expression, the $f(a, b, c, d)$ is found by setting $w = a$, $x = b$, $y = c$ and $z = d$, which is different.

Example 9.1. Find the specified value of the given function.

(a) Find $f(-2, 3)$ where $f(x, y) = x^2 + 2y$.

(b) For $h(x, y, z) = xy + yz - 2xy^2z$, what is $h(1, -1, -2)$?

(c) If $f(x_1, x_2, x_3, x_4) = 2x_1^2x_2 - \dfrac{\sqrt{x_3x_4}}{x_2}$, evaluate $f(3, 2, 4, 1)$.

Solution:
(a) We use $x = -2$ and $y = 3$ to evaluate $f(x, y) = x^2 + 2y$. We get

$$f(-2, 3) = (-2)^2 + 2(3) = 4 + 6 = 10$$

(b) Here, we replace x by 1, y by -1 and z by -2 in $h(x, y, z) = xy + yz - 2xy^2z$ to get

$$h(1, -1, -2) = (1)(-1) + (-1)(-2) - 2(1)(-1)^2(-2) = -1 + 2 - (2)(1)(-2) = 1 + 4 = 5$$

(c) With $x_1 = 3$, $x_2 = 2$, $x_3 = 4$ and $x_4 = 1$ in $f(x_1, x_2, x_3, x_4) = 2x_1^2x_2 - \dfrac{\sqrt{x_3x_4}}{x_2}$ we get

$$f(3, 2, 4, 1) = 2(3)^2(2) - \frac{\sqrt{(4)(1)}}{2} = 4(9) - \frac{2}{2} = 36 - 1 = 35$$

Example 9.2. For $f(x, y) = x\sqrt{y}e^x$, find $f(t, t^2)$ and $f(x + h, y) - f(x, y)$.

Solution:
We just replace x and y by the specified values, so for $f(t, t^2)$ we replace x by t, and y by t^2, in the expression which for $f(x, y)$. We get $f(t, t^2) = t\sqrt{t^2}e^t = t|t|e^t$. (Remember, in $\sqrt{t^2}$, if $t < 0$ the negative is lost by taking the positive square root of t^2, so $\sqrt{t^2} = |t|$. And notice that $t|t| \neq t^2$, and likewise $t|t| \neq |t^2|$, since $|t^2| = t^2$ is strictly non-negative, whereas $t|t|$ is negative whenever $t < 0$. For instance, $-1| - 1| = (-1)(1) = -1$. So there is no shorter way to express $t|t|$. We would need to use a piecewise definition: $f(t, t^2) = t^2e^t$ if $t \geq 0$ and $f(t, t^2) = -t^2e^t$ if $t < 0$.)

Likewise, to find $f(x + h, y) - f(x, y)$, we replace x by $x + h$ to find $f(x + h, y)$, and then subtract $f(x, y)$ and simplify as much as possible. We get:

$$\begin{aligned} f(x+h, y) - f(x, y) &= (x+h)\sqrt{y}e^{x+h} - x\sqrt{y}e^x \\ &= (x+h)\sqrt{y}e^xe^h - x\sqrt{y}e^x \\ &= e^x\sqrt{y}\left[(x+h)e^h - x\right] \end{aligned}$$

These kinds functions often arise in real life. In fact, they arise in real life more often than functions of a single variable do, because real life tends to be complicated, with the value of something being determined by multiple factors rather than just a single other thing. And you're actually quite familiar with some functions of 2 or more variables, although you may not have ever thought of them that way. For instance, in calculating volumes of revolution we used the fact that the volume of a cylinder is given by $V = \pi r^2h$, that is the value of the dependent variable V is a function of the 2 independent variables r and h. A cylinder may have any positive radius and any positive height, so for any values $r > 0$ and $h > 0$, the volume of a cylinder with radius r and height h is given by the function $V(r, h) = \pi r^2 h$.

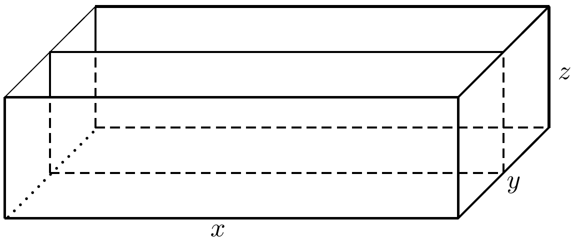
Example 9.3. An open rectangular box with length x cm, width y cm, and height z cm, is to be constructed, with one lengthwise partition. The material used to make the box has negligible thickness (e.g. very thin sheet metal or cardboard, say).

(a) Find formulas for the surface area S of the interior of the box, and the volume V of the box.

(b) Find a formula for the total cost C, in dollars, of making the box if the material for the bottom of the box costs 25 cents per square cm, the material for all sides costs 15 cents per square cm and the material for the partition costs 10 cents per square cm.

(c) Find the volume and cost for a box 5 cm long, 4 cm wide and 2 cm high.

Solution:
We start by drawing a picture, so we can see what's going on.



(a) The volume V of the box is easiest to determine, so we'll start with that. Since the partition has negligible thickness, we ignore it. The volume of the box is simply given by length times width times height and we have length x cm, width y cm and height z cm so the volume of the box, measured in cubic centimetres, is given by

$$V(x, y, z) = xyz$$

For the surface area S of the interior of the box, we think carefully. Since the partition is of negligible thickness, the surface area of the bottom of the box is given by length times width, xy. And since it is an open box, i.e. has no top, then there is only one piece which has this shape. Similarly, the surface area of each (interior) end of the box is given by width times height, yz. And there are two of these, so together they contribute $2yz$ to S. Now, we have the inside of the front of the box, and the inside of the back, and also *both* sides of the partition. (For instance, if we were going to paint the interior of the box, we would need to paint both sides of the partition, so in determining the surface area to be painted, we count the partition twice.) Each of these 4 "sides" has identical size given by length by height, xz, so together these contribute $4xz$ to S. And of course the total surface area of the interior of the box is the sum of these various interior surfaces, so S, measured in square centimetres, is given by

$$S(x, y, z) = xy + 2yz + 4xz$$

(b) We need to find C, the function giving the total cost of the box, given the cost per square centimetre of the various materials used to construct the box. The box has one bottom (and no top), 2 ends, a front, a back and a partition. The 2 ends, the front and the back are all "sides" of the box. Notice that this situation is different than when we determined the surface area of the interior of the box in that we only count the partition once. That is, we may have to paint both sides of the partition, but we only need to pay for one partition.

The bottom of the box measures xy square cm and is made from material which costs \$0.25 per square cm, so the cost for the bottom is $\$0.25xy$. The four "sides" of the box are all made from material costing \$0.15 per square cm. There are the 2 ends, each measuring yz square cm, and the front and back, which each measure xz square cm, so the total cost of the "sides" is $\$0.15(2yz + 2xz) = \$0.30yz + \$0.30xz$. And the one partition costs \$0.10 per square cm and also measures xz square cm and will therefore cost $\$0.10xz$. So the total cost in dollars is

$$C(x, y, z) = .25xy + .3yz + .3xz + .1xz = .25xy + .3yz + .4xz$$

(c) We are asked to find both the volume and the cost for a box with is 5 cm long by 4 cm wide by 2 cm high. That is, we need to evaluate both $V(x, y, z)$ and $C(x, y, z)$ at $(x, y, z) = (5, 4, 2)$ (being careful to check that the order in which the values of the variables are given to us is the same as the order in which we listed them in defining V and C). Using $V(x, y, z) = xyz$ and $C(x, y, z) = .25xy + .3yz + .4xz$ we get

$$\begin{aligned} V(5,4,2) &= 5 \times 4 \times 2 = 40 \\ C(5,4,2) &= .25(5)(4) + .3(4)(2) + .4(5)(2) = 5 + 2.4 + 4 = 11.4 \end{aligned}$$

The box will have volume 40 cubic cm and will cost \$11.40 to produce. (Goodness! It's quite small, and the materials didn't sound very expensive, but it sure adds up!)

Partial Derivatives

Now that we understand about what a function of 2 or more variables is, and how to use the notation and evaluate the function, we can talk about doing Calculus with these functions. We'll only be doing *differential calculus* with these functions, i.e. calculating and using derivatives. *Integral calculus* of this kind of function, i.e. working with integrals, is beyond the scope of this course.

When we have a function of only one variable, *the derivative* of the function *clearly means* the derivative with respect to that one independent variable. For $y = f(x)$, when we call "the derivative" $\frac{dy}{dx}$, we're explicitly saying "the derivative with respect to x". But we also call "the derivative" $f'(x)$, and when we do so it's *implied* that we mean that we're differentiating with respect to x, since that's the variable that f is a function of.

What about a function of more than one variable? Does, for instance, $f(x, y, z)$ have a derivative? Well, no, there is no "the derivative" for a function of more than one variable. But that's because there are *several* derivatives, depending on which (independent) variable we differentiate with respect to.

Recall that for $y = f(x)$, $\frac{dy}{dx} = f'(x)$ tells us what happens to the value of y as the value of x changes. That is, $\frac{dy}{dx}$ is the rate of change in y with respect to x. Because y is dependent on x, y changes whenever x changes. $\frac{dy}{dx}$ tells us about *how* it changes. For a function like $w = f(x, y, z)$, w is dependent on each of x, y and z. If any *one* of these changes, it causes a change in w. To understand the behaviour of w, we need to consider all the various reasons why it would change — because x changed, or because y changed, or because z changed. That is, when w changes, there could be several *parts* to the change — the part that was caused by x changing, the part that was caused by y changing, and/or the part that was caused by z changing.

To examine how w changes with respect to x, i.e. the part of any change in w which is caused by x changing, we examine what happens to w when **only** x changes, i.e. when x changes, but y and z remain fixed, i.e. remain **constant**. Because this explains only *part* of how w changes, we call it a **partial derivative**. And to remind us that it's different than "the derivative" for a function of only a single variable, we use slightly different notation.

Definition 9.1. If $w = f(x, y, z)$, that is if the value of w depends on x and y and z, then $\dfrac{\partial w}{\partial x}$ denotes the **partial derivative** of w with respect to x, which is the rate of change in w when *only* x changes, and the other independent variables remain constant. Similarly, $\dfrac{\partial w}{\partial y}$ and $\dfrac{\partial w}{\partial z}$ denote the partial derivatives with respect to y and z, respectively.

Notes:

1. We often just say *partials* instead of *partial derivatives.*
2. In this definition, we have explicity defined only the partial derivatives for a function of 3 independent variables. However the same idea applies to any function of 2 or more variables. If we have a function of only 2 variables, there will be only 2 of these partial derivatives. For instance, the "partials" of $z = f(x, y)$ are $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. And in general, for a function of n independent variables there are n of these partials: one for each of the independent variables.
3. ∂ is just a fancy form of the letter d.
4. When we're working with functions of a single variable, we pronounce $\frac{dy}{dx}$ as "dee y dee x". Just as we have a fancier way of writing the letter d when we're talking about partial derivatives, we also have a fancier way of pronouncing it. We pronounce $\frac{\partial w}{\partial x}$ as "dye w dye x".

So how do we calculate these partial derivatives? Oh dear, will we have to learn a whole slew of new differentiation techniques? New "Rules for Partials"? Well, no. And we've already discussed why. If only one of the independent variables is changing, the others stay fixed, i.e. remain constant. And so when we calculate the partial with respect to one particular variable, we simply *treat all of the other independent variables as* **constants**. And we already know how to deal with constants when differentiating. So we have only this one very easy rule for calculating partials:

> **Rule**
> For any function of more than one variable, to find the partial derivative of that function with respect to any one particular independent variable, apply the same differentiation rules we have already learnt, treating all of the other independent variables as constants.

So for instance if we have $w = f(x, y, z)$ then when we calculate $\frac{\partial w}{\partial x}$ we treat each of y and z the same way we would treat a 2, or a π, or an e. And when we calculate $\frac{\partial w}{\partial y}$, it is x and z which are treated as constants. And for $\frac{\partial w}{\partial z}$, it's x and y that are treated that way. Only the variable with respect to which we're differentiating is treated like a variable.

Example 9.4. Find the partial derivatives of the function $w = xyz$.

Solution:
When we differentiate xyz with respect to x, we treat y and z as constants. This is exactly like differentiating $w = kx$ with respect to x. For $w = kx$ we would use the constant multiplier rule to get

$$\frac{dw}{dx} = \frac{d}{dx}(kx) = k\left[\frac{d}{dx}(x)\right] = k(1) = k$$

and so for finding the partial of $w = xyz$ with respect to x we treat yz as a constant multiplier and get

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(xyz) = yz\left[\frac{\partial}{\partial x}(x)\right] = (yz)(1) = yz$$

Similarly, we have

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(xyz) = xz(1) = xz \qquad \text{and} \qquad \frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(xyz) = xy(1) = xy$$

Of course, for $f(x, y, z)$ we want to have notation for the partial derivatives that involves the function name f, like the f' notation we're used to using. But we can't just use f', because that doesn't tell us *which* partial derivative it is. So instead of using the $'$ superscript, we use a *subscript* corresponding to the variable we're differentiating with respect to.

Definition 9.2. For $w = f(x, y, z)$, we define that

$$f_x(x, y, z) = \frac{\partial w}{\partial x} \qquad \text{and} \qquad f_y(x, y, z) = \frac{\partial w}{\partial y} \qquad \text{and} \qquad f_z(x, y, z) = \frac{\partial w}{\partial z}$$

Example 9.5. Find f_x and f_y if $f(x, y) = x^3y^2 + x^2y + x - y^5 - x^y$.

Solution:
When we calculate $f_x(x, y)$, we treat y as a constant, and so all constant powers of y are also constants. For a power of x, we use the constant multiplier rule if there are y's multiplying it, and just use the power rule on the x terms. And for the x^y term we have the variable x raised to a constant power, so we use the power rule for that as well. We get

$$\begin{aligned}
f_x(x, y) &= \frac{\partial}{\partial x}[x^3y^2 + x^2y + x - y^5 - x^y] \\
&= y^2\left[\frac{\partial}{\partial x}(x^3)\right] + y\left[\frac{\partial}{\partial x}(x^2)\right] + \frac{\partial}{\partial x}(x) - 0 - \frac{\partial}{\partial x}(x^y) \\
&= y^2(3x^2) + y(2x) + 1 - yx^{y-1} \\
&= 3x^2y^2 + 2xy + 1 - x^{y-1}y
\end{aligned}$$

Similarly, when we calculate $f_y(x, y)$, we treat all constant powers of x as constants and use the power rule for the constant powers of y. Also, x^y is a constant base raised to a variable power, i.e. an exponential function, so we use the rule that for any base b and any variable z, the derivative of b^z is $b^z \ln b$. So we get

$$\begin{aligned}
f_y(x, y) &= \frac{\partial}{\partial y}[x^3y^2 + x^2y + x - y^5 - x^y] \\
&= x^3\left[\frac{\partial}{\partial y}(y^2)\right] + x^2\left[\frac{\partial}{\partial y}(y)\right] + 0 - \frac{\partial}{\partial y}(y^5) - \frac{\partial}{\partial x}(x^y) \\
&= x^3(2y) + x^2(1) - 5y^4 - x^y \ln x \\
&= 2x^3y + x^2 - 5y^4 - x^y \ln x
\end{aligned}$$

Example 9.6. Find $C_x(5, 4, 2)$, $C_y(5, 4, 2)$ and $C_z(5, 4, 2)$ for the function $C(x, y, z)$ found in Example 9.3(b). What is the meaning of each of these numbers?

Solution:
This was the example about the box, and C was the cost function. In that example, we found that $C(x, y, z) = .25xy + .3yz + .4xz$. First, we find the partials of this function, and then evaluate each at $(x, y, z) = (5, 4, 2)$. That is, to find $f'(a)$ for some constant a when f is a function of only 1 variable we need to first find the function f' and then evaluate that function at a. Similarly, here we find each of the partial derivative functions and then evaluate those functions at the particular

values of the independent variables that we're interested in. For the partial derivative functions, as before we differentiate with respect to each variable separately, treating the other two variables as constants, which gives

$$\begin{aligned}
C_x(x,y,z) &= \frac{\partial}{\partial x}(.25xy+.3yz+.4xz) = .25y+0+.4z = .25y+.4z \\
C_y(x,y,z) &= \frac{\partial}{\partial y}(.25xy+.3yz+.4xz) = .25x+.3z+0 = .25x+.3z \\
C_x(x,y,z) &= \frac{\partial}{\partial z}(.25xy+.3yz+.4xz) = 0+.3y+.4x = .4x+.3y
\end{aligned}$$

And now evaluating each at $(x,y,z)=(5,4,2)$ we have

$$\begin{aligned}
C_x(5,4,2) &= .25(4)+.4(2) = 1.0+.8 = 1.8 \\
C_y(5,4,2) &= .25(5)+.3(2) = 1.25+.6 = 1.85 \\
C_x(5,4,2) &= .4(5)+.3(4) = 2.0+1.2 = 3.2
\end{aligned}$$

So what do these numbers mean? Recall that a derivative with respect to a certain variable gives the rate at which the function value is changing per unit change in the particular variable with respect to which we are differentiating. And with partial derivatives, that means as that variable changes and all others are held constant. So, for instance, $C_x(x,y,z)=.25y+.4z$ says that if the value of x (the length of the box) changes, while the width y and height z stay constant, the cost of the box (in dollars) will increase by .25 times the width plus .4 times the height, per cm of change in the length. So seeing that $f_x(5,4,2)=1.8$ tells us that increasing the length of a 5 cm by 4 cm by 2 cm box will cost \$1.80 for each cm of increase in length. Similarly, $f_y(5,4,2)=1.85$ tells us that increasing the width of a 5 cm by 4 cm by 2 cm box will cost \$1.85 for each cm of increase in width, and increasing the height of a 5 cm by 4 cm by 2 cm box would cost \$3.20 for each cm of increase in height. But it's important to remember that each of these tells us *only* what happens when *just* that variable changes. If we change 2 or 3 of the variables at the same time, these partials *don't* tell us what the effect of such a change will be. For instance, if both the length and the width of the box increase by 1 cm so that we have a $6\times5\times2$ box, the cost of the box is $C(6,5,2)=15.3$. The cost has increased by $C(6,5,2)-C(5,4,2)=15.3-11.4=3.9$. This change is not fully explained by considering only C_x and C_y. The cost did not only increase by $C_x(5,4,2)+C_y(5,4,2)=1.8+1.85=3.65$. There is an *interaction* effect of changing *both* x and y which is not captured by considering only the effect of changing one variable at a time.

Another Approach:
If we are *only* interested in the values of partials at a specified set of values for the independent variables, then for the partial with respect to a particular variable we can express the function in terms of *only* that variable, by substituting in the fixed values of the other variables, before differentiating. And then after differentiating we substitute in the value of the variable we differentiated with respect to. That is, if the other variables are going to be treated as constants, we might as well give them the constant values they're going to have before we differentiate (and that way they even look like constants). So for instance to find $C_x(5,4,2)$, we can start by evaluating $C(x,4,2)$ and then differentiate that with respect to x, to get $C_x(x,4,2)$, from which we get $C_x(5,4,2)$ by setting $x=5$. Similarly, we can find $C(5,y,2)$ and $C(5,4,z)$ and then differentiate them by y and by z, respectively, before substituting for the remaining variable. Starting from $C(x,y,z)=.25xy+.3yz+.4xz$ each time, we get

$$\begin{aligned}
C(x,4,2) &= .25x(4)+.3(4)(2)+.4x(2) = 1.0x+2.4+.8x = 1.8x+2.4 \\
\Rightarrow C_x(x,4,2) &= \frac{\partial}{\partial x}(1.8x+2.4) = 1.8 \qquad \Rightarrow \qquad C_x(5,4,2) = 1.8
\end{aligned}$$

$$\begin{aligned}
C(5,y,2) &= .25(5)y+.3y(2)+.4(5)(2) = 1.25y+.6y+4 = 1.85y+4\\
\Rightarrow C_y(5,y,2) &= \frac{\partial}{\partial y}(1.85y+4) = 1.85 \qquad \Rightarrow \qquad C_y(5,4,2) = 1.85\\
C(5,4,z) &= .25(5)(4)+.3(4)z+.4(5)z = 5+1.2z+2z = 3.2z+5\\
\Rightarrow C_z(5,4,z) &= \frac{\partial}{\partial z}(3.2z+5) = 3.2 \qquad \Rightarrow \qquad C_z(5,4,2) = 3.2
\end{aligned}$$

Notice: In this case, each of the functions, once 2 of the variables were fixed at specific values, was a linear function of the one remaining variable, so that the partial with respect to that variable was constant, i.e. that variable disappeared before we substituted in the value of the variable. For instance, since $C_x(x,4,2)$ has the value 1.8 for all values of x, substituting in $x=5$ doesn't change its value. But in another situation, if there was, for instance, an x^2 in $C(x,4,2)$ then after differentiating with respect to x the partial would still have x appearing in it, i.e. the value of the partial would depend on the value of x, so we *would* need to substitute $x=5$ in order to find $C_x(5,4,2)$.

Of course, we can also have more complicated functions, for which we need to use more complicated differentiation rules to find the partials.

Example 9.7. If $w = e^{x^2y^3z} + \dfrac{3\sqrt{x}}{yz^2} - y\ln(xz)$, find $\dfrac{\partial w}{\partial x}$, $\dfrac{\partial w}{\partial y}$ and $\dfrac{\partial w}{\partial z}$.

Solution:
For each partial derivative, we apply the normal differentiation rules for the variable that we're differentiating with respect to, while holding the other variables constant. But now we're going to need the chain rule for some of the differentiation. We take it slowly and carefully, being sure to use all, and only, the rules we need, and not getting confused by the things which look like variables but are, for our purposes while finding partials, constants.

For $\frac{\partial w}{\partial x}$ we have:

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{\partial}{\partial x}\left[e^{x^2y^3z} + \frac{3\sqrt{x}}{yz^2} - y\ln(xz)\right]\\
&= \frac{\partial}{\partial x}\left[e^{x^2y^3z}\right] + \frac{3}{yz^2}\left[\frac{\partial}{\partial x}\left(x^{1/2}\right)\right] - y\left[\frac{\partial}{\partial x}(\ln(xz))\right]\\
&= e^{x^2y^3z}\left[\frac{\partial}{\partial x}(x^2y^3z)\right] + \frac{3}{yz^2}\left[\left(\frac{1}{2}\right)x^{-1/2}\right] - y\left[\frac{\frac{\partial}{\partial x}(xz)}{xz}\right]\\
&= e^{x^2y^3z}(y^3z)\left[\frac{\partial}{\partial x}(x^2)\right] + \left(\frac{3}{yz^2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{x^{1/2}}\right) - y\left(\frac{z}{xz}\right)\\
&= e^{x^2y^3z}(y^3z)(2x) + \frac{3}{(yz^2)(2)(\sqrt{x})} - y\left(\frac{1}{x}\right)\\
&= 2xy^3ze^{x^2y^3z} + \frac{3}{2\sqrt{x}\,yz^2} - \frac{y}{x}
\end{aligned}$$

Of course, we used the chain rule when we differentiated the form e^u and also for the form $\ln u$ (where u was something more complicated than just x).

