2.5

Problem 1. a) Prove the identity:

$$\tanh(\ln x) = \frac{x^2 - 1}{x^2 + 1}$$

(Hint: recall that $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\tanh x = \frac{\sinh x}{\cosh x}$)

$\boxed{e^{\ln x} = x}$

2

$$\tanh(\ln x) = \frac{\sinh(\ln x)}{\cosh(\ln x)}$$

$$= \frac{\frac{e^{\ln x} - e^{-\ln x}}{2}}{\frac{e^{\ln x} + e^{-\ln x}}{2}}$$

$$= \frac{e^{\ln x} - e^{-\ln x}}{e^{\ln x} + e^{-\ln x}} = \frac{e^{2\ln x} - 1}{e^{2\ln x} + 1}$$

0.5

$$= \frac{e^{\ln x^2} - 1}{e^{\ln x^2} + 1} = \frac{x^2 - 1}{x^2 + 1}$$

b) Compute the derivative of $f(x) = \frac{e^x + \sinh x}{2}$. Simplify as much as possible.

2

1.5

$$f'(x) = \frac{1}{2}(e^x + \cosh x)$$

$$= \frac{1}{2}\left(e^x + \frac{e^x + e^{-x}}{2}\right)$$

0.5

$$= \frac{1}{2}\left(\frac{2e^x + e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{4}\left(3e^x + e^{-x}\right)$$

$\boxed{2}$

Problem 1. a) Prove the identity:

$$\sinh 2x = 2 \sinh x \cosh x$$

(Hint: recall that $\sinh x = \frac{e^x - e^{-x}}{2}$)

1.5

$$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$$
$$= \frac{(e^x - e^{-x})(e^x + e^{-x})}{2}$$
$$= 2 \cdot \frac{e^x - e^{-x}}{2} \cdot \frac{e^x + e^{-x}}{2}$$

0.5

$$= 2 \sinh x \cosh x, \text{ where } \cosh x = \frac{e^x + e^{-x}}{2}$$

b) Compute the derivative of $f(x) = x(\sinh x - e^x)$. Simplify as much as possible.

2

$$f'(x) = \sinh x - e^x + x(\cosh x - e^x)$$
$$= \sinh x + x \cosh x - e^x - x e^x$$
$$= \frac{e^x - e^{-x}}{2} + x \frac{e^x + e^{-x}}{2} - e^x - x e^x$$

0.5

$$= \frac{1}{2}\left(e^x - e^{-x} + x e^x + x e^{-x} - 2e^x - 2x e^x\right)$$
$$= \frac{1}{2}\left(-e^x - e^{-x} - x e^x + x e^{-x}\right)$$
$$= -\frac{(e^x + e^{-x})}{2} - x\left(\frac{e^x - e^{-x}}{2}\right)$$
$$= -\cosh x - x \sinh x$$
$$= -(\cosh x + x \sinh x)$$

#2a) $$\underbrace{\lim_{x\to 0^+} \frac{x+\sin x}{x+\cos x} = \frac{0+0}{0+1}}_{1.75} = \underbrace{\frac{0}{1} = 0 \quad \text{L'Hospital's Rule does not apply}}_{0.5}$$

#2b) let $y = \left(1+\frac{2}{x}\right)^{3x}$

1.75

$$\Rightarrow \ln y = 3x \ln\left(1+\frac{2}{x}\right)$$

$$\Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty} 3x \ln\left(1+\frac{2}{x}\right) = \lim_{x\to\infty} \frac{3\ln(1+2/x)}{1/x}$$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{3\,\frac{1}{(1+2/x)}\left(-2/x^2\right)}{\left(-1/x^2\right)}$$

0.5

$$= 6$$

$$\Rightarrow y = e^6$$

b) $$\lim_{x\to 0^+} \sin x \cdot \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/\sin x}$$

$$\overset{LH}{=} \lim_{x\to 0^+} \frac{1/x}{\frac{-\cos x}{\sin^2 x}} = -\lim_{x\to 0^+} \frac{\sin^2 x}{x\cos x}$$ 1.75

$$= -\lim_{x\to 0^+} \frac{\sin x}{x} \cdot \tan x$$ 0.5

$$= -1 \cdot 0$$

$$= 0$$

$$\int_1^e (\ln x)^3 \, dx$$

$$= x(\ln x)^3 \Big|_1^e - 3\int_1^e (\ln x)^2 \, dx$$

1.5 marks for using identity correctly

$$= x(\ln x)^3 \Big|_1^e - 3\left[x(\ln x)^2 \Big|_1^e - 2\int_1^e \ln x \, dx \right]$$

$$= \left[e(\ln e)^3 - 1(\ln 1)^3 \right] - 3\left[e(\ln e)^2 - \ln(1)^2 - 2\left[x\ln x - x \right]_1^e \right]$$

2 marks for integrating correctly

$$= e - 3\left[e - 2\left(e\cancel{\ln}(e) - \cancel{e} - (\ln(1) - 1) \right) \right]$$

$$= e - 3\left[e - 2(1) \right]$$

0.5 marks for substitution

$$= e - 3e + 6$$

$$= -2e + 6$$

$$\approx 0.563436$$

0.5 marks for solving for the final answer

$$\int_0^{\pi/3} \tan^3 x \, dx$$

$$= \left.\frac{\tan^2 x}{2}\right|_0^{\pi/3} - \int_0^{\pi/3} \tan x \, dx$$

1.5 marks for using identity correctly

$$= \left.\frac{\tan^2 x}{2}\right|_0^{\pi/3} + \ln(\cos(x))\Big|_0^{\pi/3}$$

2 marks for integrating correctly by whichever method you chose

$$= \left(\frac{\tan^2(\pi/3)}{2} - \frac{\tan^2(0)}{2} \right) + \left(\ln(\cos(\pi/3)) - \ln(\cos(0)) \right)$$

