Solution To Test 2 (Version A)
MAT1300-3X, Summer 2013
Total = 20 marks

1. (3 marks) Find the slope of the tangent line of the graph of the equation $xy^2 + 3x^2 - 3y = 1$ at the point (1, 2).

Solution. Use the chain rule to find the derivative on both sides with respect to x, taking y as an intermediate variable, which is a function of x. Then we have $y^2 + 2xyy' + 6x - 3y' = 0$. Plug in $x = 1$, $y = 2$, and solve this equation for y': $4 + 4y' + 6 - 3y' = 0$. $y' = -10$. The slope of the tangent line is -10.

2. (4 marks) Suppose the price p and the demanded quantity x of a product are related by the equation $5x^2 + p^3 = 53$. What is the rate of change of x if the current price p is \$2 per unit and the price is increasing at a rate 0.08 / year.

Solution. The quantity and the price are functions of time t. $x = x(t)$ and $p = p(t)$. Taking the derivative of the equation $5x^2 + p^3 = 53$ with respect to t, with x and p as intermediate variables, we have $10xx' + 3p^2p' = 0$. When $p = 2$, $5x^2 + 8 = 53$, $x^2 = 9$, $x = 3$. Since $p' = 0.08$, $30x' + 12 \times 0.08 = 0$. $30x' = -0.96$. $x' = -0.032$. The quantity demanded is deceasing at a rate 0.032 unit each year.

3. (3 marks) (a) Find the critical point(s) of the function $y = \dfrac{x+1}{x^2}$, $x \neq 0$.

(b) In which interval(s) is this function decreasing?

Solution. (a) $y' = \dfrac{x^2 - 2x(x+1)}{x^4} = \dfrac{x - 2(x+1)}{x^3} = -\dfrac{x+2}{x^3}$. The only critical point is $x = -2$.

(b) The critical point $x = -2$ separates the domain of the function $(-\infty, 0)$ and $(0, \infty)$ into three intervals: $(-\infty, -2)$, $(-2, 0)$, and $(0, \infty)$. Since $y' < 0$ in $(-\infty, -2)$ and $(2, \infty)$, this function is decreasing in these intervals.

4. (3 marks) In which interval(s) is the graph of the function $y = x^4 + 4x^3 - 18x^2 + x$ concave up?

Solution. $y' = 4x^3 + 12x^2 - 36x + 1$, $y'' = 12x^2 + 24x - 36 = 12(x^2 + 2x - 3)$. Let $y'' = 0$. Then $x^2 + 2x - 3 = 0$. $x = 1$, $x = -3$. Since $y'' > 0$ when $x < -3$ or $x > 1$, the graph of the function is concave up if $x < -3$ or $x > 1$.

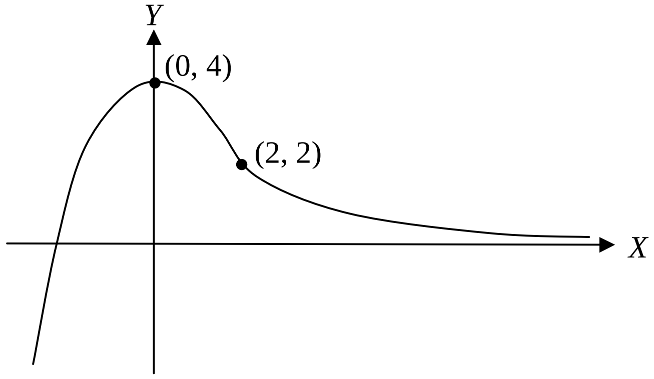
5. (3 marks) Suppose the graph of a function $y = f(x)$ satisfies the following conditions:

(a) $f'(x) > 0$ when $x < 0$, and $f'(x) < 0$ when $x > 0$.
(b) $f''(x) < 0$ when $x < 2$, and $f''(x) > 0$ when $x > 2$.
(c) $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to-\infty} f(x) = -\infty$.

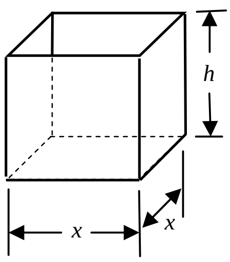
(d) $f(0) = 4$, $f(2) = 2$.

Sketch the graph of the function.

Solution.



6. (4 marks) An open rectangular box has capacity 750 cm^3. The base of the box is a square. The price of the material to make the bottom is 3 cents per cm^2, and the price of the material to make the four vertical sides is 2 cents per cm^2. Find the dimensions of the box to minimize the cost and find the minimum cost. Explain why the dimensions that you found yield the absolute minimum cost.



Solution. Let the side length of the bottom be x and let the height be h, then $hx^2 = 750$ and $h = 750 / x^2$. The cost of the material is $C = 8hx + 3x^2 = 6000 / x + 3x^2$, $0 < x < \infty$. Let $C' = -6000 / x^2 + 6x = 0$. $x^3 = 1000$. $x = 10$, and $h = 750 / 100 = 7.5$ cm. The cost is 900 cents = \$9.

Since $C' < 0$ when $x < 10$ and $C' > 0$ when $x > 10$, C attains a relative minimum at $x = 10$. When x approaches 0 or x approaches infinity, C approaches infinity. This relative minimum is an absolute minimum.