Similarly for $\frac{\partial}{\partial y}$ we have

$$\begin{aligned}
\frac{\partial w}{\partial y} &= \frac{\partial}{\partial y}\left[e^{x^2y^3z} + \frac{3\sqrt{x}}{yz^2} - y\ln(xz)\right] = \left[\frac{\partial}{\partial y}(e^{x^2y^3z})\right] + \frac{3\sqrt{x}}{z^2}\left[\frac{\partial}{\partial y}\left(\frac{1}{y}\right)\right] - [\ln(xz)]\left[\frac{\partial}{\partial y}(y)\right] \\
&= e^{x^2y^3z}\left[\frac{\partial}{\partial y}(x^2y^3z)\right] + \frac{3\sqrt{x}}{z^2}\left[\frac{\partial}{\partial y}(y^{-1})\right] - [\ln(xz)](1) \\
&= e^{x^2y^3z}(x^2z)(3y^2) + \left(\frac{3\sqrt{x}}{z^2}\right)\left(-\frac{1}{y^2}\right) - \ln(xz) = 3x^2y^2ze^{x^2y^3z} - \frac{3\sqrt{x}}{y^2z^2} - \ln(xz)
\end{aligned}$$

This time we only needed the chain rule in the first term, since that was the only one which had y embedded in some more complicated function (other than a constant multiple of y or a power of y). And finally for $\frac{\partial w}{\partial z}$ we have

$$\begin{aligned}
\frac{\partial w}{\partial z} &= \frac{\partial}{\partial z}\left[e^{x^2y^3z} + \frac{3\sqrt{x}}{yz^2} - y\ln(xz)\right] = \left[\frac{\partial}{\partial z}(e^{x^2y^3z})\right] + \frac{3\sqrt{x}}{y}\left[\frac{\partial}{\partial z}\left(\frac{1}{z^2}\right)\right] - y\left[\frac{\partial}{\partial z}(\ln(xz))\right] \\
&= e^{x^2y^3z}\left[\frac{\partial}{\partial z}(x^2y^3z)\right] + \frac{3\sqrt{x}}{y}\left[\frac{\partial}{\partial z}(z^{-2})\right] - y\left[\frac{\frac{\partial}{\partial z}(xz)}{xz}\right] \\
&= e^{x^2y^3z}(x^2y^3)(1) + \left(\frac{3\sqrt{x}}{y}\right)\left[-2\left(\frac{1}{z^3}\right)\right] - y\left(\frac{x}{xz}\right) = x^2y^3e^{x^2y^3z} - \frac{6\sqrt{x}}{yz^3} - \frac{y}{z}
\end{aligned}$$

Notice: In both $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial z}$, we found the partial of $y\ln(xz)$ to be just y times "1 over the variable we were differentiating with respect to". That's because the other variable in xz was just a constant multiplier, and we know that, for instance, $\frac{d}{dx}(\ln kx) = \frac{1}{x}$ for any constant k.

Higher Partials

Of course, a partial derivative is a function and so it, too, has a partial derivative with respect to each of the independent variables. The terminology and notation are probably what you would expect, based on what you've learned about the terminology and notation for partial derivatives, and combining that with your previous knowledge of the terminology and notation of higher derivatives of functions of one variable — except there are some complications. Of course we have notation for partials of partials for both kinds of notation for partials.

Definition 9.3. For the multivariate function $z = f(x, y)$,

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \quad \text{or} \quad f_{xx}(x,y) = \frac{\partial}{\partial x}\left[f_x(x,y)\right] \\
\text{and } \frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \quad \text{or} \quad f_{yy}(x,y) = \frac{\partial}{\partial y}\left[f_y(x,y)\right]
\end{aligned}$$

are called the **second partials** of z (or of f) with respect to x and with respect to y.

Also,

$$\begin{aligned}
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \quad \text{or} \quad f_{xy}(x,y) = \frac{\partial}{\partial y}\left[f_x(x,y)\right] \\
\text{and } \frac{\partial z^2}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \quad \text{or} \quad f_{yx}(x,y) = \frac{\partial}{\partial x}\left[f_y(x,y)\right]
\end{aligned}$$

are called the **mixed second partials** of z or of f.

Of course, a function of more than 2 variables has more of these — one second partial for each independent variable, plus one mixed second partial for each different ordering of 2 of the independent variables. And since the second partials of a function are themselves functions, they too have partial derivatives and so we can go on to find third partials, and so forth. For instance, a function $f(x, y)$ has third partials f_{xxx}, f_{xxy}, f_{xyx}, f_{xyy}, f_{yxx}, f_{yxy}, f_{yyx} and f_{yyy}. Goodness! That's 8 third partials for a multivariate function of only 2 variables. And for a function of 3 variables, there would be 27 third partials! Fortunately, in this course we will not be concerned with any higher partials than the second partials (including mixed second partials) defined above.

Example 9.8. Find all second partials of $f(x, y) = x^3y^2 + x^2y + x - y^5 - x^y$.

Solution:
In Example 9.5 we found the "first" partials of this function (although at the time we just called them the partials). We have $f_x(x, y) = 3x^2y^2 + 2xy + 1 - x^{y-1}y$ and $f_y(x, y) = 2x^3y + x^2 - 5y^4 - x^y \ln x$. Now we need to differentiate each of these first partials with respect to each of x and y. From $f_x(x, y)$ we get the second partial with respect to x and also one of the mixed partials:

$$\begin{aligned}
f_{xx}(x,y) &= \frac{\partial}{\partial x}\left[3x^2y^2 + 2xy + 1 - x^{y-1}y\right] = 3y^2\left[\frac{\partial}{\partial x}(x^2)\right] + 2y\left[\frac{\partial}{\partial x}(x)\right] + \frac{\partial}{\partial x}(1) - y\left[\frac{\partial}{\partial x}(x^{y-1})\right] \\
&= 3y^2(2x) + 2y(1) + 0 - y[(y-1)x^{y-2}] = 6xy^2 + 2y - x^{y-2}(y^2 - y)
\end{aligned}$$

$$\begin{aligned}
\text{and } f_{xy}(x,y) &= \frac{\partial}{\partial y}\left[3x^2y^2 + 2xy + 1 - x^{y-1}y\right] \\
&= 3x^2\left[\frac{\partial}{\partial y}(y^2)\right] + 2x\left[\frac{\partial}{\partial y}(y)\right] + \frac{\partial}{\partial y}(1) - \left[y\left(\frac{\partial}{\partial y}(x^{y-1})\right) + x^{y-1}\left(\frac{\partial}{\partial y}(y)\right)\right] \\
&= 3x^2(2y) + 2x(1) + 0 - y\left[(x^{y-1}\ln x)\left(\frac{\partial}{\partial y}(y-1)\right)\right] - x^{y-1}(1) \\
&= 6x^2y + 2x - x^{y-1}y(\ln x)(1) - x^{y-1} = 6x^2y + 2x - x^{y-1}[y(\ln x) - 1]
\end{aligned}$$

Of course, in finding f_{xy} we had to use the product rule for the term that was a product of two terms which both contain y. And we also needed to recognize that for purposes of differentiating with respect to y, x^{y-1} is a constant base raised to a variable power, of the form b^u where u is a function of y.

Likewise, from $f_y(x, y) = 2x^3y + x^2 - 5y^4 - x^y \ln x$ we get the second partial with respect to y and the other mixed partial.

$$\begin{aligned}
f_{yy}(x,y) &= \frac{\partial}{\partial y}\left[2x^3y + x^2 - 5y^4 - x^y\ln x\right] = 2x^3\left[\frac{\partial}{\partial y}(y)\right] + \frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial y}(5y^4) - (\ln x)\left[\frac{\partial}{\partial y}(x^y)\right] \\
&= 2x^3(1) + 0 - 5(4y^3) - (\ln x)\,(x^y\ln x) = 2x^3 - 20y^3 - x^y(\ln x)^2
\end{aligned}$$

$$\begin{aligned}
\text{and } f_{yx}(x,y) &= \frac{\partial}{\partial x}\left[2x^3y + x^2 - 5y^4 - x^y\ln x\right] \\
&= 2y\left[\frac{\partial}{\partial x}(x^3)\right] + \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial x}(5y^4) - \left[(\ln x)\frac{\partial}{\partial x}(x^y) + (x^y)\frac{\partial}{\partial x}(\ln x)\right] \\
&= 2y(3x^2) + 2x - 0 - (\ln x)(yx^{y-1}) - (x^y)\left(\frac{1}{x}\right) = 6x^2y + 2x - x^{y-1}(y\ln x) - \frac{x^y}{x}
\end{aligned}$$

But look again at the final expression. We can simplify the last term, because $\dfrac{x^y}{x} = x^yx^{-1} = x^{y-1}$.

So we get

$$f_{yx} = 6x^2y + 2x - x^{y-1}(y\ln x) - x^{y-1} = 6x^2y + 2x - x^{y-1}[y(\ln x) - 1]$$

Oh look! In this example, it turned out that $f_{yx}(x, y) = f_{xy}(x, y)$. That is, when we differentiated $f(x, y)$ first with respect to x and then with respect to y we got the same function as when we differentiated $f(x, y)$ first with respect to y and then with respect to x. So the two mixed second partials are the same! Could it be that this is always true? Conveniently, yes. As long as the mixed second partials are both continuous, they are identical.

Theorem 9.1. *For any function of 2 or more variables, if the mixed second partial obtained by first differentiating with respect to one particular variable, and then with respect to another, and the mixed second partial obtained when the order of differentiation is reversed, are both continuous, then these mixed second partials are equal.*

That is, if f *is a function of* x *and* y *(and perhaps also other variables), then whenever they are both continuous it will always be true that* $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ *, i.e.* $f_{xy} = f_{yx}$.

That means that for a function for which we know the partials must be continuous on any interval on which the function itself is continuous, it is only necessary to calculate the mixed partial(s) by differentiating with respect to two variables, in turn, in one ordering. That is, we need only calculate one of f_{xy} or f_{yx}, never both. It doesn't matter in which order we differentiate with respect to two independent variables.

Example 9.9. Find all second partials of the function $w = xyz$.

Solution:
In Example 9.4 we found that for $w = xyz$ the first partials are:

$$\frac{\partial w}{\partial x} = yz \qquad \text{and} \qquad \frac{\partial w}{\partial y} = xz \qquad \text{and} \qquad \frac{\partial w}{\partial z} = xy$$

To find the second partials, we differentiate *each* of these first partials with respect to *each* of x, y and z. However, we will use the theorem above to reduce the amount of differentiation we actually do.

Using $\frac{\partial w}{\partial x} = yz$ we get

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}(yz) = yz\left[\frac{\partial}{\partial x}(1)\right] = 0 \qquad \frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial y}(yz) = z\left[\frac{\partial}{\partial y}(y)\right] = z(1) = z$$

$$\text{and } \frac{\partial^2 w}{\partial x \partial z} = \frac{\partial}{\partial z}(yz) = y\left[\frac{\partial}{\partial z}(z)\right] = y(1) = y$$

Using $\frac{\partial w}{\partial y} = xz$, we know that $\dfrac{\partial^2 w}{\partial y \partial x} = \dfrac{\partial^2 w}{\partial x \partial y} = z$, and for the others we get

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}(xz) = xz\left[\frac{\partial}{\partial y}(1)\right] = 0$$

$$\text{and } \frac{\partial^2 w}{\partial y \partial z} = \frac{\partial}{\partial z}(xz) = x\left[\frac{\partial}{\partial z}(z)\right] = x(1) = x$$

Finally, using $\frac{\partial w}{\partial z} = xy$, we know that $\dfrac{\partial^2 w}{\partial z \partial x} = \dfrac{\partial^2}{\partial x \partial z} = y$ and $\dfrac{\partial^2 w}{\partial z \partial y} = \dfrac{\partial^2}{\partial y \partial z} = x$, and for the other one we have

$$\frac{\partial^2 w}{\partial z^2} = \frac{\partial}{\partial z}(xy) = xy\left[\frac{\partial}{\partial z}(1)\right] = 0$$

That is, we see that if we differentiate with respect to the same variable twice, since for this function the first partial with respect to that variable is just a constant when only that variable is changing, the second partial is 0. And if we differentiate with respect to two different variables, then since the function w has each of the variables appearing only as itself (i.e. to the power 1), we are left with only the variable which was a constant multiplier during both differentiations.

Notice: It shouldn't be too hard for you to realize, if you think about it for a moment, that for this example the third partials will all be constant, either 0 or 1, with

$$\frac{\partial^3 w}{\partial x \partial y \partial z} = \frac{\partial^3 w}{\partial x \partial z \partial y} = \frac{\partial^3 w}{\partial y \partial x \partial z} = \frac{\partial^3 w}{\partial y \partial z \partial x} = \frac{\partial^3 w}{\partial z \partial x \partial y} = \frac{\partial^3 w}{\partial z \partial y \partial x} = 1$$

and all of the other third partials being 0.

Of course, the pattern we observed in this example was a feature of only this particular function. Ordinarily there will be more variety among the second partials. (Of the 9 second partials of a function of 3 variables, there will be three pairs of identical mixed partials, usually different for each pair, and the 3 "non-mixed" partials will all be different, giving 6 different second partial derivative functions.) For instance, consider the following example.

Example 9.10. Find all second partials of $f(x, y, z) = x^2y^3z^4$.

Solution:
We must find the first partials before we can find the second partials. We have

$$\begin{aligned}
f_x(x,y,z) &= \frac{\partial}{\partial x}(x^2y^3z^4) = y^3z^4\left[\frac{\partial}{\partial x}(x^2)\right] = 2xy^3z^4 \\
f_y(x,y,z) &= \frac{\partial}{\partial y}(x^2y^3z^4) = x^2z^4\left[\frac{\partial}{\partial y}(y^3)\right] = 3x^2y^2z^4 \\
f_z(x,y,z) &= \frac{\partial}{\partial z}(x^2y^3z^4) = x^2y^3\left[\frac{\partial}{\partial x}(z^4)\right] = 4x^2y^3z^3
\end{aligned}$$

And now we can find the second partials. (Again, since there are clearly no continuity issues with functions involving only positive powers of variables, we'll use the theorem to reduce the work we need to do.)

$$\begin{aligned}
f_{xx}(x,y,z) &= \frac{\partial}{\partial x}(2xy^3z^4) = 2y^3z^4\left[\frac{\partial}{\partial x}(x)\right] = 2y^3z^4 \\
f_{xy}(x,y,z) &= \frac{\partial}{\partial y}(2xy^3z^4) = 2xz^4\left[\frac{\partial}{\partial y}(y^3)]\right] = (2xz^4)(3y^2) = 6xy^2z^4 \\
f_{xz}(x,y,z) &= \frac{\partial}{\partial z}(2xy^3z^4) = 2xy^3\left[\frac{\partial}{\partial z}(z^4)\right] = 8xy^3z^3
\end{aligned}$$

$$\begin{aligned}
f_{yx}(x,y,z) &= f_{xy}(x,y,z) = 6xy^2z^4 \\
f_{yy}(x,y,z) &= \frac{\partial}{\partial y}(3x^2y^2z^4) = 3x^2z^4\left[\frac{\partial}{\partial y}(y^2)\right] = 6x^2yz^4 \\
f_{yz}(x,y,z) &= \frac{\partial}{\partial z}(3x^2y^2z^4) = 3x^2y^2\left[\frac{\partial}{\partial z}(z^4)\right] = 12x^2y^2z^3 \\
f_{zx}(x,y,z) &= f_{xz}(x,y,z) = 8xy^3z^3 \\
f_{zy}(x,y,z) &= f_{yz}(x,y,z) = 12x^2y^2z^3 \\
f_{zz}(x,y,z) &= \frac{\partial}{\partial z}(4x^2y^3z^3) = 4x^2y^3\left[\frac{\partial}{\partial z}(z^3)\right] = 12x^2y^3z^2
\end{aligned}$$

Let's look at one last example.

Example 9.11. Find all first and second partials of $f(x,y) = \sin(xy) + x\ln y$.

Solution:
We need the chain rule when we differentiate $\sin(xy)$.

$$\begin{aligned}
f_x(x,y) &= \frac{\partial}{\partial x}\left[\sin(xy) + x\ln y\right] = \frac{\partial}{\partial x}\left[\sin(xy)\right] + (\ln y)\left[\frac{\partial}{\partial x}(x)\right] \\
&= [\cos(xy)]\left[\frac{\partial}{\partial x}(xy)\right] + \ln y = y\cos(xy) + \ln y \\
f_y(x,y) &= \frac{\partial}{\partial y}\left[\sin(xy) + x\ln y\right] = \frac{\partial}{\partial y}\left[\sin(xy)\right] + x\left[\frac{\partial}{\partial y}(\ln y)\right] \\
&= [\cos(xy)]\left[\frac{\partial}{\partial y}(xy)\right] + x\left(\frac{1}{y}\right) = x\cos(xy) + \frac{x}{y} \\
f_{xx}(x,y) &= \frac{\partial}{\partial x}\left[y\cos(xy) + \ln y\right] = y\left[\frac{\partial}{\partial x}[\cos(xy)]\right] + 0 = y\left[-\sin(xy)\right]\left[\frac{\partial}{\partial x}(xy)\right] = -y^2\sin(xy) \\
f_{xy}(x,y) &= \frac{\partial}{\partial y}\left[y\cos(xy) + \ln y\right] = [\cos(xy)]\left[\frac{\partial}{\partial y}(y)\right] + y\left[\frac{\partial}{\partial y}[\cos(xy)]\right] + \frac{\partial}{\partial y}(\ln y) \\
&= [\cos(xy)] + y[-\sin(xy)]\left[\frac{\partial}{\partial y}(xy)\right] + \frac{1}{y} = \cos(xy) - xy\sin(xy) + \frac{1}{y} \\
f_{yx}(x,y) &= \frac{\partial}{\partial x}\left[x\cos(xy) + \frac{x}{y}\right] = [\cos(xy)]\left[\frac{\partial}{\partial x}(x)\right] + x\left[\frac{\partial}{\partial x}[\cos(xy)]\right] + \frac{1}{y}\left[\frac{\partial}{\partial x}(x)\right] \\
&= [\cos(xy)] + x\left[-\sin(xy)\right]\left[\frac{\partial}{\partial x}(xy)\right] + \frac{1}{y} = \cos(xy) - xy\sin(xy) + \frac{1}{y} \\
f_{yy}(x,y) &= \frac{\partial}{\partial y}\left[x\cos(xy) + \frac{x}{y}\right] = x\left[\frac{\partial}{\partial y}[\cos(xy)]\right] + x\left[\frac{\partial}{\partial y}(y^{-1})\right] \\
&= x[-\sin(xy)]\left[\frac{\partial}{\partial y}(xy)\right] + x\left[-y^{-2}\right] = -x^2\sin(xy) - \frac{x}{y^2}
\end{aligned}$$

Of course, we see that $f_{yx}(x,y) = f_{xy}(x,y)$. (The second partials have the same domain as f.)

Math 1225A/B

Unit 10:
Optimization

(text reference: Sections 10.3 and 10.5

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10 Optimization

You will remember that for a function $y = f(x)$, we use the term *relative* or *local extreme values* to refer to the relative maximum or relative minimum values of the function, that is, the values attained as f reaches a value which is the highest, or the lowest, anywhere "nearby". These relative extrema can only occur at *critical numbers* of the function, which are values of the independent variable x (in the domain of f) at which the first derivative, $f'(x)$, has the value 0, or is not defined. However, not all critical numbers of f are places at which a relative extremum occurs. Sometimes there is neither a relative maximum nor a relative minimum at $x = c$, even though $f'(c) = 0$.

We can look at graphs which show these various possibilities. For instance, the function $f(x) = x^2$ has $f'(0) = 0$ and has a relative or local minimum value $f(0) = 0$ at this critical number $x = 0$. Figure 1(a) shows this function and its local minimum. And $x = 0$ is also a critical number of $f(x) = 1 - x^2$, but this time the value $f(0) = 1$ is a relative or local maximum, rather than a local minimum, as shown in Figure 1(b).

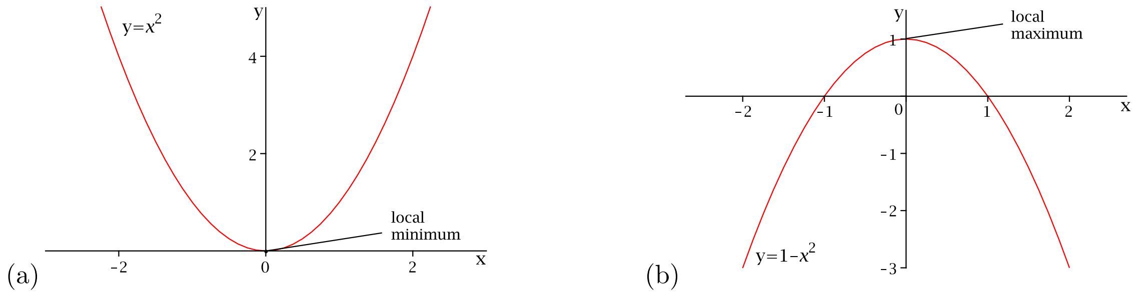

Figure 1: The graphs of $y = f(x)$ for (a) $f(x) = x^2$ and (b) $f(x) = 1 - x^2$.

The function $f(x) = x^3$ also has $x = 0$ as a critical number, since $f'(0) = 0$ for this function as well. However, $f(0)$ is not a local extreme value of f, because $f(0) = 0$ is neither a local maximum nor a local minimum. Anywhere even very slightly to the right of $x = 0$ the function value is higher than at $x = 0$, so $f(0)$ is not the highest value nearby. And anywhere even very slightly to the left of $x = 0$ the function value is lower than at $x = 0$, so $f(0)$ is also not the lowest value nearby. Figure 1(c) shows this function.

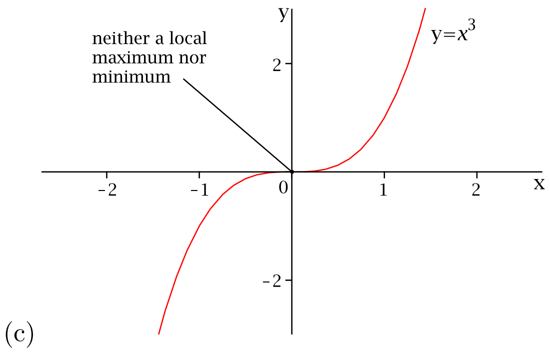

Figure 1: (c) The graph of $y = x^3$ has neither a relative maximum nor a relative minimum at $x = 0$.