0.5 marks for substitution.

$$= \frac{3}{2} + \ln\left(\frac{1}{2}\right)$$

0.5 marks for final answer

$$\approx 0.807$$

Problem 4.a) $\int x2^x dx =$

If you couldn't figure out $\int 2^x dx$, (1). If you got it, but used wrong u, dv, (1.5)

Use I.B.P:

$$\left.\begin{array}{ll} u = x & dv = 2^x dx \\ du = dx & v = \frac{1}{\ln(2)} 2^x \end{array}\right\} \quad \text{Recall } \frac{d}{dx} 2^x = \ln(2) 2^x$$

(1.5) $\to$ Recognizing it's I.B.P. & starting it off

$$\text{So } I = \int x2^x dx = uv - \int v du = \frac{x2^x}{\ln(2)} - \frac{1}{\ln(2)}\int 2^x dx$$

(1) For integrating correctly

$$\int 2^x dx = \frac{2^x}{\ln(2)}, \text{ so } I = \frac{x2^x}{\ln(2)} - \frac{2^x}{\ln^2(2)} + C = \frac{2^x}{\ln(2)}\left(x - \frac{1}{\ln(2)}\right) + C$$

$$= \frac{2^x}{\ln^2(2)}\left(x\ln(2) - 1\right) + C$$

(-1/2) +C, small errors

Problem 4.b) $\int \sin^2(3x)\cos^3(3x)dx =$ $\to$ odd power

$$= \int \sin^2(3x)\cos^2(3x)\cos(3x)dx$$

$$= \int \sin^2(3x)\left[1 - \sin^2(3x)\right]\cos(3x)dx$$

(1) $\to$ setting up, recognizing which function to apply pythag. identity to

Let $u = \sin(3x)$.

Then $du = 3\cos(3x)dx \Rightarrow \frac{du}{3} = \cos(3x)dx$

& we can substitute:

$$= \int u^2(1-u^2)\frac{du}{3}$$

$$= \frac{1}{3}\int (u^2 - u^4)du$$

$$= \frac{1}{3}\left[\int u^2 du - \int u^4 du\right]$$

$$= \frac{1}{3}\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$$

(1) $\to$ u substitution (or integrating directly)

$$= \frac{\sin^3(3x)}{9} - \frac{\sin^5(3x)}{15} + C$$

$$= \frac{5\sin^3(3x) - 3\sin^5(3x)}{45} + C$$

(-1/2) for +C, wrong sign, wrong constant, etc.

1 mark for each I.B.P.
(−1/2) for not bringing −I over to the other side, missing +C, other small mistakes.

Problem 4.a) $\int e^x \cos x dx =$

$u = e^x \quad dv = \cos x\, dx$
$du = e^x dx \quad v = \sin x$ (1)

$I = \int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$

Use I.B.P. again:
$u = e^x \quad dv = \sin x\, dx$ (1)
$du = e^x dx \quad v = -\cos x$

$I = e^x \sin x - \left[e^x \cos x - \int - e^x \cos x dx \right]$

$I = e^x \sin x + e^x \cos x - \underbrace{\int e^x \cos x dx}_{=I}$

$2I = e^x(\sin x + \cos x)$ (.5)

$\Rightarrow$

$\boxed{I = \frac{e^x(\sin x + \cos x)}{2} + C}$

$u = \cos x \quad dv = e^x dx$
$du = -\sin x dx \quad v = e^x$

$I = e^x \cos x - \int - e^x \sin x dx = e^x \cos x + \int e^x \sin x dx$

I.P.B again
$u = \sin x \quad dv = e^x dx$
$u = \cos x dx \quad v = e^x$

$I = e^x \cos x + \left[e^x \sin x - \overbrace{\int e^x \cos x dx}^{=I} \right]$

$I = e^x(\cos x + \sin x) - I$

$2I = e^x(\cos x + \sin x)$

$I = \frac{e^x(\cos x + \sin x)}{2} + C$

Problem 4.b) $\int sin^3(\pi x) \cos^2(\pi x) dx =$

Power on cos is even, on sin is odd, so use sin: $\sin^3(\pi x) = \sin^2(\pi x)\sin(\pi x)$, & $\sin^2(\pi x) = 1 - \cos^2(\pi x)$

$I = \int \sin^3(\pi x)\cos^2(\pi x) dx$

$= \int \sin^2(\pi x)\cos^2(\pi x)\sin(\pi x) dx$

(1) $= \int (1 - \cos^2(\pi x))\cos^2(\pi x)\sin(\pi x) dx$

$= \int \cos^2(\pi x)\sin(\pi x) - \cos^4(\pi x)\sin(\pi x) dx$

Let $u = \cos(\pi x)$ (1)
$du = -\sin(\pi x)\pi dx$
$\Rightarrow \sin(\pi x) dx = -\frac{du}{\pi}$

$= \int \frac{-u^2 du}{\pi} - \int \frac{-u^4 du}{\pi}$

$= \frac{-1}{\pi}\int u^2 du + \frac{1}{\pi}\int u^4 du$

$= \frac{-u^3}{3\pi} + \frac{u^5}{5\pi} + C$

$= -\frac{\cos^3(\pi x)}{3\pi} + \frac{\cos^5(\pi x)}{5\pi} + C$

(−1/2) for no plus c, wrong ⊖.
~~[illegible]~~ missing the $\frac{1}{\pi}$.