Example 1:

if pdf of X is

$$f(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

and $Y = X^2$, find the pdf of Y.

Solution:

$$X \sim \text{Uniform}(-1, 1)$$

$$\Rightarrow F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{x-(-1)}{1-(-1)} = \frac{x+1}{2} & -1 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

First, find cdf of Y.

$$F_Y(y) = P(Y \le y)$$

$$= P(X^2 \le y)$$

$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

Watch out! This step requires you to see that $-1 \le x \le 1$ and can take negative values

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$= \frac{\sqrt{y}+1}{2} - \left(\frac{-\sqrt{y}+1}{2}\right)$$

$$= \frac{\sqrt{y}+1+\sqrt{y}-1}{2} = \frac{2\sqrt{y}}{2}$$

$$= \sqrt{y}$$

$f_Y(y) = \frac{1}{2} y^{-\frac{1}{2}}$ but we need to find support of Y

Since
$$-1 \leq x \leq 1$$
$$0 \leq x^2 \leq 1$$
$$0 \leq y \leq 1$$

$$\Rightarrow \quad f_Y(y) = \begin{cases} \frac{1}{2} y^{-\frac{1}{2}} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

[Careful students will check that this is a legitimate pdf by ensuring the area under this pdf is 1.]

Example 2:

The pdf of X is

$$f_X(x) = \begin{cases} \frac{1}{2} & 1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

and $Y = X^2$. Find the pdf of Y.

Solution:

$$X \sim \text{Uniform}(1, 3)$$

$$\Rightarrow F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{x-1}{2} & 1 \leq x \leq 3 \\ 1 & x > 3 \end{cases}$$

First, find cdf of Y

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) \\ &= \frac{\sqrt{y} - 1}{2} \\ &= \frac{1}{2}\sqrt{y} - \frac{1}{2} \end{aligned}$$

Watch out! This step requires you to notice $1 \leq x \leq 3$ and <u>cannot</u> take negative values. Hence, it is wrong to say $P(-\sqrt{y} \leq X \leq \sqrt{y})$

$f_Y(y) = \frac{1}{2} \cdot \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{4} y^{-\frac{1}{2}}$ and we still need to find support of Y.

Since
$$1 \leq x \leq 3$$
$$1 \leq x^2 \leq 9$$
$$1 \leq y \leq 9$$

$$\Rightarrow \quad f_Y(y) = \begin{cases} \frac{1}{4} y^{-\frac{1}{2}} & 1 \leq y \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

[careful students will check that pdf of Y integrates to 1 to see if they've made a mistake]

Example 3

If pdf of X is

$$f(x) = \begin{cases} \frac{1}{3} & -1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

and $Y = X^2$. Find pdf of Y.

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{3} & -1 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

Solution:

Students should notice $-1 \le x \le 2$ and break that into $-1 \le x \le 1$ and $1 < x \le 2$ so that they can follow examples 1 & 2.

For $-1 \le x \le 1$

$0 \le x^2 \le 1$

$0 \le y \le 1$

Find cdf of Y

$$F_Y(y) = P(Y \le y)$$
$$= P(X^2 \le y)$$
$$= P(-\sqrt{y} \le X \le \sqrt{y})$$

Watch out x can be negative

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$
$$= \frac{\sqrt{y}+1}{3} - \left(\frac{-\sqrt{y}+1}{3}\right)$$
$$= \frac{2\sqrt{y}}{3} = \frac{2}{3} y^{\frac{1}{2}}$$

$$f_Y(y) = \frac{2}{3} \cdot \frac{1}{2} y^{-\frac{1}{2}}$$
$$= \frac{1}{3} y^{-\frac{1}{2}} \quad 0 \le y \le 1$$

For $1 < x \le 2$

$1 < x \le 4$

$1 < y \le 4$

Find cdf of Y

$$F_Y(y) = P(Y \le y)$$
$$= P(X^2 \le y)$$
$$= P(X \le \sqrt{y})$$

Watch out. x cannot be negative

$$= F_X(\sqrt{y})$$
$$= \frac{\sqrt{y}+1}{3}$$
$$= \frac{1}{3}\sqrt{y} + \frac{1}{3}$$

$$f_Y(y) = \frac{1}{3} \cdot \frac{1}{2} y^{-\frac{1}{2}}$$
$$= \frac{1}{6} y^{-\frac{1}{2}} \quad 1 < y \le 4$$

combine them

Hence,

$$f_Y(y) = \begin{cases} \frac{1}{3} y^{-\frac{1}{2}} & 0 \le y \le 1 \\ \frac{1}{6} y^{-\frac{1}{2}} & 1 < y \le 4 \\ 0 & \text{otherwise} \end{cases}$$