You will also recall that *sometimes*, the *Second Derivative Test* can be used to determine what kind of relative extremum is occurring at a critical number c of f. This test tells us that for a continuous function f, if $f'(c) = 0$ and $f''(c) > 0$ then there is a relative minimum at $x = a$, whereas if $f'(c) = 0$ and $f''(c) < 0$ then there is a relative maximum at $x = c$. For instance, for $f(x) = x^2$ we have $f''(x) = 2$, so $f''(0) > 0$ indicating a relative minimum at $x = 0$ as we have already seen. Likewise, for $f(x) = 1 - x^2$ we have $f''(x) = -2$ and so $f''(0) = -2 < 0$, indicating a relative maximum, as we saw above. But the Second Derivative Test only works sometimes, because when

$f'(c) = 0$ and $f''(c) = 0$ also, the test tells us nothing about whether f has a relative extremum at $x = c$, or if so what kind of extremum it is. For instance, the functions $f(x) = x^3$, $g(x) = x^4$ and $h(x) = -x^4$ all have both first derivative and second derivative values being 0 at $x = 0$. And f does not have a relative extremum at this critical number, as we have already seen, whereas g has a relative minimum and h has a relative maximum at $x = 0$. (You can easily confirm those for yourself.)

And of course, we use the term *extreme values* (or *extrema*), without the qualifier *relative* or *local*, to describe the *absolute* (or globally) largest and smallest function values anywhere in the domain of the function. For instance, for $f(x) = x^2$, the relative minimum $f(0) = 0$ is also *the minimum* value of this function, while for $f(x) = 1 - x^2$ the relative maximum $f(0) = 1$ is also *the maximum* value of that function. But neither of these functions has an extremum of the other type, since $f(x) = x^2$ gets large without bound and $f(x) = 1 - x^2$ runs off toward $-\infty$.

We have something similar to all of this for functions of 2 independent variables. But of course it's somewhat more complicated than for functions of a single variable. And for functions of more than 2 variables, it's even more complicated — so much so that it's beyond the scope of this course. We will only be examining the relative extrema of $z = f(x, y)$. (We will also look at finding constrained optima, and there we'll consider functions of more than 2 variables, but that's different.) We start with some definitions, which are analogous to their counterparts for functions of a single variable.

Definition 10.1. Consider the function $f(x, y)$. Let (a, b) be some point in the domain D of $f(x, y)$.

- We say that $f(a, b)$ is **a relative maximum value** of f if $f(a, b) \geq f(x, y)$ for all (x, y) in D near (a, b),
- and that $f(a, b)$ is **the maximum value** of f if $f(a, b) \geq f(x, y)$ for all (x, y) in D.
- Also, $f(a, b)$ is **a relative minimum value** of f if $f(a, b) \leq f(x, y)$ for all (x, y) in D near (a, b),
- and $f(a, b)$ is **the minimum value** of f if $f(a, b) \leq f(x, y)$ for all (x, y) in D.
- The relative maxima and minima of f are collectively referred to as the **relative extreme values**.

As stated previously, for a function $f(x)$, critical numbers are places where f' is 0 or is not defined, and there are 2 different kinds of reasons why the derivative function may not be defined at a particular value in the domain of the function: the value may be on the boundary of the domain (and of course a derivative is a limit and therefore does not exist if it cannot be approached from both sides), or there may be a discontinuity, or a corner, or other oddity occurring on the function at that x-value. This means there are actually 3 kinds of critical numbers of a function of only one variable. Similarly, there are 3 different kinds of places at which a relative extreme value of $f(x, y)$ can occur. We call these **critical points** of f.

Definition 10.2. $(x, y) = (a, b)$ is a **critical point** of the function $f(x, y)$ if (a, b) is:

(i) a boundary point of the domain,

or (ii) a point where all first partial derivatives are 0,

or (iii) a point where something bizarre is happening — for instance, a discontinuity, a corner, etc.

(*Notice:* This is exactly analagous to the definition of a critical number of a function of one variable, except that there we didn't need to state that *all* first derivatives were 0 as in type (ii) here, since a function of only one variable has only one first derivative. Also, as was probably true in your Introductory Calculus course, we will only be concerned with type (ii), and never with type (i) or type (iii) critical points.)

As was mentioned earlier, relative extrema of a function of one variable can only occur at critical numbers of the function. Again, we have an analogous result for functions of 2 variables.

Theorem 10.1. *Relative extreme values of $f(x,y)$ can only occur at critical points of f.*

As with functions of one variable, a function $z = f(x,y)$ may have a relative maximum value, a relative minimum value, or neither, at a critical point (a,b). We see each of these in the following example.

Example 10.1. Find all critical points of each of the following functions.

(a) $f(x,y) = x^2 + y^2$ (b) $g(x,y) = xy - x^2 - y^2$ (c) $h(x,y) = x^3 + y^3$

Solution:
(a) For $f(x,y) = x^2 + y^2$ we get first partials:

$$f_x(x,y) = \frac{\partial}{\partial x}(x^2+y^2) = 2x \qquad \text{and} \qquad f_y(x,y) = \frac{\partial}{\partial y}(x^2+y^2) = 2y$$

Setting each, in turn, equal to 0 we have

$$\begin{array}{lllll} f_x(x,y) = 0 & \Rightarrow & 2x = 0 & \Rightarrow & x = 0 \\ f_y(x,y) = 0 & \Rightarrow & 2y = 0 & \Rightarrow & y = 0 \end{array}$$

We see that the first partial with respect to x is 0 whenever $x = 0$, no matter what the value of y is, and similarly the first partial with respect to y is 0 whenever $y = 0$, for any value of x. But we only have *both* first partials being 0 at the same time when x and y are both 0. So $(0,0)$ is the only critical point of f.

We can look at a graph of $z = f(x,y)$.

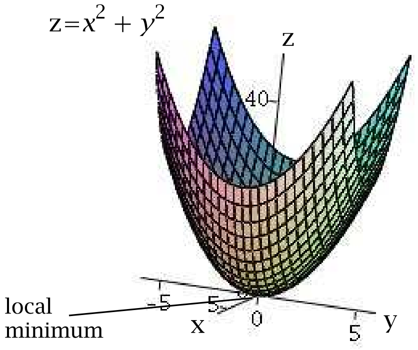

Figure 2: (a) The graph of $z = x^2 + y^2$.

We see that at the critical point, where $x = y = 0$, the function has a local, i.e. a relative, (and absolute) minimum value of $f(0,0) = 0$.

(b) For $g(x,y) = xy - x^2 - y^2$ we get first partials:

$$g_x(x,y) = \frac{\partial}{\partial x}(xy - x^2 - y^2) = y - 2x \qquad \text{and} \qquad g_y(x,y) = \frac{\partial}{\partial y}(xy - x^2 - y^2) = x - 2y$$

Setting $g_x(x,y) = 0$ and $g_y(x,y) = 0$ we get

$$\begin{array}{lllll} g_x(x,y) = 0 & \Rightarrow & y - 2x = 0 & \Rightarrow & y = 2x \\ g_y(x,y) = 0 & \Rightarrow & x - 2y = 0 & \Rightarrow & x = 2y \end{array}$$

We see that the first partial with respect to x is 0 whenever $x = 2y$, so for any value of y there is a corresponding x-value for which this partial has the value 0. Similarly the first partial with respect to y is 0 whenever $y = 2x$, and for any value of x there is a corresponding y-value for which f_y has the value 0. But we only have *both* first partials being 0 at the same time when $y = 2x$ while $x = 2y$. That is,

$$f_x(x,y) = f_y(x,y) = 0 \Rightarrow x = 2y \text{ and } y = 2x \Rightarrow x = 2y = 2(2x) = 4x \Rightarrow x = 4x \Rightarrow x = 0$$

which means also that $y = 2x = 0$. Again, $(0,0)$ is the only critical point of f.

We can look at a graph of $z = g(x,y)$.

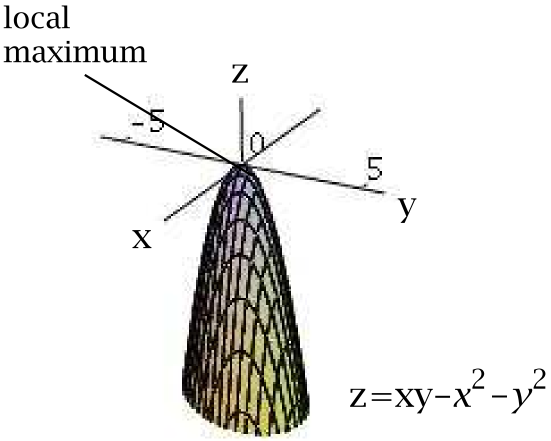

Figure 2: (b) The graph of $z = xy - x^2 - y^2$.

This time we see that at the critical point, where $x = y = 0$, the function has a local, i.e. a relative (and absolute) *maximum* value of $f(0,0) = 0$.

(c) Now we have $h(x) = x^3 + y^3$, which has first partials

$$h_x(x,y) = \frac{\partial}{\partial x}(x^3 + y^3) = 3x^2 \qquad \text{and} \qquad h_y(x,y) = \frac{\partial}{\partial y}(x^3 + y^3) = 3y^2$$

For $h_x(x,y) = 0$ and $h_y(x,y) = 0$ we get

$$\begin{array}{lllll} h_x(x,y) = 0 & \Rightarrow & 3x^2 = 0 & \Rightarrow & x = 0 \\ h_y(x,y) = 0 & \Rightarrow & 3y^2 = 0 & \Rightarrow & y = 0 \end{array}$$

As with f, we have $h_x(x,y) = 0$ whenever $x = 0$, no matter what the value of y is, and similarly $h_y(x,y) = 0$ whenever $y = 0$, for any value of x, but we only have *both* first partials being 0 at the same time when x and y are both 0. So once again $(0,0)$ is the only critical point of f.

But

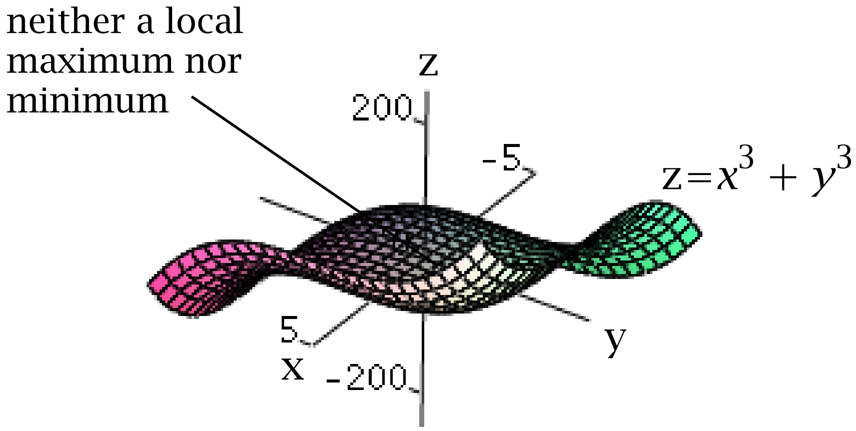

Figure 2: (c) The graph of $z = x^3 + y^3$.

This time the *surface* $z = h(x,y)$ is much less regularly shaped than before. And we see that at the critical point, $h(0,0) = 0$ is neither a local minimum value nor a local maximum value. For $h(x,y) = x^3 + y^3$, increasing either x or y even the slightest bit gives a larger function value than at $(0,0)$, whereas decreasing either independent variable gives a smaller function value than $h(0,0)$. So $h(0,0)$ is neither the largest nor the smallest function value "nearby".

With functions of 2 variables, there's another possibility that can occur, that isn't possible with functions of only one variable. We can have something which is, in a sense, both a relative maximum and a relative minimum at the same time. From one perspective, the function value will get bigger if we look at what's happening anywhere, in either direction, along a particular line in xy-space, but if we look at what's happening in either direction along the line perpendicular to that line in xy-space, the function value will decrease. We see this circumstance in the next example.

Example 10.2. Find all critical points of $s(x,y) = x^2 - y^2$.

Solution:
The first partials are:

$$s_x(x,y) = \frac{\partial}{\partial x}(x^2 - y^2) = 2x \qquad \text{and} \qquad s_y(x,y) = \frac{\partial}{\partial y}(x^2 - y^2) = -2y$$

As before, we see that both first partials are 0 only when $x = y = 0$, so once again $(x,y) = (0,0)$ is the only critical point.

Let's see what the graph of $s(x,y) = x^2 - y^2$ looks like. When you look at the graph (on the next page), be sure that you understand that the z-axis is in front of the part of the graph that's sort of obscuring it and the x-axis is mostly blocked from view. The lower part at the front/left is on the "positive x" side of where the z-axis comes up at the origin, and the higher part at the back/right is behind where the z-axis comes up, on the "negative x" side of the origin.

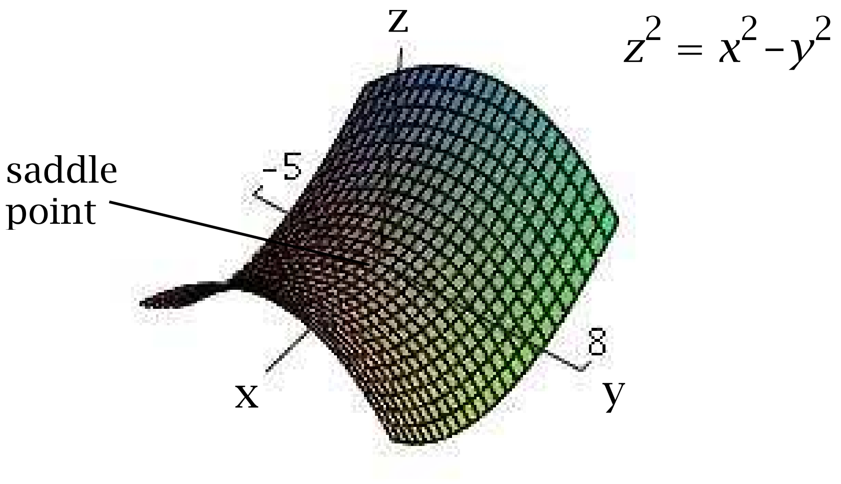

Figure 3: The graph of $z = x^2 - y^2$.

Notice that at the critical point, where $x = y = 0$ so that $s(0, 0) = 0$, there is neither a relative maximum nor a relative minimum. If we change the value of x slightly, while holding y constant at 0, the function value, $x^2 - y^2$, will increase (to the square of the new x-value). But if we change the value of y slightly, while holding x constant at 0, the function value will decrease (to the negative of the square of the new y-value).

Let's think a bit more about that. Imagine that we slice the function $s(x, y)$ into two pieces, cutting along the x-axis, which has $y = 0$. Now imagine looking at the cut edge (which goes through the critical point). Everywhere along the cut edge we have $y = 0$ and so we would see the function value increasing no matter which way we move from the critical point along this cut edge. Along this edge, it would appear that there's a relative minimum at $(x, y) = (0, 0)$.

But now imagine that we slice the function into 2 pieces, cutting along the y-axis instead. This time, if we were to look at the cut edge (which again goes through the critical point), we would see the function value decreasing no matter which way we move away from the critical point along this cut edge, because everywhere along this edge we have $x = 0$. So along this slice, it appears that there's a relative maximum at $(x, y) = (0, 0)$.

We have a name for a critical point at which this kind of thing is happening. Have you ever seen a horse's saddle? There's a picture of one on the front cover of the text. Think about what the centre of the saddle looks like. (Think of the saddle being on a horse, so that we can describe which perspective we're looking from.) If you stand behind the horse (not so close you get kicked, of course), you can see the saddle going down on the left and on the right, as it fits the rounded shape of the horse's back. But now move around to the side of the horse. From this perspective, you see that the saddle goes up, both towards the front of the horse and towards the back. So looked at from the "behind the horse" perspective, the centre point of the saddle is higher than the points on the saddle to either side. But looked at from the "side of the horse" perspective, the centre point of the saddle is lower than the points on the saddle ahead or behind. Just like what was happening with the function in Figure 3 at the critical point. (But here, we didn't slice the horse into two pieces, of course! We could slice the saddle, but we don't need to.) So we call this kind of place on a function of 2 variables a **saddle point**, because it's looks like a horse's saddle.

Definition 10.3. A point on the surface $z = f(x, y)$ is called a **saddle point** if it appears to be a relative maximum when viewed from one perspective, but appears to be a relative minimum when viewed from another (usually perpendicular) perspective.

We know how to find critical points of a function $f(x, y)$, and we know that relative extreme values of the function can only occur at these critical points. But how do we determine whether there's a relative maximum, a relative minimum, a saddle point, or maybe none of these, occurring at a particular critical point of the function? Well, we have a test for that. It's sort of the function-of-2-variables-equivalent of the Second Derivative Test for functions of a single variable. And we call it the Second Partials Test, or just the Second Derivative Test, as our text does. And like the Second Derivative Test for functions of one variable, it only works *sometimes.* In other situations, the test simply gives us no information about what the function is doing.

The Second Partials Test or Second Derivative Test

Consider any function $f(x, y)$ which has continuous second partial derivatives nearby a critical point (a, b) which has $f_x(a, b) = f_y(a, b) = 0$. Define the test function

$$D(x, y) = [(f_{xx}(x, y))\,(f_{yy}(x, y))] - (f_{xy}(x, y))^2$$

Then:

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, $f(a, b)$ is a relative maximum value.
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, $f(a, b)$ is a relative minimum value.
3. If $D(a, b) < 0$, f had a saddle point at (a, b).
4. If $D(a, b) = 0$, the test is inconclusive (i.e. gives no information about $f(a, b)$).

So after we have found a critical point (a, b) at which both first partials are 0, we calculate the value of this test function $D(a, b)$ using the formula given. (It's not hard to remember.) If $D(a, b)$ is positive, then we know that f has a relative extreme value at (a, b). (Notice that it's not possible for $D(a, b)$ to be positive when $f_{xx}(a, b) = 0$, so cases 1. and 2. are the only possibilities when $D(a, b)$ is positive.) We use the second partial with respect to x to determine whether this relative extreme value is a minimum or a maximum, just like in the Second Derivative Test for functions of one variable. (Notice also that if $D(a, b)$ is positive then f_{xx} and f_{yy} must have the same sign at (a, b) so it doesn't really matter which of the "unmixed" second partials we look at.) Effectively, we're looking at the concavity of $f(x, b)$ and using the same rule as in the original Second Derivative Test. But if $D(a, b)$ is negative, then there isn't a relative extreme value at (a, b), because there's a saddle point there. And if $D(a, b) = 0$, then the test is simply unable to give us any information (just like when the second derivative is 0 at a critical number, using the original Second Derivative Test).

Now, we just need to practice using this test. First, let's see what the Second Partials Test tells us about the functions we've already looked at.

Example 10.3. Apply the Second Partials Test to the critical point $(0, 0)$ for each of the functions in Examples 10.1 and 10.2.

Solution:
In Example 10.1 we had the functions $f(x,y) = x^2+y^2$, $g(x,y) = xy-x^2-y^2$ and $h(x,y) = x^3+y^3$, and in Example 10.2 we were looking at the function $s(x,y) = x^2-y^2$. We found that $(x,y) = (0,0)$ is the only critical point for each of these functions. We consider the functions one by one and apply the Second Partials Test.

For $f(x,y) = x^2+y^2$ we found the first partials to be:

$$f_x(x,y) = 2x \qquad \text{and} \qquad f_y(x,y) = 2y$$

and we can easily find the second partials:

$$f_{xx}(x,y) = 2 \qquad \text{and} \qquad f_{yy}(x,y) = 2 \qquad \text{and} \qquad f_{xy}(x,y) = 0$$

This gives $D(x,y) = (f_{xx}(x,y))\,(f_{yy}(x,y)) - (f_{xy}(x,y))^2 = (2)(2) - (0)^2 = 4$ so $D(0,0) = 4 > 0$, and since we also have $f_{xx}(0,0) = 2 > 0$ then we have situation 1 in the Second Partials Test, which tells us that $f(0,0) = 0$ is a relative minimum value, as we saw on the graph.

For $g(x,y) = xy - x^2 - y^2$ we found the first partials to be:

$$g_x(x,y) = y - 2x \qquad \text{and} \qquad g_y(x,y) = x - 2y$$

and we find the second partials:

$$g_{xx}(x,y) = -2 \qquad \text{and} \qquad g_{yy}(x,y) = -2 \qquad \text{and} \qquad g_{xy}(x,y) = 1$$

This gives $D(x,y) = (g_{xx}(x,y))\,(g_{yy}(x,y)) - (g_{xy}(x,y))^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3$ so that $D(0,0) = 3 > 0$, and since we also have $g_{xx}(0,0) = -2 < 0$ then we have situation 2 in the Second Partials Test, which tells us that $f(0,0) = 0$ is a relative maximum value, which again is what we saw on the graph.

For $h(x,y) = x^3 + y^3$ we found the first partials to be:

$$h_x(x,y) = 3x^2 \qquad \text{and} \qquad h_y(x,y) = 3y^2$$

and the second partials are:

$$h_{xx}(x,y) = 6x \qquad \text{and} \qquad h_{yy}(x,y) = 6y \qquad \text{and} \qquad h_{xy}(x,y) = 0$$

This gives $D(x,y) = (h_{xx}(x,y))\,(h_{yy}(x,y)) - (h_{xy}(x,y))^2 = (6x)(6y) - 0^2 = 36xy$, so $D(0,0) = 0$. This time we have situation 4 in the Second Partials Test, which tells nothing about what's happening at $h(0,0)$, which isn't surprising since looking at the graph we saw that there is no relative extreme value at $h(0,0)$, and we can see that there isn't a saddle point there either.

Finally, for $s(x,y) = x^2 - y^2$ we found the first partials to be:

$$s_x(x,y) = 2x \qquad \text{and} \qquad s_y(x,y) = -2y$$

which gives the second partials as:

$$s_{xx}(x,y) = 2 \qquad \text{and} \qquad s_{yy}(x,y) = -2 \qquad \text{and} \qquad s_{xy}(x,y) = 0$$

We get $D(x,y) = (s_{xx}(x,y))\,(s_{yy}(x,y)) - (s_{xy}(x,y))^2 = (2)(-2) - (0)^2 = -4$, so $D(0,0) = -4 < 0$. Ah good. We have situation 3 in the Second Partials Test, which tells us that there is a saddle point at $(0,0)$, as we have already discussed.

Those functions had pretty straightforward (i.e. boring) partials. Let's look at a more interesting function.

Example 10.4. Find all critical points of $f(x,y) = \frac{x^2}{2} + 6x + xy - y^2 + y^3$ and determine whether each is a relative maximum, a relative minimum or a saddle point, if possible.

Solution:
Notice that $f(x,y)$ is defined and is continuous everywhere. Thus there are no boundary points and no places at which the function behaves peculiarly. So the only critical points will be those which occur when both first partials are 0. We need to find the first partials of f:

$$\begin{aligned} f_x(x,y) &= \frac{\partial}{\partial x}\left(\frac{x^2}{2} + 6x + xy - y^2 + y^3\right) = x + 6 + y \\ f_y(x,y) &= \frac{\partial}{\partial y}\left(\frac{x^2}{2} + 6x + xy - y^2 + y^3\right) = x - 2y + 3y^2 \end{aligned}$$

To find the critical points, we set each of these equal to 0. As often happens (with more interesting first partials) this allows us to express one variable in terms of the other:

$$\begin{aligned} x + 6 + y = 0 \quad &\Rightarrow \quad x = -y - 6 \\ x - 2y + 3y^2 = 0 \quad &\Rightarrow \quad x = 2y - 3y^2 \end{aligned}$$

To find the places at which *both* are 0, we equate these two expressions for x. That is, we have seen that $x = -y - 6$ whenever $f_x = 0$, and that $x = 2y - 3y^2$ whenever $f_y = 0$, so it must be true that $-y - 6 = 2y - 3y^2$ whenever both f_x and f_y are 0.

$$-y - 6 = 2y - 3y^2 \Rightarrow 3y^2 - 3y - 6 = 0 \Rightarrow 3(y^2 - y - 2) = 0 \Rightarrow (y-2)(y+1) = 0$$

We see that both first partials are 0 when $y = 2$, which gives $x = -y - 6 = -8$ and also when $y = -1$, which gives $x = -y - 6 = -5$. So the critical points of f are $(x,y) = (-8,2)$ and $(x,y) = (-5,-1)$.

To determine whether each is a (relative) maximum or minimum, or is a saddle point, we need to do The Second Partials Test, for which we need to find the second partials. We have

$$\begin{aligned} f_{xx}(x,y) &= \frac{\partial}{\partial x}(x + 6 + y) = 1 \\ f_{yy}(x,y) &= \frac{\partial}{\partial y}\left(x - 2y + 3y^2\right) = -2 + 6y \\ f_{xy}(x,y) &= \frac{\partial}{\partial y}(x + 6 + y) = 1 \end{aligned}$$

We see that the test function is

$$D(x,y) = (f_{xx}(x,y))\,(f_{yy}(x,y)) - (f_{xy}(x,y))^2 = (1)(-2+6y) - (1)^2 = 6y - 3$$

Notice that since $f_{xx}(x,y) = 1 > 0$ for all (x,y) in the domain of f, f cannot have any relative maxima (because relative maxima have $D > 0$ and $f_{xx} < 0$). Now we evaluate $D(a,b)$ at each of the critical points to see if we can determine whether each is a relative minimum or a saddle point.

For $(-8,2)$, we have $D(-8,2) = 6(2) - 3 = 9 > 0$ so we see that (with $D > 0$ and $f_{xx} > 0$) this critical point is a relative minimum of f.

For $(-5,-1)$, we have $D(-5,-1) = 6(-1) - 3 = -9 < 0$ so we see that (because $D < 0$) this critical point is a saddle point of f.

Of course, sometimes optimization problems need to be set up before we can start solving them. Real world problems don't tend to have the form "find the maximum (or minimum) of this function". Consider the following example.

Example 10.5. A company makes 2 kinds of widgets - *type 1* and *type 2*. If the company produces some quantity x of the *type 1* widget and some quantity y of the *type 2* widget, they will be able to sell each *type 1* widget for a price of $\$110 - \$5x$, and each *type 2* widget for a price of $\$85 - \$3y$. Producing x of the *type 1* widgets and y of the *type 2* widgets will cost $C(x, y) = 6x + 5y - 2xy$ dollars. Find the production levels of the 2 types of widgets which will maximize the company's profit.

Solution:
Of course, profit = revenue − cost. We know the cost, $C(x, y)$, of producing x *type 1* widgets and y *type 2* widgets. The revenue from selling x *type 1* widgets at a price of $\$110 - \$5x$ each is $x(110 - 5x) = 110x - 5x^2$ dollars. Similarly, the revenue from selling y *type 2* widgets at a price of $\$85 - \$3y$ each is $y(85 - 3y) = 85y - 3y^2$ dollars. So the total revenue (in \$'s) from producing x *type 1* widgets and y *type 2* widgets will be $R(x, y) = 110x - 5x^2 + 85y - 3y^2$. Thus the company's profit, in dollars, is given by:

$$\begin{aligned} P(x,y) &= R(x,y) - C(x,y) \\ &= \left(110x - 5x^2 + 85y - 3y^2\right) - (6x + 5y - 2xy) \\ &= 104x - 5x^2 + 80y - 3y^2 + 2xy \end{aligned}$$

We need to find the maximum of this function. We see that:

$$\begin{aligned} P_x(x,y) &= 104 - 10x + 2y \\ \text{and } P_y(x,y) &= 80 - 6y + 2x \\ \text{so that } P_x(x,y) = 0 &\Rightarrow 104 - 10x + 2y = 0 \Rightarrow y = \frac{10x - 104}{2} = 5x - 52 \\ \text{and } P_y(x,y) = 0 &\Rightarrow 80 - 6y + 2x = 0 \Rightarrow x = \frac{6y - 80}{2} = 3y - 40 \end{aligned}$$

To have $x = 3y - 40$ while $y = 5x - 52$, we must have

$$x = 3(5x - 52) - 40 \Rightarrow x = 15x - 156 - 40 \Rightarrow 14x = 196 \Rightarrow x = 14$$

which gives $y = 5(14) - 52 = 18$. So $(x, y) = (14, 18)$ as the only critical point of this function that has $P_x = P_y = 0$.

We can easily confirm that this is a relative maximum, using the Second Partials Test. We have:

$$P_{xx}(x,y) = -10, \qquad P_{yy}(x,y) = -6 \qquad \text{and } P_{xy}(x,y) = 2$$

so that $D(x, y) = (P_{xx}(x, y))(P_{yy}(x, y)) - (P_{xy}(x, y))^2 = (-10)(-6) - (2)^2 = 56$. Therefore $D(14, 18) = 56 > 0$ and since $P_{xx}(14, 18) = -10 < 0$, it is indeed a relative maximum which occurs at the point $(x, y) = (14, 18)$.

Constrained Optimization

Recall that using single variable calculus it is sometimes possible to find the *constrained optimum* of a 2-variable function, provided that the constraint can be used to express one variable in terms of the other, to reduce the problem to one of finding the unconstrained optimum of a single variable problem. We can do something similar, sometimes, when we have a 3-variable constrained optimization problem. That is, if we have a function of 3 variables, and a constraint which allows us to express one of the variables in terms of the other two, then we can reduce the constrained 3-variable problem to an unconstrained 2-variable problem, to which we can apply the Second Partials Test, which may allow us to solve the problem (i.e. find the optimum value we're looking for). For instance, consider the following problem.

Example 10.6. Find the dimensions of an open rectangular box with one lengthwise partition, as described in Example 9.3 on page 135, with costs as given in part (b) of that example, which minimize the cost of making the box, subject to the constraint that the volume of the box must be 30 cubic cm.

Solution:
In the earlier example we found the volume of the box to be given by $V(x, y, z) = xyz$ and the cost of making the box to be $C(x, y, z) = .25xy + .3yz + .4xz$, where x is the length of the box, y is the width and z is the height. Now, we want to find the minimum possible value of $C(x, y, z)$ for which we have $V(x, y, z) = 30$. We use the constraint that $xyz = 30$ to express z in terms of x and y. (It doesn't have to be z that we express in terms of the other two variables. We could eliminate any one of the variables.) We have

$$xyz = 30 \qquad \Rightarrow \qquad z = \frac{30}{xy}$$

We use this to express the cost function in terms of only x and y. The new cost function is:

$$f(x, y) = .25xy + .3y\left(\frac{30}{xy}\right) + .4x\left(\frac{30}{xy}\right) = .25xy + \frac{9}{x} + \frac{12}{y} = \frac{xy}{4} + \frac{9}{x} + \frac{12}{y}$$

That is, we have $f(x, y) = \frac{xy}{4} + 9x^{-1} + 12y^{-1}$. The domain of the function is $x > 0, y > 0$ (since none of length, width or height can be negative and the constraint will not allow any of them to be 0, either). We can use the Second Partials Test to find the minimum value of this function on its domain.

First, we find the first partials, and then set them both equal to zero:

$$\begin{aligned}
f_x(x, y) &= \frac{y}{4}(1) + 9(-x^{-2}) + 0 = \frac{y}{4} - \frac{9}{x^2} \\
f_y(x, y) &= \frac{x}{4}(1) + 0 + 12(-y^{-2}) = \frac{x}{4} - \frac{12}{y^2} \\
\text{so } f_x(x, y) = 0 &\Rightarrow \frac{y}{4} - \frac{9}{x^2} = 0 \Rightarrow y = \frac{36}{x^2} \\
\text{and } f_y(x, y) = 0 &\Rightarrow \frac{x}{4} - \frac{12}{y^2} = 0 \Rightarrow x = \frac{48}{y^2}
\end{aligned}$$

Since we need both $f_x(x, y) = 0$ and $f_y(x, y) = 0$ at the same time, we must have $x = \frac{48}{y^2}$ while $y = \frac{36}{x^2}$. Thus we need

$$x = \frac{48}{\left(\frac{36}{x^2}\right)^2} = (48)\left(\frac{x^2}{(36)}\right)^2 = (4 \times 12)\frac{x^4}{36 \times 36} = \frac{4x^4}{3 \times 36} = \frac{x^4}{3 \times 9} = \frac{x^4}{27}$$

Of course, $x = \dfrac{x^4}{27}$ gives $x^4 = 27x$ and since $x \neq 0$ we get $x^3 = 27 = 3^3$ so $x = 3$. And so we also need $y = \dfrac{36}{x^2} = \dfrac{36}{3^2} = \dfrac{36}{9} = 4$. That is, the only critical point is $(x, y) = (3, 4)$. We need to check that $f(3,4)$ is a relative minimum. We find the second partials:

$$\begin{aligned} f_x(x,y) = \frac{y}{4} - 9x^{-2} \qquad &\Rightarrow \qquad f_{xx}(x,y) = 0 - 9(-2x^{-3}) = \frac{18}{x^3} \\ \text{and also} \quad & f_{xy}(x,y) = \frac{1}{4} - 0 = \frac{1}{4} \\ \text{and } f_y(x,y) = \frac{x}{4} - 12y^{-2} \qquad &\Rightarrow \qquad f_{yy}(x,y) = 0 - 12(-2y^{-3}) = \frac{24}{y^3} \end{aligned}$$

We see that at the critical point $(x,y) = (3,4)$ we have $f_{xx}(3,4) = \frac{18}{3^3} = \frac{2}{3}$ and $f_{yy}(3,4) = \frac{24}{4^3} = \frac{6}{16} = \frac{3}{8}$, while $f_{xy}(3,4) = \frac{1}{4}$. Therefore we get

$$D(3,4) = (f_{xx}(3,4))\,(f_{yy}(3,4)) - (f_{xy}(3,4))^2 = \left(\frac{2}{3}\right)\left(\frac{3}{8}\right) - \left(\frac{1}{4}\right)^2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16} > 0$$

Since $D(3,4) > 0$ and also $f_{xx}(3,4) > 0$ we see that $f(3,4)$ really is a relative minimum value. And since this is the only critical point of f, which is continuous throughout its domain, this must also be the (absolute) minimum value of f. This minimum value is

$$f(x,y) = \frac{xy}{4} + \frac{9}{x} + \frac{12}{y} \qquad \Rightarrow \qquad f(3,4) = \frac{3(4)}{4} + \frac{9}{3} + \frac{12}{4} = 3 + 3 + 3 = 9$$

and is obtained when $x = 3$ and $y = 4$ so that $z = \frac{30}{xy} = \frac{30}{3(4)} = \frac{10}{4} = 2.5$. That is, a box which is 3 cm long, 4 cm wide and 2.5 cm high provides volume 30 cubic cm at the minimum possible cost (for this volume) of only \$9.

For *some* constrained optimization problems, the method we used above works. But *only* when the constraint has a form which allows one variable to be expressed in terms of the others. But constraints can be more complicated than that. Even in a problem in which there are only 2 variables, it may not be possible to untangle the variables so that the constraint can be eliminated, i.e. incorporated into the function to be optimized, by using it to replace one of the variables in terms of the other. So we need another way to approach constrained optimization problems.

There's another approach we can use, which works for *any* constrained optimization problem which has only one constraint. It doesn't matter how complicated the constraint is, or even how many variables there are. The same method can be used for a two-variable problem, a three-variable problem, or even a problem with more variables, as long as there's only one constraint. This approach is called **Lagrange's Method**, or the **Method of Lagrange Multipliers**.

The way this approach works is certainly counter-intuitive. Rather than trying to reduce the size of the problem by eliminating a variable, instead we *add a new variable*, making a two-variable problem into a three-variable problem, or a three-variable problem into a four-variable problem, etc., but the larger problem is an *unconstrained* problem. That is, we once again eliminate the constraint by incorporating it into the function to be optimized. And then, (since the proof that this method works is way beyond the scope of this course, we take Mr. Lagrange's word for it that) the optimal solution (if there is one) will be among the type 2 critical points of the new function. That is, the optimal solution will be one of the points for which all first partials are 0, so we simply need to identify all such points and determine which of them provide(s) the optimal (largest or smallest, depending on the problem) function value. (Assuming that the problem *does* have an optimum.)

We will express this method as it applies to a two-variable problem. But then later we'll see how to extend it to three-variable problems, and it can be extended to more variables in exactly the same way. The more variables there are, the more first partials there are which need to be set equal to zero, but otherwise the approach is exactly the same.

Lagrange's Method
Consider the problem of optimizing some function $f(x, y)$ subject to some constraint of the form $g(x, y) = 0$. Define the **Lagrange function**

$$F(x, y, \lambda) = f(x, y) + \lambda[g(x, y)]$$

The relative extreme values of the original constrained optimization problem will be found at points (x, y) which are among the points which satisfy:

$$\begin{aligned} F_x(x, y, \lambda) &= 0 \\ \text{and } F_y(x, y, \lambda) &= 0 \\ \text{and } F_\lambda(x, y, \lambda) &= 0 \end{aligned}$$

And if the problem has absolute extrema, they will be among these relative extrema, so we can determine the optimum of f by finding all candidate points (x, y) which satisfy all 3 of the above equations and determine which give(s) the optimal (i.e. largest or smallest, depending on the problem) value for f.

Notice: The constraint *must* have right-hand-side 0. But that just means that if there's a different constant on the right hand side, we move it to the left hand side. That is, for a constraint of the form $h(x, y) = c$ for some constant $c \neq 0$, we use the constraint function $g(x, y) = h(x, y) - c$ so that the constraint is $g(x, y) = 0$.

Also note: The funny looking thing is the Greek letter "lambda" (the b is silent, as in "lamb"). We always use lambda (λ) as the multiplier on the constraint function in Lagrange's method.

Since we want to start by applying this approach to two-variable problems, we won't use it, yet, to see another way to solve the constrained problem we already did. We'll do that later, to see how Lagrange's method is used for a three-variable problem. For now, we'll start by applying it to a two-variable problem that we know how to solve by the "old" method, so we can see that this new approach gives the same answer.

Example 10.7. Find the minimum of $f(x, y) = x^2 + y^2 + xy - 6x$ subject to $x - y = 10$:

(a) by eliminating one of the variables to reduce the problem to a single-variable unconstrained problem and solving it using the Second Derivative Test (as in your Introductory Calculus course)

(b) using the Method of Lagrange Multipliers.

Solution:
(a) We have only 2 variables, and a friendly constraint which easily allows us to express one variable in terms of the other. We need $x - y = 10$, so we let $y = x - 10$ and use this to replace y in the function which is to be minimized. We get:

$$x^2+y^2+xy-6x = x^2+(x-10)^2+x(x-10)-6x = x^2+(x^2-20x+100)+(x^2-10x)-6x = 3x^2-36x+100$$

We'll call this new function $h(x)$. So we have transformed the problem
minimize $f(x, y) = x^2 + y^2 + xy - 6x$ subject to $x - y = 10$
into that of finding the unconstrained minimum of $h(x) = 3x^2 - 36x + 100$.

We need to find the critical numbers of h, by finding the values of x for which $h'(x) = 0$. We have:

$$h'(x) = 6x - 36 = 6(x - 6)$$

and so we see that $h'(x) = 0$ only when $x = 6$, so this is the only critical number of h. And we have $h''(x) = 6 > 0$, so there is a relative minimum at $x = 6$, i.e. $h(6) = 3(6)^2 - 36(6) + 100 = -3(36) + 100 = -108 + 100 = -8$ is a relative minimum value. And since h is a quadratic, this relative minimum is also the global (i.e. absolute) minimum.

Therefore the minimum value of $f(x, y)$ is -8, obtained when $x = 6$ and $y = x - 10 = -4$. (*Note:* It never hurts to check your arithmetic by calculating $f(6, -4)$ to make sure it really does give the value -8.)

(b) We have the function to be minimized: $f(x, y) = x^2 + y^2 + xy - 6x$, and the constraint: $x - y = 10$ which we rearrange to $x - y - 10 = 0$, so for the Lagrange method the constraint function is $g(x, y) = x - y - 10$. We form the Lagrange function:

$$F(x, y, \lambda) = f(x, y) + \lambda[g(x, y)] = x^2 + y^2 + xy - 6x + \lambda(x - y - 10)$$

Now we find the points (x, y) at which all first partials of F are 0. Since we now have 3 variables (x, y and λ), there are 3 first partials to consider:

$$\begin{aligned}
F_x(x, y, \lambda) &= \frac{\partial}{\partial x}[x^2 + y^2 + xy - 6x + \lambda(x - y - 10)] = 2x + y - 6 + \lambda(1) = 2x + y - 6 + \lambda \\
F_y(x, y, \lambda) &= \frac{\partial}{\partial y}[x^2 + y^2 + xy - 6x + \lambda(x - y - 10)] = 2y + x + \lambda(-1) = 2y + x - \lambda \\
F_\lambda(x, y, \lambda) &= \frac{\partial}{\partial \lambda}[x^2 + y^2 + xy - 6x + \lambda(x - y - 10)] = 1(x - y - 10) = x - y - 10
\end{aligned}$$

Notice: Because of the form of the Lagrange function, we always get "plus λ times the first partial of g" in the first partial with respect to each of the original variables, and the first partial with respect to the new variable λ is always just g.

Next, we need to solve for the values of x and y which make all first partials equal 0. When we do this, again because of the form of the Lagrange function, the first partials with respect to the original variables can always be rearranged to the form "λ equals" some expression involving only the original variables. This allows us to eliminate λ, leaving us with just x and y. That is, in this case we need

$$F_x(x, y, \lambda) = 0 \qquad \Rightarrow \qquad 2x + y - 6 + \lambda = 0 \qquad \Rightarrow \qquad \lambda = 6 - 2x - y$$

and we also need

$$F_y(x, y, \lambda) = 0 \qquad \Rightarrow \qquad 2y + x - \lambda = 0 \qquad \Rightarrow \qquad \lambda = x + 2y$$

Since both of these must be true at the same time, we can equate these 2 expressions for λ:

$$6 - 2x - y = x + 2y \qquad \Rightarrow \qquad -3y = 3x - 6 \qquad \Rightarrow \qquad y = -x + 2 = 2 - x$$

This last equation, $y = 2 - x$, together with the equation from setting the first partial with respect to λ equal to 0, which does not contain λ and which we have not yet used, gives 2 equations in the 2 unknowns x and y. That is, we need $F_\lambda(x, y, \lambda) = 0$, which gives

$$F_\lambda(x, y, \lambda) = 0 \qquad \Rightarrow \qquad x - y - 10 = 0 \qquad \Rightarrow \qquad x - y = 10$$

and since we also need $y = 2 - x$ we get

$$x - (2 - x) = 10 \qquad \Rightarrow \qquad 2x = 12 \qquad \Rightarrow \qquad x = 6 \qquad \Rightarrow \qquad y = 2 - x = 2 - 6 = -4$$

Notice that we could find the corresponding value of λ, using either of the expressions for λ we found, but *we just don't care what the value of λ is.*

Since $(x, y) = (6, -4)$ is the only point satisfying $F_x = F_y = F_\lambda = 0$, then this must be the only relative extremum of the original problem. So is it a relative minimum or a relative maximum? Well, we evaluate $f(6, -4) = -8$ (as before) and then we can check any other point on the constraint function, e.g. when $x = 0$, $x - y = 10 \Rightarrow y = -10$ and $f(0, -10) = 0^2 + (-10)^2 + 0 - 0 = 100 > -8$. Since f is a quadratic, the relative extremum must also be a global extremum, and we can see that it's not the global maximum (since we found a higher value elsewhere) and so the relative extremum we found must be a minimum. Therefore, as before, the minimum value of $f(x, y)$ is $f(6, -4) = -8$.

In this situation, the solution using Lagrange's method was not quicker, but it wasn't hard. And in other situations it will be the only method available. For instance, what if the function to be optimized and the constraint function were interchanged in this problem? If we wanted to find the maximum or minimum of $x-y$ subject to the constraint that $x^2+y^2+xy-6x = c$ for some constant c, we would not be able to use the constraint to express one variable in terms of the other. The x's and y's in $x^2 + y^2 + xy - 6x = c$ are too intertwined to isolate one of the variables. So for a problem like that, we *have to* use Lagrange's method. We don't know any other way to solve the constrained optimization problem. As in the next example.

Example 10.8. Find the minimum of $x - y$ subject to the constraint $x^2 + y^2 + xy - 6x = 4$.

Solution:
We have $f(x, y) = x - y$ and we rearrange the constraint to have 0 on the right hand side, which gives the constraint function $g(x, y) = x^2 + y^2 + xy - 6x - 4$. So the Lagrange function is

$$F(x, y) = x - y + \lambda(x^2 + y^2 + xy - 6x - 4)$$

We find all first partials:

$$\begin{aligned} F_x(x, y, \lambda) &= 1 + \lambda(2x + y - 6) \\ F_y(x, y, \lambda) &= -1 + \lambda(2y + x) \\ F_\lambda(x, y, \lambda) &= x^2 + y^2 + xy - 6x - 4 \end{aligned}$$

We set F_x equal to 0 and rearrange to isolate λ, and then do the same with F_y:

$$F_x(x, y, \lambda) = 0 \quad \Rightarrow \quad 1 + \lambda(2x + y - 6) = 0 \quad \Rightarrow \quad \lambda = -\tfrac{1}{2x+y-6}$$

$$F_y(x, y, \lambda) = 0 \quad \Rightarrow \quad -1 + \lambda(2y + x) = 0 \quad \Rightarrow \quad \lambda = \tfrac{1}{2y+x}$$

Now, we equate the two expressions for λ and rearrange:

$$\begin{aligned} \lambda = -\frac{1}{2x + y - 6} \text{ and } \lambda = \frac{1}{2y + x} \quad &\Rightarrow \quad -\frac{1}{2x + y - 6} = \frac{1}{2y + x} \\ &\Rightarrow \quad -(2y + x) = 2x + y - 6 \\ &\Rightarrow \quad -2y - x = 2x + y - 6 \\ &\Rightarrow \quad -3y = 3x - 6 \\ &\Rightarrow \quad y = 2 - x \end{aligned}$$

Of course, we also need to set F_λ equal to zero. We do that, and substitute for y using what we found above:

$$\begin{aligned} F_\lambda(x, y, \lambda) = 0 \quad &\Rightarrow \quad x^2 + y^2 + xy - 6x - 4 = 0 \\ &\Rightarrow \quad x^2 + (2 - x)^2 + x(2 - x) - 6x - 4 = 0 \\ &\Rightarrow \quad x^2 + (4 - 4x + x^2) + (2x - x^2) - 6x - 4 = 0 \\ &\Rightarrow \quad x^2 - 8x = 0 \\ &\Rightarrow \quad x(x - 8) = 0 \end{aligned}$$

We see that all first partials will be 0 when $x = 0$ and also when $x = 8$. And since $y = 2 - x$ we see that when $x = 0$, $y = 2$, while $x = 8$ gives $y = -6$. That is, the candidate points are $(0, 2)$ and $(8, -6)$. And the minimum of f, which is what we're looking for, has to occur at one of these. Also, there was nothing in what we did so far that took into consideration that we're looking for a minimum, so if we were looking for a maximum, that would also have to occur at one of these. So if the two points have different values for f, then the lower of these values is the minimum value (and the higher is the maximum value). That is, once we have found the candidate points, we simply evaluate f at each candidate point and select the one that gives the smallest value, if we're looking for a minimum, or the one that gives the largest value, if we're looking for a maximum. But remember, we're talking about values of the function f, that we needed to optimize, not of the constraint function g or the Lagrange function F.

In this case, with $f(x, y) = x - y$ we get $f(0, 2) = 0 - 2 = -2$ and $f(8, -6) = 8 - (-6) = 14$. Therefore the minimum value of $f(x, y) = x - y$, subject to the constraint that $x^2 + y^2 + xy - 6x = 4$, is -2, which is obtained at $(x, y) = (0, 2)$.

Note: It can't hurt to check, at this point, that the candidate points do satisfy the contstraint equation. If one or more of them doesn't, that means you made an arithmetic mistake somewhere along the line. (Surely not a calculus mistake! But that's possible too ...)

Example 10.9. Find the maximum value of $f(x, y) = x^2 + y^2$ subject to the constraint $x^2 + xy + y^2 = 12$.

Solution:
We have $f(x, y) = x^2 + y^2$ and rearranging the constraint we get $g(x, y) = x^2 + xy + y^2 - 12$, so the Lagrange function is

$$F(x, y, \lambda) = x^2 + y^2 + \lambda(x^2 + xy + y^2 - 12)$$

The first partials of this function are:

$$\begin{aligned} F_x(x, y, \lambda) &= 2x + \lambda(2x + y) \\ F_y(x, y, \lambda) &= 2y + \lambda(x + 2y) \\ \text{and } F_\lambda(x, y, \lambda) &= x^2 + xy + y^2 - 12 \end{aligned}$$

We set $F_x(x, y, \lambda) = 0$ and $F_y(x, y, \lambda) = 0$ and isolate λ in each:

$$2x + \lambda(2x + y) = 0 \quad \Rightarrow \quad \lambda(2x + y) = -2x \quad \Rightarrow \quad \lambda = -\tfrac{2x}{2x+y}$$

$$2y + \lambda(x + 2y) = 0 \quad \Rightarrow \quad \lambda(x + 2y) = -2y \quad \Rightarrow \quad \lambda = -\tfrac{2y}{x+2y}$$

Next, we equate these to solve for y in terms of x (or vice versa).

$$\begin{aligned} -\frac{2x}{2x + y} = -\frac{2y}{x + 2y} \quad &\Rightarrow \quad (x + 2y)(2x) = (2x + y)(2y) \\ &\Rightarrow \quad 2x^2 + 4xy = 4xy + 2y^2 \\ &\Rightarrow \quad 2x^2 = 2y^2 \\ &\Rightarrow \quad x^2 = y^2 \\ &\Rightarrow \quad y = \pm x \end{aligned}$$

Now, we use $F_\lambda(x, y, \lambda) = 0$ to find the candidate points. We have the equation $x^2 + xy + y^2 - 12 = 0$, with $y = \pm x$.

When $y = x$, we have $x^2 + x(x) + (x)^2 - 12 = 0$, so $3x^2 = 12$ and $x^2 = 4$, which gives $x = \pm 2$. This gives us two candidate points: $(2, 2)$ and $(-2, -2)$ (because we have $y = x$).

When $y = -x$, we have $x^2 + x(-x) + (-x)^2 - 12 = 0$, so $x^2 = 12$ which gives $x = \pm\sqrt{12} = \pm 2\sqrt{3}$. This gives us two more candidate points: $(2\sqrt{3}, -2\sqrt{3})$ and $(-2\sqrt{3}, 2\sqrt{3})$ (remember: in this case we have $y = -x$).

We have 4 candidate points for giving the maximum value of $f(x, y) = x^2 + y^2$. We see that

$$f(2, 2) = f(-2, -2) = 4 + 4 = 8$$

and that

$$f(2\sqrt{3}, -2\sqrt{3}) = f(-2\sqrt{3}, 2\sqrt{3}) = (2\sqrt{3})^2 + (2\sqrt{3})^2 = 12 + 12 = 24$$

Clearly, 24 is the maximum value and 8 is the minimum value. So the maximum value of $x^2 + y^2$ subject to $x^2 + xy + y^2 = 12$ is 24, occurring at both $(2\sqrt{3}, -2\sqrt{3})$ and $(-2\sqrt{3}, 2\sqrt{3})$.

Notice: We must be very careful about finding the candidate points, especially in the kind of situation we had in this example. Or ones that you might confuse with it. Here we had $y = \pm x$, and we considered two separate cases, one where $y = x$ and the other where $y = -x$. In finding the candidate points for each case, we had to remember which case it was we were working on. Because within each case, the sign of y relative to x is determined. But in other situations, it is sometimes the case that the two cases give the same x values, so that in fact any combination of a sign for x and a sign for y gives a candidate point. But that's not true here. For instance, $f(2\sqrt{3}, 2\sqrt{3}) = 24$ as well, but this point *doesn't satisfy the constraint* and so it is not one of the candidate points. As was stated earlier, it *never* hurts to take a moment and check that your candidate points do satisfy the constraint.

In fact, the kind of situation that you might confuse with this arises when the constraint function has the kind of form that the function to be maximized had in the example above, with *only* x^2 and y^2, but no xy term. So it's worth looking at an example like that.

Example 10.10. Find the maximum of $f(x, y) = x^2 + xy + y^2$ subject to the constraint that $x^2 + y^2 = 32$.

Solution:
We have $f(x, y) = x^2 + xy + y^2$ and rearranging the constraint we get $g(x, y) = x^2 + y^2 - 32$, so the Lagrange function is

$$F(x, y, \lambda) = x^2 + xy + y^2 + \lambda(x^2 + y^2 - 32)$$

The first partials of this function are:

$$\begin{aligned} F_x(x, y, \lambda) &= 2x + y + \lambda(2x) \\ F_y(x, y, \lambda) &= x + 2y + \lambda(2y) \\ \text{and } F_\lambda(x, y, \lambda) &= x^2 + y^2 - 32 \end{aligned}$$

We set $F_x(x, y, \lambda) = 0$ and $F_y(x, y, \lambda) = 0$ and isolate λ in each:

$$\begin{array}{lclcl} 2x + y + \lambda(2x) = 0 & \Rightarrow & 2x\lambda = -2x - y & \Rightarrow & \lambda = -1 - \frac{y}{2x} \\ x + 2y + \lambda(2y) = 0 & \Rightarrow & 2y\lambda = -x - 2y & \Rightarrow & \lambda = -\frac{x}{2y} - 1 \end{array}$$

Next, we equate these to solve for y in terms of x (or vice versa).

$$\begin{aligned} -1 - \frac{y}{2x} = -\frac{x}{2y} - 1 \quad &\Rightarrow \quad -\frac{y}{2x} = -\frac{x}{2y} \\ &\Rightarrow \quad 2y^2 = 2x^2 \\ &\Rightarrow \quad y^2 = x^2 \\ &\Rightarrow \quad y = \pm x \end{aligned}$$

Next we will use $F_\lambda(x, y, \lambda) = 0$ to find the candidate points. But notice ... We have the constraint function $g(x, y) = x^2 + y^2 - 32$. In this function, each of x and y **only** appears squared. So we can simply use the fact that we need $y^2 = x^2$. We don't need to consider 2 cases, with $y = x$ and with $y = -x$. We could do so, but we would just be doing the same thing each time, because we would immediately square all the x's, which would take away the one thing that was distinguishing between the two cases, i.e. the negative sign, or lack thereof. Recall that in the previous example, we did have one term in the constraint function, and therefore in F_λ, in which the x's and y's weren't squared, and that's why that situation was different than the current one.

Here, we simply substitute $y^2 = x^2$ into the equation stating that $F_\lambda(x, y, \lambda) = 0$ and solve for x. But after that we will need to think carefully again. We have:

$$y^2 = x^2 \text{ and } x^2 + y^2 - 32 = 0 \Rightarrow x^2 + x^2 - 32 = 0 \Rightarrow 2x^2 = 32 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$$

So we have $x = \pm 4$, with $y^2 = x^2$ so that $y = \pm x$. Using $x = 4$ and $y = x$ we get the candidate point $(4, 4)$. Using $x = 4$ with $y = -x$ we get the candidate point $(4, -4)$. Using $x = -4$ and $y = x$ we get the candidate point $(-4, -4)$. And using $x = -4$ with $y = -x$ we get the candidate point $(-4, 4)$. So this time we do have *every* possible combination of a sign for x and a sign for y giving a candidate point.

Evaluating $f(x, y) = x^2 + xy + y^2$ at the various candidate points, we get:

$$\begin{aligned} f(4,4) &= (4)^2 + (4)(4) + (4)^2 = 16 + 16 + 16 = 48 \\ f(4,-4) &= (4)^2 + (4)(-4) + (-4)^2 = 16 - 16 + 16 = 16 \\ f(-4,4) &= (-4)^2 + (-4)(4) + (4)^2 = 16 - 16 + 16 = 16 \\ f(-4,-4) &= (-4)^2 + (-4)(-4) + (-4)^2 = 16 + 16 + 16 = 48 \end{aligned}$$

We see that the maximum value of $x^2 + xy + y^2$, subject to the constraint that $x^2 + y^2 = 32$, is 48, which is obtained both at $(x, y) = (4, 4)$ and at $(x, y) = (-4, -4)$.

As stated previously, Lagrange's Method can easily be extended to more variables. An n variable constrained optimization problem is transformed into an $n + 1$ variable unconstrained optimization by forming the Lagrange function in the usual way. That is, given the problem of finding the maximum or minimum value of $f(x_1, ..., x_n)$ subject to the constraint that $g(x_1, ..., x_n) = 0$, we let $F(x_1, ..., x_n, \lambda) = f(x_1, ..., x_n) + \lambda[g(x_1, ..., x_n)]$. The optimal solution to the original constrained problem will then be found among the points $(x_1, ..., x_n)$ which satisfy the requirement that all first partials of F are equal to 0. F has $n + 1$ first partials, so this gives $n + 1$ equations (all with right hand side 0). We use the first n of them (i.e. the first partials with respect to each of the original variables) to obtain n different expressions for λ. We then equate pairs of these expresions for λ to one another, using $n - 1$ different pairs which together use all n expressions for λ (for instance, equate the first expression for λ to each of the others). This gives $n - 1$ relationships between one of the variables and the others. Substituting these into the equation stating that the first partial with respect to λ must equal 0 (which as always is just the original constraint) allows us to find the possible values of one of the variables, and then the $n - 1$ relationships give us the corresponding possible values of the other variables, giving the various candidate points. We finally evaluate f at each of the candidate points. The largest (or smallest) f-value found among the candidate points is the maximum (or minimum) value we needed to find.

We will see how this works by re-solving the problem in Example 10.6, which is a 3-variable problem, using Lagrange's method.

Example 10.11. Use Lagrange's Method to find the dimensions of an open rectangular box with one lengthwise partition, as described in Example 9.3 on page 135, with costs as given in part (b) of that example, which minimize the cost of making the box, subject to the constraint that the volume of the box must be 30 cubic cm.

Solution:
In Example 9.3 we found the volume of the box to be given by $V(x,y,z) = xyz$ and the cost of making the box to be $C(x,y,z) = .25xy + .3yz + .4xz$, where x is the length of the box, y is the width and z is the height. Now, we want to find the minimum possible value of $C(x,y,z)$ for which we have $V(x,y,z) = 30$, so the constraint function for Lagrange's method is $g(x,y,z) = xyz - 30$. (The cost function, C, plays the role of f in this situation, since it is the function whose optimum value we need to find.)

We start by forming the Lagrange function:

$$F(x,y,z,\lambda) = C(x,y,z) + \lambda g(x,y,z) = .25xy + .3yz + .4xz + \lambda(xyz - 30)$$

and then find all first partials of this function:

$$\begin{aligned}
F_x(x,y,z,\lambda) &= .25y + 0 + .4z + \lambda(yz) = .25y + .4z + yz\lambda \\
F_y(x,y,z,\lambda) &= .25x + .3z + 0 + \lambda(xz) = .25x + .3z + xz\lambda \\
F_z(x,y,z,\lambda) &= 0 + .3y + .4x + \lambda(xy) = .4x + .3y + xy\lambda \\
F_\lambda(x,y,z,\lambda) &= xyz - 30
\end{aligned}$$

We equate each of the first 3 first partials to 0 and rearrange to isolate λ:

$$F_x(x,y,z,\lambda) = 0 \quad\Rightarrow\quad .25y + .4z + yz\lambda = 0 \quad\Rightarrow\quad yz\lambda = -.25y - .4z \quad\Rightarrow\quad \lambda = -\frac{.25y + .4z}{yz}$$

$$F_y(x,y,z,\lambda) = 0 \quad\Rightarrow\quad .25x + .3z + xz\lambda = 0 \quad\Rightarrow\quad xz\lambda = -.25x - .3z \quad\Rightarrow\quad \lambda = -\frac{.25x + .3z}{xz}$$

$$F_z(x,y,z,\lambda) = 0 \quad\Rightarrow\quad .4x + .3y + xy\lambda = 0 \quad\Rightarrow\quad xy\lambda = -.4x - .3y \quad\Rightarrow\quad \lambda = -\frac{.4x + .3y}{xy}$$

Notice that the constraint $xyz = 30$ does not allow any of these variables to be 0, so we were able to divide through by x, y and/or z without having to worry about "or that variable equals 0" giving us other situations that needed to be considered. We will need that again in the next step, as well.

This gives 3 different expressions for λ, which can be paired up in 3 different ways. However, if we use any 2 different pairings and require that the pair of expressions be equal for each, then we have in fact required that all three expressions be equal. For instance, if we require that the first and second expressions for λ be equal, and then require that the first also equal the third, then the second and third expressions must also be equal to one another, since each is equal to the first. Therefore we only need to use two pairings. (Any two of the 3 pairings will do.)

Equating the expressions for λ obtained from F_x and F_y we get:

$$\begin{aligned}
-\frac{.25y+.4z}{yz} = -\frac{.25x+.3z}{xz} \quad &\Rightarrow \quad xz(.25y+.4z) = yz(.25x+.3z)\\
&\Rightarrow \quad .25xyz + .4xz^2 = .25xyz + .3yz^2\\
&\Rightarrow \quad .4xz^2 = .3yz^2\\
&\Rightarrow \quad .4x = .3y\\
&\Rightarrow \quad y = \left(\frac{4}{3}\right)x
\end{aligned}$$

And then equating the expressions for λ obtained from F_x and F_z we get:

$$\begin{aligned}
-\frac{.25y+.4z}{yz} = -\frac{.4x+.3y}{xy} \quad &\Rightarrow \quad xy(.25y+.4z) = yz(.4x+.3y)\\
&\Rightarrow \quad .25xy^2 + .4xyz = .4xyz + .3y^2z\\
&\Rightarrow \quad .25xy^2 = .3y^2z\\
&\Rightarrow \quad .25x = .3z\\
&\Rightarrow \quad z = \left(\frac{25}{30}\right)x\\
&\Rightarrow \quad z = \left(\frac{5}{6}\right)x
\end{aligned}$$

Now, we use these relationships to substitute for y and z in terms of x in the equation requiring that the first partial of F with respect to λ must also equal 0:

$$F_\lambda(x,y,z,\lambda) = 0 \Rightarrow xyz - 30 = 0 \Rightarrow x\left[\left(\frac{4}{3}\right)x\right]\left[\left(\frac{5}{6}\right)x\right] = 30 \Rightarrow x^3 = 30\left(\frac{3}{4}\right)\left(\frac{6}{5}\right) \Rightarrow x^3 = 27$$

We see that the only possible value for x is $x = 3$, which gives $y = \left(\frac{4}{3}\right)x = \left(\frac{4}{3}\right)(3) = 4$ and $z = \left(\frac{5}{6}\right)x = \left(\frac{5}{6}\right)(3) = \frac{5}{2}$. Therefore $(x,y,z) = (3,4,2.5)$ is the only candidate point, and so the minimum cost of a box with volume 30 cubic cm is (in dollars)

$$C(3,4,2.5) = .25xy + .3yz + .4xz = \left(\frac{1}{4}\right)(3)(4) + \left(\frac{3}{10}\right)(4)\left(\frac{5}{2}\right) + \left(\frac{2}{5}\right)(3)\left(\frac{5}{2}\right) = 3+3+3 = 9$$

That is, a box 3 cm long, 4 cm wide and 2.5 cm high should be constructed, at a cost of \$9. Of course, this is the same answer as we got in Example 10.6, when we solved this problem the other way.

Let's look at one more 3-variable problem, in which we have to think carefully about which signs on variables go together to give candidate points, as we saw in some of the 2-variable problems.

Example 10.12. Find all maximum and minimum values of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint that $xyz = 64$.

Solution:
The Lagrange function is:

$$F(x,y,z,\lambda) = x^2 + y^2 + z^2 + \lambda(xyz - 64)$$

which has first partials:

$$\begin{aligned}
F_x(x,y,z,\lambda) &= 2x + yz\lambda\\
F_y(x,y,z,\lambda) &= 2y + xz\lambda\\
F_z(x,y,z,\lambda) &= 2z + xy\lambda\\
F_\lambda(x,y,z,\lambda) &= xyz - 64
\end{aligned}$$

Setting each of these equal to zero we get:

$$F_x(x,y,z,\lambda)=0 \quad\Rightarrow\quad 2x+yz\lambda=0\Rightarrow\lambda=-\frac{2x}{yz} \tag{1}$$

$$F_y(x,y,z,\lambda)=0 \quad\Rightarrow\quad 2y+xz\lambda=0\Rightarrow\lambda=-\frac{2y}{xz} \tag{2}$$

$$F_z(x,y,z,\lambda)=0 \quad\Rightarrow\quad 2z+xy\lambda=0\Rightarrow\lambda=-\frac{2z}{xy} \tag{3}$$

$$F_\lambda(x,y,z,\lambda)=0 \quad\Rightarrow\quad xyz-64=0\Rightarrow xyz=64 \tag{4}$$

From (1) and (2) we get:

$$-\frac{2x}{yz}=-\frac{2y}{xz}\Rightarrow -2x^2z=-2y^2z\Rightarrow x^2=y^2\Rightarrow y=\pm x$$

From (1) and (3) we get:

$$-\frac{2x}{yz}=-\frac{2z}{xy}\Rightarrow -2x^2y=-2yz^2\Rightarrow x^2=z^2\Rightarrow z=\pm x$$

Now, since each of y and z may be either x or $-x$,we have 4 different cases to consider: (i) $y=z=x$, (ii) $y=x$ and $z=-x$, (iii) $y=-x$ and $z=x$, and (iv) $y=z=-x$. But the 4 cases produce only 2 patterns. In cases (i) and (iv), that is if either $y=z=x$ or $y=z=-x$, so that y and z have the same sign, then $xyz=x^3$ and so from (4), the one equation we haven't used yet (the requirement that $F_\lambda=0$), we get $x^3=64$ which gives $x=4$. This gives the 2 candidate points: $(4,4,4)$ and $(4,-4,-4)$. On the other hand, in the second and third cases (i.e. if y and z have different signs so that one is equal to x and the other is equal to $-x$) then $xyz=-x^3$ and so from (4) we get $-x^3=64$, so $x^3=-64$ and we must have $x=-4$. This gives 2 more candidate points: $(-4,-4,4)$ and $(-4,4,-4)$. (*Notice:* We can have either 0 or exactly 2 negative signs among the three variables, since the product must be 64, a positive number.)

We calculate the value of $f(x,y,z)=x^2+y^2+z^2$ corresponding to each candidate point:

$$\begin{aligned} f(4,4,4) &= (4)^2+(4)^2+(4)^2=16+16+16=48\\ f(4,-4,-4) &= (4)^2+(-4)^2+(-4)^2=16+16+16=48\\ f(-4,-4,4) &= (-4)^2+(-4)^2+(4)^2=16+16+16=48\\ f(-4,4,-4) &= (-4)^2+(4)^2+(-4)^2=16+16+16=48 \end{aligned}$$

We see that all of the candidate points give the same function value, so under the given constraint, f has only one extreme value. Is it a maximum or a minimum? We check the function value at any other point on the constraint to see whether it is larger or smaller. For instance, the point $(x,y,z)=(8,8,1)$ also satisfies the constraint $xyz=64$, and for this point we get $f(8,8,1)=8^2+8^2+1^2=129$. We see that f can have values larger than 48 on the constraint, so the extreme value 48 must be a minimum. That is, the minimum value of $f(x,y,z)=x^2+y^2+z^2$ subject to the constraint $xyz=64$ is 48, which is obtained in 4 different places: at $(4,4,4)$, at $(4,-4,-4)$, at $(-4,-4,4)$ and at $(-4,4,-4)$. And f has no maximum value when subjected to this constraint.

(*Notice:* In this example, there *is* no maximum value of $f(x,y,z)=x^2+y^2+z^2$ subject to the constraint that $xyz=64$. We can satisfy the constraint by making one or two of the variables as large as we like and making the other one or two variables very small to compensate. For instance, $y=z=\frac{8}{\sqrt{x}}$ has $xyz=x\left(\frac{8}{\sqrt{x}}\right)\left(\frac{8}{\sqrt{x}}\right)=\frac{64x}{x}=64$ no matter what (non-zero) value of x we choose, and so $x^2+y^2+z^2=x^2+\frac{64}{x}+\frac{64}{x}=x^2+\frac{128}{x}$ can be made arbitrarily large by choosing arbitrarily large values of x. So for example for $x=10{,}000$ we get $y=z=\frac{8}{100}=.08$ with $f(10000,.08,.08)=(10{,}000)^2+\frac{128}{10{,}000}=100{,}000{,}000.0128$. And putting more 0's on the end of x will give an even bigger function value, i.e. will just put more 0's both before and after the decimal place.)

Math 1225A/B

Unit 11:
First-Order Differential Equations

(text reference: Section 7.1 and 7.4

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11 First-Order Differential Equations

The last topic of the course is *Differential Equations.* This is a topic that can get quite complex in a higher level Calculus course, but we will be dealing with only the most basic kind of differential equation, called *First-Order* differential equations.

Definition 11.1. A **differential equation** (often abbreviated DE) is an equation which relates some unknown function, often y, to one or more of its derivatives.

In the expression $\int f(x)\,dx$, the dx can be called a *differential.* So $\frac{dy}{dx}$ can be thought of as a *ratio of differentials.* Likewise, $\frac{d^2y}{dx^2}$ is a ratio of the *second order* differentials d^2y and dx^2. So a differential equation involves a function, and differentials. The *order* of a differential equation refers to the *highest order* derivative, or differentials, involved in the equation.

Definition 11.2. A **first order differential equation** involves only the first derivative of the function. A **higher order differential equation** involves higher derivatives.

So a differential equation relating a function y to $\frac{dy}{dx}$ (and to x, but not to any other derivatives of y) is a first order DE. An equation relating y to its second derivative, and perhaps also to its first derivative, is a second order DE. An equation relating y to its third, and perhaps also first and/or second, derivative is called a third order DE.

For instance, $y\frac{dy}{dx} = x$ and $y - 3\frac{dy}{dx} + 4 = 0$ are examples of first order differential equations. And $y + \frac{dy}{dx} - \frac{d^2y}{dx^2} = 3x^2 - 2$ is and example of a second order differential equation. In this course we will *only* be working with *first order DE's.*

When working with a differential equation, the objective is to use all available information to find the unknown function, i.e. to find an expression which *doesn't* involve differentials which describes the function as accurately as possible. Sometimes, this means finding a family of functions. Other times, a single function can be found. It depends on the information available. So we *solve* a DE by finding all *solutions*, i.e. all expressions for the unknown function, which satisfy the given information about the function, which would include the DE and any other information, i.e. *side conditions*, provided.

Definition 11.3. A **solution** to a differential equation involving the unknown function y is a function of the form $y = f(x)$ which satisfies the DE. We often use $\boldsymbol{y(x)}$ to denote the solution to a DE. An **implicit solution** to a DE is an equation that relates the function (but not any of its derivatives) to the independent variable, but which cannot be expressed in $y = f(x)$ form. To **solve** a DE means to find all solutions (or implicit solutions) to that DE. If no other information is available, this means finding the **general solution** to the DE, i.e. the form which all solutions to the DE must have. If other information is available, then a **particular solution** is found.

For instance, the first order DE $\frac{dy}{dx} = y$ is satisfied by the function $y(x) = e^x$, so this is a solution to the DE. However, this DE is also satisfied by $y = 2e^x$, and $y = -37e^x$, and even by $y(x) = 0$. In fact, the DE is satisfied by any function of the form $y(x) = Ae^x$ for some constant A, so this is the *general solution* to the DE. And *solving* the DE means finding *all* solutions, which in the absence of any additional information means the general solution, so $y(x) = Ae^x$ is what needs to be found if we are asked to solve the DE $\frac{dy}{dx} = y$. Likewise, for any value C, $y^2 = x^2 + C$ is an implicit solution to the first order DE $y\frac{dy}{dx} = x$, and is the general solution required in solving this DE. Similarly, $\sin y = x + C + \cos x$ is the implicit general solution to the DE $\cos y \frac{dy}{dx} = 1 - \sin x$. Each of these general solutions expresses a family of functions, one for each value of the arbitrary constant. (Note that this is similar to the idea of "the general antiderivative" of a function. So we have seen this idea of a "general solution" before.) For the first order DE $\frac{dy}{dx} = 3y$, if it is also known that $y(0) = 2$ we get the *particular solution* $y(x) = 2e^{3x}$. (Again, this is similar to finding a "particular or specific antiderivative" when we have a "side condition" or "initial value" that allows us to find the particular value of the arbitrary constant of integration.)

We are going to learn about two different kinds of first order differential equations – *separable* DE's and *linear* DE's. These two types of DE's require very different approaches to solve them, although some DE's actually fit both definitions and can be solved by either approach. We start by learning about separable first order DE's.

Separable First Order Differential Equations

Definition 11.4. A first order differential equation is called **separable** if may be put into one of the following forms:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \quad \textit{or} \quad \frac{dy}{dx} = \frac{g(y)}{f(x)} \quad \textit{or} \quad \frac{dy}{dx} = f(x)g(y)$$

Notice that these are really all the same form, which we could characterise as $\frac{dy}{dx} = (f(x))(g(y))$. For instance, $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ has the form $\frac{dy}{dx} = (f(x))(h(y))$ where $h(y) = \frac{1}{g(y)}$.

The important feature of a separable DE is that the y's and the x's can be *separated*, so that $\frac{dy}{dx}$ can be expressed as the product (or quotient) of (1) a function involving only x and (2) a function involving only y. This allows us to *separate* the functions, i.e. to rearrange the DE to the form "(some expression involving only y) dy = (some expression involving only x) dx". If we can get to this form, then we can solve the DE by integrating both sides of this equation, integrating with respect to y on the LHS, and integrating with respect to x on the RHS.

Notice that the equation $\frac{dy}{dx} = f(x)$ could be said to have this form, where $g(y) = 1$. To solve the equation $\frac{dy}{dx} = f(x)$, we separate it to $dy = f(x)dx$, which gives $\int dy = \int f(x)dx$ so we have $y = F(x) + C$, where F is any antiderivative of f and C is any constant. We may not have thought of that as what we were doing when we "solved" $\frac{dy}{dx} = f(x)$, but we can describe it that way.

Example 11.1. Solve the equation $\frac{dy}{dx} = x^2$.

Solution:
We have $\frac{dy}{dx} = x^2$, which we can "separate" to $dy = x^2 dx$. We integrate both sides of the separated equation to get:

$$\int dy = \int x^2\,dx \qquad \Rightarrow \qquad y = \frac{x^3}{3} + C$$

We have the general solution to $\frac{dy}{dx} = x^2$ being $y = \frac{x^3}{3} + C$. That is, any function of this form is a solution to the equation we started with.

Any time we need to solve a separable first order DE, we follow these same basic steps. This gives the following procedure.

Procedure For Solving a Separable Differential Equation:

Step 1: Recognize the problem as a separable DE
e.g. recognize something like $\frac{dy}{dx} = \frac{f(x)}{g(y)}$

Step 2: Separate the equation into the form
(terms involving y)(dy) = (terms not involving y)(dx)
e.g. rearrange $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ to $g(y)dy = f(x)dx$

Step 3: Integrate both sides. (Only one arbitrary constant is needed – apply to RHS.)
i.e. evaluate $\int g(y)dy = \int f(x)dx$ as $G(y) = F(x) + C$ where G is any antiderivative of g and F is any antiderivative of f.

Step 4: Solve for y if possible.
i.e. express in the form y = some expression involving only x, if possible.

Note that only one arbitrary constant is needed because if we add an arbitrary constant to each side, for instance adding C_1 on the LHS and adding C_2 on the RHS, we could then rearrange the equation subtracting C_1 from both sides, and replace both constants with the single arbitrary constant $C = C_2 - C_1$.

Also note that if we can express the solution in the form y = some function of x then we have found the general solution to the DE. But if we can't, i.e. if we have "function of y = function of x" and cannot isolate y, then what we have is an *implicit general solution* to the DE.

Example 11.2. Solve $\dfrac{dy}{dx} = \dfrac{4x}{3y^3}$.

Solution:
Step 1:
We have $\frac{dy}{dx}$ equal to something with only x's over something with only y's, i.e. a function of x over a function of y, so this is a separable DE.

Step 2: We separate:

$$\frac{dy}{dx} = \frac{4x}{3y^3} \qquad \Rightarrow \qquad 3y^3\,dy = 4x\,dx$$

Step 3: Integrate both sides:

$$3y^3\,dy = 4x\,dx \;\;\Rightarrow\;\; \int 3y^3\,dy = \int 4x\,dx \;\;\Rightarrow\;\; 3\int y^3\,dy = 2\int 2x\,dx \;\;\Rightarrow\;\; 3\left(\frac{y^4}{4}\right) = 2(x^2) + C$$

Step 4: We solve for y:

$$3\left(\frac{y^4}{4}\right) = 2(x^2) + C \qquad \Rightarrow \qquad y^4 = \left(\frac{4}{3}\right)(2x^2 + C) \qquad \Rightarrow \qquad y^4 = \frac{8x^2}{3} + \frac{4C}{3}$$

Of course, $\frac{4C}{3}$ is still just an arbitrary constant. We can use something different, such as C_1 or D to denote it, or we could just call it C, since that's what we usually use to denote an arbitrary constant. So we have the implicit general solution:

$$y^4 = \frac{8x^2}{3} + C$$

With y being raised to an even power, y could be either the positive or the negative of the fourth root of the RHS, so if we want to express this in "$y =$" form, we must indicate that. Also notice that it is *the whole RHS* which we would take the fourth root of. And the addition of a constant *cannot move out of a root!* So the solution could be expressed as:

$$y = \pm\sqrt[4]{\frac{8x^2}{3} + C}$$

However, since we can't express y as just a *single* function of x, we usually leave it as the implicit solution:

$$y^4 = \frac{8x^2}{3} + C$$

Notice: It is very important to realize that the $+C$ would be *inside* the fourth root above. This is very different than if it were outside, i.e. after, it. In finding the general solution to a DE, whenever we have to do some manipulation to get to the form $y =$ some function of x, we *almost always* end up with the arbitrary constant being *inside* the function of x, rather than just added to it.

Example 11.3. Solve $e^y \dfrac{dy}{dx} = 4x^3 + 2x$.

Solution:
We recognize that this as a separable DE, in which some of the separating has already been done. That is, this equation could have been expressed as $\frac{dy}{dx} = \frac{4x^3+2x}{e^y}$, in which case we would recognize that it is a separable DE, and the first thing we would do is multiply through by e^y to get the form shown above. So we just need to finish separating and then integrate. We get:

$$e^y \frac{dy}{dx} = 4x^3 + 2x \Rightarrow e^y\, dy = (4x^3 + 2x)dx \Rightarrow \int e^y\, dy = \int (4x^3 + 2x)dx \Rightarrow e^y = x^4 + x^2 + C$$

This is an implicit general solution, but it's possible to express this in the form $y = f(x)$, so we're not finished until we've done so. Taking natural logarithms of both sides we have:

$$e^y = x^4 + x^2 + C \quad\Rightarrow\quad \ln e^y = \ln(x^4 + x^2 + C) \quad\Rightarrow\quad y = \ln(x^4 + x^2 + C)$$

Notice that here the $+C$ is *inside the natural logarithm!*, and there is no way to get it out, since there is no other way to express the ln of a sum. Also notice that the value of the arbitrary constant affects the domain of the function. That is, for any particular value of C, the domain contains only those values of x for which $x^4 + x^2 + C > 0$ since we can only take the natural logarithm of a strictly positive number.

Example 11.4. Solve $\dfrac{dy}{dx} = \dfrac{\cos x}{y + e^y}$.

Solution:
We recognize that this is a separable DE, so we separate and integrate:

$$\frac{dy}{dx} = \frac{\cos x}{y + e^y} \quad\Rightarrow\quad (y + e^y)dy = \cos x\, dx \quad\Rightarrow\quad \int (y + e^y)dy = \int \cos x\, dx \quad\Rightarrow\quad \frac{y^2}{2} + e^y = \sin x + C$$

In this case, we cannot isolate y to get to the form $y = f(x)$. What we have is an implicit general solution, and we must leave it that way. However, we would probably multiply through by 2, to eliminate the pesky denominator. And when we multiply the RHS by 2, that means multiplying the arbitrary constant by 2, but 2 times an arbitrary constant is just an arbitrary constant, so as before we can still express the arbitrary constant as C. (But it's a different C than the one above.) We get the implicit general solution:

$$y^2 + 2e^y = 2\sin x + C$$

Differential Equations with Side Conditions

We have previously seen the idea of using side conditions, i.e. other information which has to be satisfied, to find the specific one of a family of functions, in integration. We use that same idea with differential equations, whenever we have information about a particular value of the unknown function. We find the general solution in the usual way, and then use the information given to find the particular value of the arbitrary constant and identify the specific solution. The only difference is that with the general solution to a DE, the constant is often inside the function, whereas previously it was always just a "$+C$" on the end.

Example 11.5. Solve $\dfrac{dy}{dx} = \dfrac{y}{x^2}$, subject to the condition that when $x = \frac{1}{2}$, $y = e$.

Solution:
We recognize that this is a separable DE. For $\dfrac{dy}{dx} = \dfrac{y}{x^2}$, as long as $y \neq 0$, we can separate by dividing through by y. (But we will have to remember that $y = 0$ is another possibility, which does satisfy the DE.) That is, we have:

$$\begin{aligned} \frac{dy}{dx} = \frac{y}{x^2} \quad &\Rightarrow \quad \frac{1}{y}\,dy = \frac{1}{x^2}\,dx && \text{or } y = 0 \\ &\Rightarrow \quad \int \frac{1}{y}\,dy = \int (x^{-2})\,dx && \text{or } y = 0 \\ &\Rightarrow \quad \ln|y| = -x^{-1} + C && \text{or } y = 0 \end{aligned}$$

We need to get the y out of the ln function, which we do by taking $e^{\ln|y|} = e^{-x^{-1}+C}$ or $y = 0$ so we have $|y| = e^{-x^{-1}+C}$ or $y = 0$. Of course, we know that $e^{-x^{-1}+C} = e^{-x^{-1}}e^C$, and e^C is another arbitrary constant, whose value can be any positive number, so we have $y = 0$ or

$$\begin{aligned} |y| = e^C e^{-x^{-1}} \quad &\Rightarrow \quad |y| = Be^{-x^{-1}} && \text{for any } B > 0 \\ &\Rightarrow \quad y = \pm Be^{-x^{-1}} && \text{for any } B > 0 \end{aligned}$$

But now $\pm B$ may have any non-zero value, and since we could also have $y = 0 = 0e^{-x^{-1}}$, we can replace the arbitrary non-zero constant $\pm B$ by the arbitrary constant A, which may have *any* value (and stop remembering the "or $y = 0$" part). So we have

$$y = \pm Be^{-x^{-1}} \text{ or } y = 0 \Rightarrow y = Ae^{-x^{-1}} \Rightarrow y = \frac{A}{e^{1/x}}$$

So the general solution to the DE is $y(x) = \dfrac{A}{e^{1/x}}$ where A is an arbitrary constant. Now we use the side condition. We know that when $x = \frac{1}{2}$ we must have $y = e$, i.e. that $y\left(\frac{1}{2}\right) = e$, so we evaluate the general solution at $x = \frac{1}{2}$ and solve for the value of A that gives the value e. Of course, when $x = \frac{1}{2}$ we have $\frac{1}{x} = 2$ and so we get

$$\begin{aligned} y\left(\tfrac{1}{2}\right) &= e \\ \Rightarrow \qquad \frac{A}{e^2} &= e \\ \Rightarrow \quad A = e^3 & \end{aligned}$$

That is, the only way to satisfy the side condition that $y\left(\frac{1}{2}\right) = e$ is to have the value of the constant A be e^3, which gives

$$y(x) = \frac{e^3}{e^{1/x}} = e^{3-\frac{1}{x}} = e^{3-x^{-1}}$$

so the specific solution to the DE, i.e. the function we are looking for, is

$$y(x) = e^{3-x^{-1}}$$

Notice that earlier we said that the solution to a DE is often expressed as $y(x) = f(x)$ for some function f, but this is the first time we've used that notation. We most often use the $y(x)$ style of expressing the solution in the presence of a side condition.

Example 11.6. Find the solution to $\dfrac{dy}{dx} = e^{x-y}$ for which $y(0) = \ln 2$.

Solution:
We can rewrite the DE as

$$\frac{dy}{dx} = e^{x-y} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{e^x}{e^y}$$

which we recognize as being separable. As before, we first need to find the general solution to this DE:

$$\frac{dy}{dx} = \frac{e^x}{e^y} \qquad \Rightarrow \qquad e^y dy = e^x dx \qquad \Rightarrow \qquad \int e^y dy = \int e^x dx \Rightarrow e^y = e^x + C$$

We need to get the y out of the exponents, so we take natural logarithms of both sides:

$$\ln\left(e^y\right) = \ln\left(e^x + C\right) \qquad \Rightarrow \qquad y = \ln\left(e^x + C\right)$$

That is, the general solution to the DE is $y(x) = \ln\left(e^x + C\right)$. Now, we use the condition that $y(0) = \ln 2$ to find the value of C:

$$y(x) = \ln\left(e^x + C\right) \qquad \Rightarrow \qquad y(0) = \ln\left(e^0 + C\right) = \ln(1 + C)$$

We need $\ln(1+C) = \ln 2$, and so we must have $1 + C = 2$, which gives $C = 1$. Therefore the specific solution to $\frac{dy}{dx} = e^{x-y}$ subject to $y(0) = \ln 2$ is

$$y(x) = \ln(e^x + 1)$$

We said earlier that there are two different kinds of first order DE's we will be studying. We have seen how to solve the first kind, separable DE's. The second kind are called *linear* DE's.

Linear First Order Differential Equations

Definition 11.5. A **linear differential equation** is a DE which can be put into the form

$$\frac{dy}{dx} + y\left[P(x)\right] = Q(x)$$

for some functions $P(x)$ and $Q(x)$.

Of course, p and q are functions which involve only x, so we see that a linear DE has the form "$\frac{dy}{dx}$ plus y times some function of x is equal to some function of x".

The approach we use for solving a linear DE is very different from that used for a separable DE. It involves finding something we call an *integrating factor.*

Definition 11.6. For a linear DE $\frac{dy}{dx} + y[P(x)] = Q(x)$, the **integrating factor**, denoted by $\boldsymbol{u(x)}$, is the function

$$u(x) = e^{\int P(x)\,dx}$$

Note that for the integrating factor $u(x)$ we can use *any* antiderivative of $P(x)$, so we always assume the value of the integration constant is 0. That is, when we find $\int P(x)\,dx$, we don't worry about the $+C$.

Okay, so when we have a linear DE we need this function called Phi, which we refer to as an "integrating factor". But why do we need it? How do we use it?

Suppose that we have some linear DE

$$\frac{dy}{dx} + y[P(x)] = Q(x)$$

First, we multiply through by the integrating factor, $u(x) = e^{\int P(x)\,dx}$, which gives

$$e^{\int P(x)\,dx}\left[\frac{dy}{dx}\right] + y\left[P(x)e^{\int P(x)\,dx}\right] = Q(x)e^{\int P(x)\,dx}$$

Now, we need to use some non-obvious intuition. What is the derivative, with respect to x, of $y\,(u(x))$? We use implicit differentiation and the product rule (with the chain rule) to see that:

$$\begin{aligned}\frac{d}{dx}[y(u(x))] &= \frac{d}{dx}\left[ye^{\int P(x)\,dx}\right]\\ &= \left(\frac{dy}{dx}\right)\left(e^{\int P(x)\,dx}\right) + y\left[e^{\int P(x)\,dx}\left(\frac{d}{dx}\int P(x)\,dx\right)\right]\\ &= e^{\int P(x)\,dx}\left[\frac{dy}{dx}\right] + y\left[e^{\int P(x)\,dx}P(x)\right]\end{aligned}$$

because the derivative of the integral of some function is just the original function, so we have $\frac{d}{dx}\int P(x)dx = P(x)$.

And this last line above is exactly what we have on the left hand side of the equation we got by multiplying through the DE by $u(x)$. So after applying the integrating factor, the linear DE becomes:

$$\frac{d}{dx}\left[ye^{\int P(x)\,dx}\right] = Q(x)e^{\int P(x)dx}$$

Next, we integrate both sides of the equation *with respect to x*:

$$\int \frac{d}{dx}\left[ye^{\int P(x)\,dx}\right]dx = \int\left(Q(x)e^{\int P(x)\,dx}\right)dx$$

Using the fact that the integral of the derivative of a function is just the original function (plus an integration constant, but we only need one integration constant, so as before we will apply that when we integrate the RHS) we get:

$$ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)\,dx}dx$$

At this point, if we were solving a particular linear DE, we would of course carry through the integration on the RHS as well. And then to isolate y, i.e. to find the general solution to the linear DE, we simply need to divide through by $e^{\int P(x)\,dx}$, i.e. by $u(x)$, to get the form $y = f(x)$ for some function f involving only x. That is, we see that:

$$y = e^{-\int P(x)\,dx}\int Q(x)e^{\int P(x)\,dx}dx \qquad \Rightarrow \qquad y = \frac{1}{u(x)}\int Q(x)u(x)dx$$

Notice: Having gone through these steps once, for a non-specific linear DE, to see what the result is, we don't need to remember these steps and carry them out each time. Instead, we just need to remember the formula we obtained, and use that to find the general solution to any linear DE.

Theorem 11.1. *For any linear differential equation*

$$\frac{dy}{dx} + y[P(x)] = Q(x)$$

let $u(x) = e^{\int P(x)\,dx}$, *with integration constant* $C = 0$. *Then the general solution to the DE is given by:*

$$y = \frac{1}{u(x)} \int Q(x)u(x)\,dx$$

Notice: A *constant* multiplier can be carried through (into or out of) an integral sign, but $u(x)$ is **not a constant**. That is, although it's true that for some constant k,

$$\frac{1}{k} \int kQ(x)\,dx = \int \left(\frac{1}{k}\right)(k)\,Q(x)\,dx = \int Q(x)\,dx$$

i.e. the $\frac{1}{k}$ and the k cancel out, what we have in the solution to a linear DE is a *very different situation.* Since $u(x)$ is *not* a constant, the $\frac{1}{u(x)}$ multiplier which is *outside* the integral **does not** cancel out the $u(x)$ multiplier which is *inside* the integral, so

$$\boldsymbol{\frac{1}{u(x)} \int Q(x)u(x)\,dx} \qquad \textbf{is not the same as} \qquad \boldsymbol{\int Q(x)\,dx}$$

Example 11.7. Solve $\frac{dy}{dx} + \frac{4y}{x} = 7x^2$.

Solution:
By writing the given equation as:

$$\frac{dy}{dx} + y\left(\frac{4}{x}\right) = 7x^2$$

we recognize that it is a linear first order DE, with $P(x) = \frac{4}{x}$ and $Q(x) = 7x^2$. According to Theorem 11.1 we just need to find the integrating factor and then apply the formula. We see that

$$\int P(x)\,dx = \int \frac{4}{x}\,dx = 4\int \frac{1}{x}\,dx = 4\ln|x| = \ln|x|^4 = \ln x^4$$

(Remember, we use $C = 0$ for this part.) We were able to eliminate the absolute value signs because we have an even power. So the integrating factor is given by

$$u(x) = e^{\int P(x)dx} = e^{\ln(x^4)} = x^4$$

Substituting this into the formula for the general solution of a linear DE we have:

$$y = \frac{1}{u(x)} \int Q(x)u(x)\,dx = \frac{1}{x^4} \int (7x^2)(x^4)\,dx = \left(\frac{1}{x^4}\right)\left(x^7 + C\right) = \frac{x^7 + C}{x^4} = x^3 + \frac{C}{x^4}$$

We see that the general solution to this linear DE is

$$y = x^3 + \frac{C}{x^4}$$

Notice: The $+C$ is part of what is multiplied by $\dfrac{1}{u(x)}$. And since $\frac{1}{x^4}$ is not a constant, it **cannot** be absorbed into the arbitrary constant. This is **always** the case in solving a linear DE.

We can summarize what we did in the following procedure. We just follow these same steps every time we need to solve a linear DE.

Procedure For Solving a Linear Differential Equation:

Step 1: Recognize that we have a linear DE. Identify $P(x)$ and $Q(x)$.

Step 2: Find the integrating factor $u(x) = e^{\int P(x)\,dx}$ (using $C = 0$).

Step 3: Set $y = \dfrac{1}{u(x)} \displaystyle\int Q(x)u(x)\,dx$.

Step 4: Integrate, then simplify.

Example 11.8. Solve: $\dfrac{dy}{dx} + y = e^x$

Solution:
Step 1: We see that we have a linear DE with $P(x) = 1$ and $Q(x) = e^x$.

Step 2: We find the integrating factor:

$$u(x) = e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^x$$

Step 3: Use the formula:

$$y = \frac{1}{e^x}\int (e^x)(e^x)dx = e^{-x}\int e^{2x}\,dx$$

Step 4: Integrate and simplify:

$$y = e^{-x}\left(\frac{e^{2x}}{2} + C\right) = \frac{e^{2x-x}}{2} + Ce^{-x} = \frac{e^x}{2} + \frac{C}{e^x}$$

We see that the general solution to this DE is $y = \dfrac{e^x}{2} + \dfrac{C}{e^x}$.

Example 11.9. Solve $\dfrac{dy}{dx} - \dfrac{y}{x} = x$.

Solution:
We recognize that this is a linear DE with $P(x) = -\dfrac{1}{x}$ and $Q(x) = x$. The integrating factor is

$$u(x) = e^{\int\left(-\frac{1}{x}\right)dx} = e^{-\int \frac{1}{x}\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$$

This gives:

$$y = \frac{1}{u(x)}\int (Q(x))(u(x))dx = \frac{1}{1/x}\int (x)\left(\frac{1}{x}\right)dx = x\int \frac{x}{x}\,dx = x\int dx = x(x+C) = x^2 + Cx$$

Therefore the general solution to $\dfrac{dy}{dx} - \dfrac{y}{x} = x$ is $y = x^2 + Cx$.

Example 11.10. Solve $\dfrac{dy}{dx} - \dfrac{x}{y} = x$.

Solution:
At first glance, this looks very much like the previous example. But in fact it is *very* different. In Example 11.9, the second term on the LHS of the DE was $-\frac{y}{x} = y\left(-\frac{1}{x}\right)$, which gave the DE the form of a linear DE. This time, the x and the y have switched places in that second term. We have $\frac{x}{y}$ instead of $\frac{y}{x}$, which means that we *don't* have the form of a linear DE. For a linear DE, we must have y times some function of x, and we don't have that here.

In fact, what we have here is a separable DE, as we can see if we rearrange the equation:

$$\frac{dy}{dx} - \frac{x}{y} = x \qquad \Rightarrow \qquad \frac{dy}{dx} = x + \frac{x}{y} \qquad \Rightarrow \qquad \frac{dy}{dx} = x\left(1 + \frac{1}{y}\right)$$

So we just need to separate and integrate:

$$\frac{dy}{dx} = x\left(1 + \frac{1}{y}\right) \quad \Rightarrow \quad \left(\frac{1}{1+\frac{1}{y}}\right) dy = x\,dx \quad \Rightarrow \quad \left(\frac{y}{y+1}\right) dy = x\,dx \quad \Rightarrow \quad \int \left(\frac{y}{y+1}\right) dy = \int x\,dx$$

We need to restate $\dfrac{y}{y+1}$ before it will be easy to integrate. We could perform a substitution, but we can get to the same place by simply reconnizing that $y = (y+1) - 1$. So we have

$$\frac{y}{y+1} = \frac{y+1-1}{y+1} = \frac{y+1}{y+1} - \frac{1}{y+1} = 1 - \frac{1}{y+1}$$

This gives

$$\int \frac{y}{y+1}\,dy = \int x\,dx \qquad \Rightarrow \qquad \int \left(1 - \frac{1}{y+1}\right) dy = \int x\,dx \qquad \Rightarrow \qquad y - \ln|y+1| = \frac{x^2}{2} + C$$

We cannot isolate y in this equation, so the implicit solution to $\frac{dy}{dx} - \frac{x}{y} = x$ is $y - \ln|y+1| = \frac{x^2}{2} + C$.
Note: With a linear DE we never end up with an implicit general solution, because the formula directly gives us $y = f(x)$ form.

Of course, we can have side conditions with linear DE's, just as we did with separable DE's, which allow us to find the specific solution by solving for the value of C.

Example 11.11. Solve $\frac{dy}{dx} = (\sin x)(1-y)$, subject to the condition that $y\left(\frac{\pi}{2}\right) = 4$.

Solution:
We have:

$$\frac{dy}{dx} = (\sin x)(1-y) \qquad \Rightarrow \frac{dy}{dx} = \sin x - y\sin x \qquad \Rightarrow \qquad \frac{dy}{dx} + y(\sin x) = \sin x$$

Now we see that we can solve this as a linear DE, with $P(x) = \sin x$ and also $Q(x) = \sin x$. The integrating factor is:

$$u(x) = e^{\int P(x)\,dx} = e^{\int \sin x\,dx} = e^{-\cos x}$$

Using the formula, we get:

$$y = \frac{1}{u(x)} \int Q(x)u(x)dx = \frac{1}{e^{-\cos x}} \int (\sin x)\, e^{-\cos x} dx = e^{\cos x} \int (\sin x) e^{-\cos x} dx$$

To solve the integral, we need the Substitution Rule. Let $t = -\cos x$ so that $dt = -(-\sin x)dx = \sin x\, dx$. Then we have:

$$\int (\sin x)e^{-\cos x} dx = \int e^t\, dt = e^t + C = e^{-\cos x} + C$$

so we get:

$$y = e^{\cos x}\left(e^{-\cos x} + C\right) = e^{\cos x - \cos x} + Ce^{\cos x} = e^0 + Ce^{\cos x} = 1 + Ce^{\cos x}$$

We have the general solution $y(x) = 1 + Ce^{\cos x}$, and we know that $y\left(\frac{\pi}{2}\right) = 4$. We evaluate the general solution at $x = \frac{\pi}{2}$ to get:

$$y\left(\frac{\pi}{2}\right) = 1 + Ce^{\cos\frac{\pi}{2}} = 1 + Ce^0 = 1 + C$$

and so we need $1 + C = 4$, which gives $C = 3$. Therefore the particular solution we're looking for is $y(x) = 1 + 3e^{\cos x}$.

Notice: $\dfrac{dy}{dx} = (\sin x)(1 - y)$, which we just solved as a linear DE, is also a *separable* DE. Let's see what we get when we approach it that way.

Example 11.11. *Revisited:*
Solve $\dfrac{dy}{dx} = (\sin x)(1 - y)$, subject to the condition that $y\left(\frac{\pi}{2}\right) = 4$.

Solution:
If $y \neq 1$ we can separate and integrate:

$$\frac{dy}{dx} = (\sin x)(1-y) \quad \Rightarrow \quad \frac{1}{1-y}\, dy = \sin x\, dx \quad \Rightarrow \quad \int \frac{1}{1-y}\, dy = \int \sin x\, dx$$

Notice that $\frac{1}{1-y} = -\frac{1}{y-1}$, and we know an antiderivative of $\frac{1}{y-1}$, since it has the form $\frac{1}{y+c}$. So we have

$$\begin{aligned}
\int \frac{1}{1-y}\, dy = \int \sin x\, dx \quad &\Rightarrow \quad -\int \frac{1}{y-1}\, dy = \int \sin x\, dx \\
&\Rightarrow \quad -\ln|y-1| = -\cos x + C \\
&\Rightarrow \quad \ln|y-1| = \cos x + B \qquad \text{(where } B = -C \text{ is still an arbitrary constant)} \\
&\Rightarrow \quad e^{\ln|y-1|} = e^{\cos x + B} \\
&\Rightarrow \quad |y-1| = e^{\cos x}e^B \\
&\Rightarrow \quad y - 1 = \pm e^B e^{\cos x} \\
&\Rightarrow \quad y - 1 = Ae^{\cos x} \qquad \text{(where } A = \pm e^B \text{ is an arbitrary non-zero constant)} \\
&\Rightarrow \quad y = 1 + Ae^{\cos x}
\end{aligned}$$

So if $y \neq 1$ we have $y = 1 + Ae^cosx$ for some non-zero constat A, and to include the possibility of $y = 1$ we simply allow A to have any value (i.e. including 0). Therefore the general solution is $y = 1 + Ae^{\cos x}$ for any value of A.

Now, as before, we use the fact that $y\left(\frac{\pi}{2}\right) = 4$:

$$1 - Ae^{\cos\frac{\pi}{2}} = 4 \quad \Rightarrow \quad -Ae^0 = 3 \quad \Rightarrow \quad A = -3$$

Once again, we get the specific solution $y(x) = 1 + 3e^{\cos x}$.

Notice: We did less work when we approached this as a linear DE than when we approached it as a separable DE, but that won't always be the case. That is, some first order DE's are both separable and linear, and often one approach will be easier than the other. But which is the easier approach depends on the particular functions involved.

Example 11.12. Solve $\frac{dy}{dx} + 2xy = x^3$, subject to $y(1) = 0$.

Solution:
We have a linear DE with $P(x) = 2x$ and $Q(x) = x^3$. We get

$$u(x) = e^{\int P(x)\,dx} = e^{\int 2x\,dx} = e^{(x^2)}$$

So the general solution for this linear DE is:

$$y = \frac{1}{u(x)} \int Q(x)u(x)\,dx = \frac{1}{e^{(x^2)}} \int x^3 e^{(x^2)} dx = e^{-x^2} \int x^3 e^{(x^2)}\,dx$$

To find $\int x^3 e^{(x^2)} dx$ we need to use integration by parts.

Notice that if we had $\int x^3 e^x\,dx$ we would let $u = x^3$ and $dv = e^x dx$. However, with $e^{(x^2)}$ in the integrand, we need $dv = xe^{(x^2)}dx$. That is, $v = \int dv$ doesn't work for $\int e^{(x^2)}dx$, because $\frac{d}{dx}\left(e^{(x^2)}\right) = 2xe^{(x^2)}$, so we need an x **with** the $e^{(x^2)}$.

So we let $u = x^2$ and $dv = xe^{(x^2)}dx$, which gives $du = 2x$ and $v = \frac{1}{2}e^{(x^2)}$. This gives:

$$\int x^3 e^{(x^2)}dx = x^2\left(\frac{e^{(x^2)}}{2}\right) - \int (2x)\left(\frac{e^{(x^2)}}{2}\right) dx = \frac{x^2 e^{(x^2)}}{2} - \int xe^{(x^2)}dx = \frac{x^2 e^{(x^2)}}{2} - \frac{e^{(x^2)}}{2} + C$$

(Notice that for $\int vdu$ we had the same integral as for $v = \int dv$.) So we get the general solution

$$y = \frac{1}{e^{(x^2)}} \int x^3 e^{(x^2)}dx = \frac{1}{e^{(x^2)}}\left(\frac{x^2 e^{(x^2)}}{2} - \frac{e^{(x^2)}}{2} + C\right) = \frac{x^2}{2} - \frac{1}{2} + \frac{C}{e^{(x^2)}}$$

Now, using the side condition $y(1) = 0$, we get:

$$y(1) = 0 \qquad \Rightarrow \qquad \frac{1}{2} - \frac{1}{2} + \frac{C}{e^{(1)^2}} = 0 \qquad \Rightarrow \qquad 0 + \frac{C}{e} = 0 \qquad \Rightarrow \qquad C = 0$$

Therefore the particular function we are looking for is $y = \frac{x^2}{2} - \frac{1}{2}$.

Example 11.13. Solve $\dfrac{dy}{dt} = ky$, subject to the condition that $y(0) = y_0$, for some constants k and y_0.

Solution:
This is another DE which can be considered as either a separable DE or a linear DE. We'll solve this problem using both approaches.

Approach 1: Consider as a separable DE
If $y \neq 0$ then we have:

$$\begin{aligned} \tfrac{dy}{dt} = ky \quad &\Rightarrow \quad \tfrac{1}{y}\,dy = k\,dt \\ &\Rightarrow \quad \int \frac{1}{y}\,dy = \int k\,dt \\ &\Rightarrow \quad \ln|y| = kt + C \\ &\Rightarrow \quad e^{\ln|y|} = e^{kt+C} \\ &\Rightarrow \quad |y| = e^{kt}e^{C} \\ &\Rightarrow \quad y = \pm e^{C}e^{kt} \end{aligned}$$

We can express the general solutions as $y = Ae^{kt}$, where $A = \pm e^{C}$ or $A = 0$, so A is just an arbitrary constant. (Of course $\pm e^{C} \neq 0$, but we allow $A = 0$ also, to encompass the $y = 0$ case that we had to exclude in order to separate the DE.) Now we use the side condition to find the value of A:

$$y(0) = y_0 \quad \Rightarrow \quad Ae^{k(0)} = y_0 \quad \Rightarrow \quad Ae^{0} = y_0 \quad \Rightarrow \quad A = y_0$$

Therefore the particular solution is $y(t) = y_0e^{kt}$.

Approach 2: Consider as a linear DE
We have $\dfrac{dy}{dt} = ky$ which we can rearrange to $\dfrac{dy}{dt} + y(-k) = 0$ which we recognize as a linear DE with $p(t) = -k$ and $q(t) = 0$. The integrating factor is

$$u(t) = e^{\int p(t)\,dt} = e^{-\int k\,dt} = e^{-kt}$$

This gives:

$$\begin{aligned} y = \frac{1}{u(t)}\int q(t)u(t)\,dt \quad &\Rightarrow \quad y = \frac{1}{e^{-kt}}\int (0)\left(e^{-kt}\right)dt \\ &\Rightarrow \quad y = e^{kt}\int 0dt \\ &\Rightarrow \quad y = e^{kt}(0t + C) \\ &\Rightarrow \quad y = Ce^{kt} \end{aligned}$$

Notice that the only difference between this general solution and the one we got in the previous approach is that the arbitrary constant is called C instead of A. So once again $y(0) = y_0$ gives the value of the constant as y_0 (i.e. we solve for C just as we did for A and get $C = y_0$). Therefore, just as with the previous approach, we get the particular solution $y(t) = y_0e^{kt}$.

In the next section, where we look at Applications of DE's, we will see the DE from Example 11.13 again, and come to know it quite well. This differential equation describes any situation in which the rate of change in the quantity y is proportional to y.

Math 1225A/B

Unit 12:

Applications of Differential Equations

(text reference: Sections 7.1, 5.5 and 7.4

custom text pgs. 261-265, 37-41 and 276 - 277)

12 Applications of Differential Equations

We will finish up the course by looking at various applications of Differential Equations. First, we will look at applications of a particular family of DE's. Then we'll look at several other applications of DE's.

Exponential Growth and Decay

At the end of the last unit, in Example 11.13, we solved the DE $\frac{dy}{dt} = ky$ subject to the condition that $y(0) = y_0$, for some constants k and y_o. We observed that this DE is both separable and linear, and found the specific solution $y(t) = y_0 e^{kt}$.

The differential equation $\frac{dy}{dt} = ky$ says that the rate of change in y, with respect to t (usually time), is some constant (k) times the value of y. So the rate at which the quantity y is increasing, or decreasing, is *proportional to* the amount of y present at the time, where k is the constant of proportionality. For instance, if $k = 1$, then we have the rate at which y is changing at some time t is the same as the value of y, so y is growing at a rate that would have the quantity double in one unit of time, if it kept growing at that rate, except that as soon as there's more, the rate of growth increases.

As we saw, the general solution to this DE is $y = Ae^{kt}$ for some constant A. And we also saw that the value of this constant is the value of y at time $t = 0$. Because of the form of this function, this situation is called *exponential growth*. Or, if k is negative, so that the rate of change is negative and the quantity y is decreasing, *exponential decay*.

Since we have already solved this DE, and the solution will have the same form no matter what the specifics of the situation are (in terms of the value of k and the initial quantity, y_0), we don't need to solve the DE every time. We simply use what we have already found.

Theorem 12.1. *If* $\dfrac{dy}{dt} = ky$*, then* $y(t) = y_0 e^{kt}$*, where* $y_0 = y(0)$*.*

This situation of exponential growth or decay arises often in nature. For instance, cell reproduction, which occurs by cells splitting, so that each individual cell becomes 2 cells, which then each split, and so forth, is a common situation involving exponential growth. And the deterioration of radioactive atoms is an instance of exponential decay.

For any situation involving exponential growth or decay, we simply use the theorem above. We might know the proportional constant k, and the initial quantity y_0, and simply need to find $y(t)$, or need to find some other specific value $y(a)$. Or we might not know k and/or y_0, but know other specific values of the function, such as $y(a)$ and perhaps also $y(b)$, for some known numbers a and b, which allow us to solve for y_0 and/or k.

Examples of Biological Growth

Example 12.1. Let $y(t)$ be the number of cells in a certain population at time t, measured in hours. If it is known that $\dfrac{dy}{dx} = ky$, $y(0) = 500$ and $y(5) = 5000$, find the value of k.

Solution:
We know that the general solution to the DE is $y(t) = y_0 e^{kt}$, where $y_0 = y(0)$, and we're told that

$y(0) = 500$, so we have $y(t) = 500e^{kt}$. Also, we're told that $y(5) = 5000$. Since $y(5) = 500e^{5k}$ we see that

$$500e^{5k} = 5000 \quad \Rightarrow \quad e^{5k} = \frac{5000}{500} = 10 \quad \Rightarrow \quad 5k = \ln 10 \quad \Rightarrow \quad k = \frac{\ln 10}{5}$$

This answer could be expressed as $\frac{\ln 10}{5}$ or as $.2 \ln 10$ or as $\ln 10^{.2}$ or as $\ln \sqrt[5]{10}$, etc.

Notice that with this value of k we can write the function $y(t)$ as:

$$y(t) = 500e^{kt} = 500e^{t(\ln 10)/5} = 500e^{(t/5)\ln 10} = 500e^{\ln\left(10^{t/5}\right)} = 500(10^{t/5})$$

Written this way, we can see that this population will continue to increase ten-fold every 5 hours.

Example 12.2. It is known that the cells in a certain bacterial culture divide every 2 hours (on average). By how much will the cell population have grown after 12 hours?

Solution:
Here, we have growth arising from cell division, so the rate of growth in the cell population is proportional to the size of the population. Therefore letting $y(t)$ be the number of cells present at time t we have $y(t) = y_0e^{kt}$ for some constant k, where $y_0 = y(0)$.

Since the cells divide every 2 hours, then after 2 hours each cell will have split to become 2 cells and so the population will double in size every 2 hours. That is, 2 hours after time 0 there will be $y(2) = 2y_0$ cells in the population. But of course $y(2) = y_0e^{2k}$ so we see that

$$y_0e^{2k} = 2y_0 \quad \Rightarrow \quad e^{2k} = 2 \quad \Rightarrow \quad 2k = \ln 2 \quad \Rightarrow \quad k = \frac{\ln 2}{2} = \ln \sqrt{2}$$

Therefore $y(t) = y_0e^{t \ln \sqrt{2}}$, so after 12 hours there number of cells in the population will be

$$y(12) = y_0e^{12 \ln \sqrt{2}} = y_0e^{\ln(\sqrt{2})^{12}} = y_0(\sqrt{2})^{12} = y_0[(2^{1/2})^{12}] = y_0(2^6) = 64y_0$$

That is, we see that after 12 hours, the population will be 64 times as big as it was when it started, i.e. will have grown by a factor of 64.

Notice that although it didn't seem like we'd been given very much information, it was in fact enough to find the size of the population, relative to its initial size, at any time t. But to actually determine what the initial size of the population was, we would need to know the size of the population at some particular time.

Example 12.3. For a certain population it is known that $y(t)$, the number in the population at time t, satisfies the differential equation $\frac{dy}{dt} = ky$. If it is known that at $t = \ln 2$ the number in the population was 20, and at time $t = \ln 4$ the number in the population was 50, find the initial size of the population and size of the population at time $t = \ln 8$.

Solution:
Since $\frac{dy}{dt} = ky$ then we know that $y(t)$ has the form $y(t) = y_0e^{kt}$. We are told that $y(\ln 2) = 20$ so we have

$$y_0e^{k \ln 2} = 20 \quad \Rightarrow \quad e^{\ln 2^k} = \frac{20}{y_0} \quad \Rightarrow \quad 2^k = \frac{20}{y_0}$$

Also, we know that $y(\ln 4) = 50$ and so we have

$$y_0e^{k \ln 4} = 50 \quad \Rightarrow \quad e^{\ln 4^k} = \frac{50}{y_0} \quad \Rightarrow \quad 4^k = \frac{50}{y_0}$$

But $4^k = (2^2)^k = 2^{2k} = (2^k)^2$. Therefore knowing that $2^k = \dfrac{20}{y_0}$ and $4^k = \dfrac{50}{y_0}$ tells us that

$$\frac{50}{y_0} = \left(\frac{20}{y_0}\right)^2 \qquad \Rightarrow \qquad \frac{50}{y_0} = \frac{400}{(y_0)^2} \qquad \Rightarrow \qquad \frac{(y_0)^2}{y_0} = \frac{400}{50} \qquad \Rightarrow \qquad y_0 = 8$$

We see that the initial size of the population was 8. And now $2^k = \frac{20}{y_0}$ gives $2^k = \frac{20}{8} = \frac{5}{2}$, so we see that

$$2^k = \frac{5}{2} \qquad \Rightarrow \qquad \ln 2^k = \ln\left(\frac{5}{2}\right) \qquad \Rightarrow \qquad k \ln 2 = \ln\left(\frac{5}{2}\right) \qquad \Rightarrow \qquad k = \frac{\ln(5/2)}{\ln 2}$$

Therefore the function $y(t)$ is

$$y(t) = 8e^{\left(\frac{\ln(5/2)}{\ln 2}\right)t}$$

So at $t = \ln 8 = \ln 2^3 = 3\ln 2$ the exponent is

$$\left(\frac{\ln(5/2)}{\ln 2}\right)(3\ln 2) = 3\ln\left(\frac{5}{2}\right) = \ln\left(\frac{5}{2}\right)^3 = \ln\left(\frac{125}{8}\right)$$

and so we have

$$y(\ln 8) = 8e^{\ln(125/8)} = 8\left(\frac{125}{8}\right) = 125$$

That is, at time $y = \ln 8$ there will be 125 in the population.

Radioactive Decay

As was mentioned earlier, radioactive decay is a situation involving exponential decay — like exponential growth, but with $k < 0$ so that the quantity is decreasing, rather than increasing. The rate at which a radioactive element decays is often expressed in terms of its *half-life.*

Definition 12.1. The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present to decay, i.e. for the quantity to be reduced by a factor of one half.

Example 12.4. The number of atoms of plutonium-210 remaining after t days is given by

$$y(t) = y_o e^{(-4.95\times 10^{-3})t}$$

where y_0 is the number of atoms present at time 0. Find the half-life of plutonium-210.

Solution:
The half-life is the time t at which there are $\frac{y_0}{2}$ atoms present. So we must solve $y(t) = \frac{y_0}{2}$ for the value of t. We have

$$\begin{aligned} y_0 e^{(-4.95\times 10^{-3})t} = \frac{y_0}{2} \quad &\Rightarrow \quad e^{(-4.95\times 10^{-3})t} = \frac{1}{2} \\ &\Rightarrow \quad (-4.95\times 10^{-3})t = \ln\left(\frac{1}{2}\right) = \ln 1 - \ln 2 = -\ln 2 \\ &\Rightarrow \quad t = \frac{-\ln 2}{-4.95\times 10^{-3}} = \frac{\ln 2}{.00495} \approx 140 \end{aligned}$$

We see that the half-life of plutonium is about 140 days.

Note: Of course, without a calculator, we would leave this as $\frac{\ln 2}{4.95\times 10^{-3}}$ days.

In that example, the half-life turned out to be $\frac{\ln 2}{|k|} = \frac{\ln 2}{-k}$, which is independent of y_0. This will always be true. Consider any situation involving exponential decay. That is, suppose that $\frac{dy}{dt} = ky$ for some $k < 0$, where $y_0 = y(0)$. Then as we know, $y(t) = y_0 e^{kt}$. So for the half-life we have

$$y(t) = \frac{y_0}{2} \quad \Rightarrow \quad y_0 e^{kt} = \frac{y_0}{2} \quad \Rightarrow \quad e^{kt} = \frac{1}{2} \quad \Rightarrow \quad kt = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad t = \frac{\ln 1 - \ln 2}{k} = \frac{-\ln 2}{k} = -\frac{\ln 2}{k}$$

Of course, since k is negative, then the negatives cancel to give a positive value for the half-life.

Theorem 12.2. *If the quantity, y, of a radioactive element satisfies $\dfrac{dy}{dt} = ky$, then the half-life of the element is $t = -\dfrac{\ln 2}{k}$.*

Example 12.5. Uranium-237 has a half-life of about 678 days. If there are 10 grams of Uranium-237 now, how much will be left after 2 weeks?

Solution:
We have $y_0 = 10$, so $y(t) = 10e^{kt}$ for some $k < 0$. Knowing that the half-life is about 678 days tells us that

$$-\frac{\ln 2}{k} \approx 678 \quad \Rightarrow \quad k \approx -\frac{\ln 2}{678}$$

Therefore we have

$$y(t) = 10e^{-t(\ln 2)/678}$$

and so after 2 weeks the quantity of U-237 remaining will be

$$y(14) = 10e^{(-14\ln 2)/678}$$

Using a calculator, we find that in 2 weeks there will be about 2.39 grams of U-237 remaining.

Carbon Dating

One particular use of radioactive decay is for *carbon dating*. This is the method which scientists use to determine the age of a fossil. Or to determine that a body found in a marsh somewhere isn't a current murder case because the guy was murdered 1,000 years ago.

Here's how it works. Carbon-14 is a radioactive element which is present in all living tissue. While tissue is alive, as the Carbon-14 decays it is replaced, at the same rate, by absorption from the atmosphere. In this way, the ratio of C-14 to the stable, i.e. non-radioactive, element C-12 remains constant in the tissue. But when the tissue dies, it stops absorbing C-14 from the atmosphere, so the quantity of C-14 present decreases over time, causing the ratio of C-14 to C-12 to decrease.

The half-life of C-14 is known to be about 5730 years. Using this information, scientists can determine how long a tissue sample has been dead by measuring what proportion of its C-14 is still present, i.e. by comparing the quantity of C-14 present in the tissue to the amount that living tissue of that type would contain.

Example 12.6. Find the age of a fossil which has been found to contain 90% of its original quantity of Carbon-14.

Solution:
We use $y(t) = y_0 e^{kt}$ and the fact that the half-life of C-14 is about 5730 years. We have

$$5730 = -\frac{\ln 2}{k} \qquad \Rightarrow \qquad k = -\frac{\ln 2}{5730}$$

Therefore, with time t measured in years, we have $y(t) = y_0 e^{t(-\ln 2)/5730}$. We need to find the value of t for which $y(t) = .9y_0$. We get

$$\begin{aligned} y_0 e^{-t(\ln 2)/5730} = .9y_0 \quad &\Rightarrow \quad e^{-t(\ln 2)/5730} = .9 \\ &\Rightarrow \quad \frac{-t(\ln 2)}{5730} = \ln .9 \\ &\Rightarrow \quad t = -(\ln .9) \times \frac{5730}{\ln 2} \approx 870.98 \end{aligned}$$

(Notice that since $.9 < 1$ then $\ln .9 < 0$, which cancelled out the negative.) The fossil is about 871 years old.

Notice: You will not be expected to remember that the half-life of C-14 is about 5730 years. And without a calculator, the answer would be expressed as $t = -\frac{5730 \ln .9}{\ln 2}$.

Other Applications

We now leave exponential growth and decay and look at some other applications of differential equations. We'll start by looking at a very familiar kind of problem in Calculus, to review how a mathematical problem can be expressed as a word problem, and how we go from the word problem to the solution.

Example 12.7. The slope of the tangent line to a particular curve, at any point (x, y) on the curve, is given by $\dfrac{ax^2}{y}$ for some constant a. The curve passes through the points $(0, -1)$ and $(1, 2)$. Find all points of intersection of the curve with the line $x = 2$.

Solution:
We know that the slope of the tangent line to a curve is given by $\dfrac{dy}{dx}$. So the first sentence is telling us that

$$\frac{dy}{dx} = \frac{ax^2}{y}$$

Notice that this is a separable DE, which we will need to solve. That is, we will use this to find an equation of the curve.

The points on the curve tell us y-values which correspond to certain specific x-values. That is, we know that when $x = 0$, $y = -1$ and that when $x = 1$, $y = 2$. Notice that these don't have to be the *only* y-values corresponding to these x-values, because a "curve" does not have to be a function. But we do know that $x = 0$ and $y = -1$ must satisfy any equation of the curve. Likewise, we know that $x = 1$ and $y = 2$ must satisfy the equation of the curve.

First, we need to solve the DE. As we already observed, it is a separable DE, so we separate and integrate:

$$\frac{dy}{dx} = \frac{ax^2}{y} \quad \Rightarrow \quad y\,dy = ax^2\,dx \qquad \Rightarrow \quad \int y\,dy = \int ax^2\,dx$$

$$\Rightarrow \quad \frac{y^2}{2} = \frac{ax^3}{3} + C \quad \Rightarrow \quad y^2 = \frac{2ax^3}{3} + C$$

We have found an implicit solution. So now we know that $y^2 = \dfrac{2ax^3}{3} + C$ is an equation of the curve, for some constants a and C. We use the side conditions to find the values of those constants. When $x = 0$ and $y = -1$ we have

$$(-1)^2 = \frac{2a(0)^3}{3} + C \qquad \Rightarrow \qquad 1 = 0 + C$$

so we must have $C = 1$ and now we know that the curve is $y^2 = \dfrac{2ax^3}{3} + 1$. And when $x = 1$ and $y = 2$ we get

$$(2)^2 = \frac{2a(1)^3}{3} + 1 \qquad \Rightarrow \qquad 4 = \frac{2a}{3} + 1 \qquad \Rightarrow \qquad \frac{2a}{3} = 3$$

We could continue, to find a, but it is actually $\dfrac{2a}{3}$ that we need. We see that the equation of the curve is $y^2 = 3x^3 + 1$.

But we weren't asked to find an equation of the curve. We needed to find that in order to find what we *were* asked for, but we're not done yet. What we were asked to do is find all points at which this curve intersects the line $x = 2$. That is, we need to find all points $(2, y)$ which satisfy the equation of the curve. When $x = 2$ we have

$$y^2 = 3(2^3) + 1 = 3(8) + 1 = 25$$

and $y^2 = 25$ is satisfied by both $y = 5$ and $y = -5$. Therefore the curve intersects the line $x = 2$ at the points $(2, -5)$ and $(2, 5)$.

There was nothing new there. Next, we look at a couple of real-world problems in which first order differential equations arise: *Heating and Cooling* problems, and *Mixing* problems. Even in considering only these few specific types of situations — exponential growth and decay, heating/cooling problems, mixing problems — we see a wide variety of applications of differential equations. DE's arise in many different kinds of situations and are a very useful tool for mathematical modelling.

Heating and Cooling Problems

You know that when you put your cup of hot coffee on the counter and walk away for a while, when you come back the coffee will be cooler than it was, because the air in the room is cooler than the coffee. Likewise, if you fill an ice cube tray with water and put it in the freezer, the water will gradually cool and turn to ice, because the air in the freezer is below the freezing temperature of water. Similarly, if you put a roast in a hot oven, the roast gradually heats up and cooks, because of the hot air surrounding it in the oven. These are the kinds of situations described by *Heating and Cooling problems.*

In this kind of problem, we have a substance (e.g. coffee) placed in a medium (e.g. air) of constant temperature. The temperature y of the substance is initially different than the temperature of the medium, which is denoted by M, and so the substance either warms or cools through exposure to the medium. That is, y changes over time, which we will denote as always by t, i.e. y has a rate of change with respect to t. In this situation, *Newton's Law of Cooling* gives us the differential equation

$$\frac{dy}{dt} = k(y - M) \text{ for some constant } k$$

That is, the rate of change of the temperature of the substance, over time, is proportional to the difference between the temperature of the substance and the temperature of the medium in which it is placed. (We're taking Sir Isaac's word for that. Because we know Newton was pretty smart. And I'm sure he wouldn't lie to us.)

For this kind of problem, we generally know:

1. the initial temperature of the substance, y_0
2. the constant temperature of the medium, M, and
3. some other observation of the temperature of the substance, at some known time, i.e. that at time t_1, the temperature was y_1.

These pieces of information, together with the DE from Newton's law, give us the mathematical model:

$$\begin{aligned} \frac{dy}{dt} &= k(y - M) \\ y(0) &= y_0 \\ y(t_1) &= y_1 \end{aligned}$$

We can solve this model to get $y(t)$, a specific formula for the temperature of the object at time t. Once we have this formula, we can answer any other questions of interest.

Example 12.8. A steak is removed from a freezer and put into the refrigerator to thaw. The freezer is kept at $-10°C$ and the fridge is kept at $4°C$. After 4 hours, the temperature of the steak was $-6°C$. When will the steak be thawed to $2°C$?

Solution: In this case, the substance we're interested in is the steak. The medium it is placed in is the air in the refrigerator, so we have $M = 4$. Let $y(t)$ be the temperature of the steak at time t hours after it is put into the refrigerator. Because the steak has been in the freezer, presumably for a long time, its initial temperature is the same as that of the air in the freezer. Therefore $y(0) = -10$. And we also know that $y(4) = -6$. This gives the mathematical model:

$$\begin{aligned} \frac{dy}{dt} &= k(y - 4) \\ y(0) &= -10 \\ y(4) &= -6 \end{aligned}$$

We can consider this as a separable differential equation, (although it is also a linear DE: $\frac{dy}{dt} - ky = -4k$). Assuming that $y \neq 4$ we separate to get:

$$\begin{aligned} &\tfrac{dy}{dt} = k(y-4) \;\Rightarrow\; \tfrac{1}{y-4}\,dy = k\,dt \;\Rightarrow\; \int \tfrac{1}{y-4}\,dt = \int k\,dt \;\Rightarrow\; \ln|y-4| = kt + C \\ \Rightarrow\; &|4 - y| = e^{kt+C} = e^{kt}e^{C} \;\Rightarrow\; y - 4 = \pm e^{C}e^{kt} \;\Rightarrow\; y = 4 - Ae^{kt} \end{aligned}$$

(Here $A = \pm e^C$ is any non-zero constant when we assume $y \neq 4$, and relaxing that assumption allows A to be *any* constant.)

Now we use the fact that $y(0) = -10$ to get:

$$-10 = 4 - Ae^{0k} = 4 - A \qquad \Rightarrow \qquad A = 14$$

so we see that

$$y(t) = 4 - 14e^{kt}$$

Next we use the other observation of temperature to find the value of k. That is, we have $y(4) = -6$, which gives:

$$-6 = 4 - 14e^{4k} \quad \Rightarrow \quad 14e^{4k} = 10 \quad \Rightarrow \quad e^{4k} = \tfrac{10}{14}$$

$$\Rightarrow \quad 4k = \ln\left(\tfrac{10}{14}\right) \quad \Rightarrow \quad k = \tfrac{\ln\left(\frac{5}{7}\right)}{4}$$

This gives the temperature function as $y(t) = 4 - 14e^{t[\ln(5/7)]/4}$.

Now that we have found the formula for $y(t)$, we can answer the question which was asked. We want to know when the steak will reach $2°C$. Setting $y(t) = 2$ and solving for t we get

$$\begin{aligned} 2 = 4 - 14e^{-t[\ln(5/7)]/4} \quad &\Rightarrow \quad 14e^{t[\ln(5/7)]/4} = 4 - 2 = 2 \\ &\Rightarrow \quad e^{t[\ln(5/7)]/4} = \frac{2}{14} = \frac{1}{7} \\ &\Rightarrow \quad \frac{t}{4}\ln\left(\frac{5}{7}\right) = \ln\frac{1}{7} = -\ln 7 \\ &\Rightarrow \quad t = -\frac{4\ln 7}{\ln(5/7)} = \frac{4\ln 7}{-\ln(5/7)} = \frac{4\ln 7}{\ln(7/5)} \approx 23.13 \end{aligned}$$

Since we are measuring t in hours, we see that it will take almost a full day for the steak to thaw. (Without a calculator we would leave the answer as $t = -\frac{4\ln 7}{\ln(5/7)}$ or as $t = \frac{4\ln 7}{\ln(7/5)}$.)

Notice: In any heating or cooling problem, the temperature of the substance never passes the temperature of the medium. (When $T = M$, $\frac{dy}{dt} = 0$, so the value of y stops changing.) We can see in this example that as t gets large, $e^{t[\ln(5/7)]/4} \approx e^{-0.08412t}$ gets close to 0 so that $y(t) = 4 - 14e^{t[\ln(5/7)]/4}$ gets very close to (and never exceeds) 4.

Mixing Problems

The kind of problem known as *Mixing Problems*, or *Mixture Problems*, as our text calls them, involve a "tank" of (usually) liquid (e.g. water) into which some substance is added at a certain *input rate* (measured in kg/hr or lb/min or some such units). The substance is mixed with the liquid in the tank, and the liquid containing the substance leaves the tank, so that the substance is leaving the tank at a certain *output rate* (measured in the same units). We use $y(t)$ to denote the amount of the substance in the tank at time t.

The "substance" of interest in in these problems is often already contained in liquid as it enters the tank. For instance, water containing the substance at a certain concentration enters a tank of water (which may or may not already contain some of the substance), and mixes with the water already in the tank. And simultaneously the water (which contains the substance) is draining from the tank. So the concentration of the substance in the medium (the water) when it enters the tank is generally not the same as the concentration of the substance in the medium when it leaves the tank. It is important that you understand that it is the input rate and output rate of the *substance* which are needed, and that $y = y(t)$ is measuring the amount of the *substance*, not the volume of the whole mixture, which is present in the tank at time t.

Often, what we know is the *inflow* rate of a mixture containing the substance, and the *outflow* rate at which the mixture is leaving the tank. Knowing the inflow rate and the concentration of the substance in the mixture flowing in allows us to determine the *input rate* at which the substance itself is entering the tank. We generally assume that mixing within the tank takes place instantaneously, so that the outflow concentration is simply the quantity of the substance present in the tank, divided by the total volume of the mixture in the tank. This, together with the outflow rate, allows us to determine the *output rate.*

The differential equation involved in this kind of problem arises from the following natural relationship – that the rate at which the amount of the substance in the tank is changing (over time) is simply the difference between the rate at which it is coming into the tank and the rate at which it is leaving the tank. That is, we have:

$$\frac{dy}{dt} = (\text{input rate}) - (\text{output rate})$$

For a problem of this type, we generally know:

1. the volume in the tank initially,
2. the (constant) inflow rate, and the concentration of the substance in the inflow,
3. the (constant) outflow rate, and
4. the quantity of the substance which is present in the tank initially, y_0.

We will only be looking at situations in which the inflow and outflow rates are equal, so that the total volume in the tank is constant (although the concentration of the substance changes over time).

The mathematical model for this kind of situation is therefore:

$$\begin{aligned} \frac{dy}{dt} &= (\text{input rate}) - (\text{output} \quad \text{rate}) \\ y(0) &= y_0 \end{aligned}$$

where the input rate is generally a constant, and the output rate is a function of y, the quantity present. We solve this model to find $y(t)$, the function giving the quantity of the substance present in the tank at time t.

Example 12.9. Initially, a water tower contains 1 million litres of pure water. Two valves are then opened. One valve allows a solution of water and fluoride, with a concentration of 0.1 kg of fluoride per litre of water, to flow into the tower at a rate of 80 litres per minute. The other valve allows the solution in the tank to be drained at 80 litres per minute. Assume that the solution is mixed constantly, so that we have a homogeneous fluid in the tank. That is, at any point in time, the concentration of fluoride in the water is uniform throughout the tank.

(a) Find an expression for the amount (in kg) of fluoride in the water tower t minutes after the valves are opened.

(b) Determine how long it will take for the concentration of fluoride in the water to reach .05 kg/litre.

Solution:
(a) Let $y = y(t)$ be the number of kilograms of fluoride in the tank at time t. We need to find both the input and output rates. Since water (with fluoride) is flowing both in and out at a rate of 80 litres per minute, the volume in the tank remains constant at 1,000,000 litres.

Input Rate: We have 80 litres of water per minute entering the tank, with each litre containing 0.1 kg of fluoride, so each minute we have (80)(0.1) = 8 kg of fluoride entering the tank. That is, we have

$$\textit{Input Rate} = 8 \text{ kg/min}$$

Output Rate: The entire system contains y kg of fluoride at any given time, evenly distributed throughout the one million litres in the tank. This means that each litre of water in the tank contains $\frac{y}{1{,}000{,}000}$ kg of fluoride at any given moment in time. We have 80 litres per minute leaving the system, therefore in any one minute we have $(80)(\frac{y}{1{,}000{,}000})$ kg of fluoride leaving the tank. That is, we have:

$$\textit{Output Rate} = \frac{8y}{100,000} \text{ kg/min}$$

We use the differential equation $\frac{dy}{dt}$ =(*input rate*) − (*output rate*). Substituting the rates above into this formula, we get:

$$\frac{dy}{dt} = 8 - \left(\frac{8y}{100,000}\right)$$

Also, we are told that initially, the water tower contained pure water, i.e. at $t = 0$ there was no fluoride in the tank, so $y(0) = 0$. Therefore the mathematical model is:

$$\begin{aligned} \frac{dy}{dt} &= 8 - \left(\frac{8y}{100,000}\right) \\ y(0) &= 0 \end{aligned}$$

We see that we have a linear DE:

$$\frac{dy}{dt} + 0.00008y = 8$$

with $P(t) = 0.00008$ and $Q(t) = 8$. To solve this, we first find the integrating factor:

$$u(t) = e^{\int P(t)\,dt} = e^{\int 0.00008\,dt} = e^{0.00008t}$$

Now the general solution is given by:

$$\begin{aligned} y &= e^{-0.00008t} \int 8e^{0.00008t} dt \\ &= e^{-0.00008t} \left(\frac{8e^{0.00008t}}{.00008} + C\right) \\ &= e^{-0.00008t} \left(100,000e^{0.00008t} + C\right) \\ &= 100,000 + Ce^{-0.00008t} \end{aligned}$$

Now we use the side condition, that $y(0) = 0$, to find the particular solution, by solving for C. We get:

$$0 = 100,000 + Ce^0 \qquad \Rightarrow \qquad C = -100,000$$

Therefore the function giving the quantity of fluoride in the water tower at time t is

$$y(t) = 100,000 - 100,000e^{-0.00008t} = 100,000(1 - e^{-0.00008t})$$

Notice that we found $C < 0$. In the general solution $y(t) = 100,000 + Ce^{-.000008t}$, the 100,000 constant corresponds to a concentration of .1 kg of fluoride per litre in the 1 million litre tank, which is the same concentration at which fluoride is entering the tank. When this concentration is reached, the input rate equals the output rate, so the amount of fluoride in the tank remains constant after that. That is, the 100,000 kg is the maximum amount of fluoride the water tank will ever contain and so of course the value of C had to be negative, so that until this equilibrium is achieved, the quantity of fluoride in the tank at time t is less than that maximum quantity.

(b) We need to determine how long it will take for the concentration to reach a level of .05 kg/litre. This concentration of fluoride in the 1,000,000 litres in the tank corresponds to having $(1,000,000)(.05) = 50,000$ kg of fluoride in total in the tank. Therefore we need to find the value of t that satisfies $y(t) = 50,000$.

$$\begin{aligned}
& 50,000 &=& \; 100,000(1 - e^{-0.00008t}) \\
\Rightarrow \quad & \tfrac{50,000}{100,000} &=& \; 1 - e^{-0.00008t} \\
\Rightarrow \quad & e^{-0.00008t} &=& \; 1 - \frac{1}{2} = \tfrac{1}{2} \\
\Rightarrow \quad & \ln(e^{-0.00008t}) &=& \; \ln\left(\tfrac{1}{2}\right) \\
\Rightarrow \quad & -0.00008t &=& \; -\ln 2 \\
\Rightarrow \quad & t &=& \; \tfrac{\ln 2}{0.00008} \approx 8664.34
\end{aligned}$$

We see that it will take approximately 8664.34 minutes $\approx$ 144.41 hours, or just over 6 days, for the concentration of fluoride to reach .05 kg/litre.

In more complicated mixing problems, the total volume in the tank may itself be changing over time, if the inflow rate of the substance or mixture is not the same as the outflow rate of the mixture. Solution of such a problem follows the same basic steps, but is slightly more complicated. In that kind of situation, the volume in the tank is a function of t, so the output rate involves both y and t.

Notice that we can have the same sort of situation where the “liquid” in the “tank” is something other than water. For instance, we could have blood (the liquid) in a person’s liver (the tank), with a drug (the substance) being introduced to the blood in the liver – a medical application of a mixing problem. Also, the medium in the tank, into which a substance is being input, doesn’t have to be a liquid. For instance, it could be air in the “tank”, such as when the terrorists that Jack Bauer is after introduce a toxic gas into the air system in the room where all the good-guy world leaders are gathered to negotiate world peace and economic prosperity for all.