(A)

MAT 2384 3X
DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS
MIDTERM
June 19, 2013

Instructor: Dr. Steve Desjardins

Duration: 80 minutes

Name: $\boxed{\text{Solutions}}$

Student Number:__________________

Instructions:
- Print your name and student number on this page.
- Verify that your copy of the exam has all 6 pages.
- There are 5 questions worth 6 marks each for a total of 30 marks.
- You must answer all questions.
- Write your answers in the spaces below the questions. You may use the backs of the pages if necessary.
- **No Notes or Books.**
- **Basic scientific calculators only - graphing and/or programmable calculators are NOT permitted.**

Question 1. Solve the initial value problems:

(a) $y' = \dfrac{2x\cos^2 y}{x^2+1}$, $y(0) = 0$

the DE is separable $\sec^2 y \, dy = \dfrac{2x}{x^2+1}\, dx$

integrate on both sides $\int \sec^2 y \, dy = \int \dfrac{2x}{x^2+1}\, dx + C$

to get $\tan y = \ln(x^2+1) + C$

and so $y = \arctan(\ln(x^2+1) + C)$ (general solution)

$y(0) = 0 \Rightarrow 0 = \arctan(\ln(1) + C) = \arctan C$

$\Rightarrow C = 0$

$\therefore$ the unique solution is $\boxed{y = \arctan(\ln(x^2+1))}$

(b) $y' + \dfrac{y}{x} = 4x^2$, $y(1) = 2$

first-order linear with $f(x) = \frac{1}{x}$, $r(x) = 4x^2$

the integrating factor is $\mu(x) = e^{\int f(x)dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

and so $y(x) = \dfrac{1}{\mu(x)}\left[\int \mu(x) r(x)\, dx + C\right]$

$= \dfrac{1}{x}\left[\int x(4x^2)\, dx + C\right]$

$= \dfrac{1}{x}\left(\int 4x^3\, dx + C\right)$

$= \dfrac{1}{x}(x^4 + C) = x^3 + Cx^{-1}$ (general solution)

$y(1) = 2 \Rightarrow 2 = (1)^3 + C(1)^{-1} \Rightarrow C = 1$

$\therefore$ the unique solution is $\boxed{y = x^3 + x^{-1}}$

Question 2. Solve the initial value problem: (not separable)

$$(3x^2y - 2y^2 \sin x)\,dx + (2x^3 + 6y\cos x - 6)\,dy = 0, \quad y(0) = 2\,.$$

$$M(x,y) = 3x^2y - 2y^2 \sin x \Rightarrow M_y = 3x^2 - 4y\sin x$$
$$N(x,y) = 2x^3 + 6y\cos x - 6 \Rightarrow N_x = 6x^2 - 6y\sin x$$
$\Big\}$ $M_y \neq N_x$, DE not exact

$$\frac{M_y - N_x}{M} = \frac{-3x^2 + 2y\sin x}{3x^2y - 2y^2\sin x} = \frac{-1}{y} \quad \text{(function of } y \text{ only)}$$

then $\mu(y) = e^{-\int \frac{-1}{y}dy} = e^{\ln y} = y$ and the DE becomes

$$(3x^2y^2 - 2y^3\sin x)dx + (2x^3y + 6y^2\cos x - 6y)dy = 0$$

$$M^*(x,y) = 3x^2y^2 - 2y^3\sin x \Rightarrow M^*_y = 6x^2y - 6y^2\sin x$$
$$N^*(x,y) = 2x^3y + 6y^2\cos x - 6y \Rightarrow N^*_x = 6x^2y - 6y^2\sin x$$
$\Big\}$ $M^*_y = N^*_x$, DE exact

$$F(x,y) = \int M^*(x,y)dx + g(y) = \int(3x^2y^2 - 2y^3\sin x)dx + g(y)$$
$$= x^3y^2 + 2y^3\cos x + g(y)$$

then $\frac{\partial F}{\partial y} = 2x^3y + 6y^2\cos x + g'(y) = N^*(x,y) = 2x^3y + 6y^2\cos x - 6y$

$$\Rightarrow g'(y) = -6y \Rightarrow g(y) = -3y^2$$

and so $F(x,y) = x^3y^2 + 2y^3\cos x - 3y^2$

the general solution is $x^3y^2 + 2y^3\cos x - 3y^2 = C$

then $y(0) = 2 \Rightarrow (0)^3(2)^2 + 2(2)^3\cos(0) - 3(2)^2 = C \Rightarrow C = 4$

$\therefore$ the unique solution is $\boxed{x^3y^2 + 2y^3\cos x - 3y^2 = 4}$

Question 3. Solve the initial value problems:

(a) $y'' - 2y' + 10y = 0, \quad y(0) = 2, \quad y'(0) = -1$

the characteristic equation is $\lambda^2 - 2\lambda + 10 = 0$

so $\lambda_{1,2} = \dfrac{2 \pm \sqrt{(-2)^2 - 4(10)}}{2} = \dfrac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i$

the general solution is $y(x) = C_1 e^x \cos(3x) + C_2 e^x \sin(3x)$

$y(0) = 2 \Rightarrow 2 = C_1 e^0 \cos(0) + C_2 e^0 \sin(0) \Rightarrow C_1 = 2$

$y'(x) = C_1 e^x \cos(3x) - 3C_1 e^x \sin(3x) + C_2 e^x \sin(3x) + 3C_2 e^x \cos(3x)$

$y'(0) = -1 \Rightarrow -1 = C_1 e^0 \cos(0) - 3C_1 e^0 \sin(0) + C_2 e^0 \sin(0) + 3C_2 e^0 \cos(0)$

so $C_1 + 3C_2 = -1 \Rightarrow C_2 = -1$

$\therefore$ the unique solution is $\boxed{y(x) = 2e^x \cos(3x) - e^x \sin(3x)}$

(b) $x^2 y'' + 5xy' + 4y = 0, \quad x > 0, \quad y(1) = 2, \quad y'(1) = 0$

the char. eq. is $m(m-1) + 5m + 4 = m^2 + 4m + 4 = (m+2)^2 = 0$

so $m_1 = m_2 = -2$ and the general solution is

$y(x) = C_1 x^{-2} + C_2 x^{-2} \ln x$

$y(1) = 2 \Rightarrow 2 = C_1 (1)^{-2} + C_2 (1)^{-2} \ln(1) \Rightarrow C_1 = 2$

$y'(x) = -2C_1 x^{-3} - 2C_2 x^{-3} \ln x + C_2 x^{-3}$

$y'(1) = 0 \Rightarrow 0 = -2C_1 (1)^{-3} - 2C_2 (1)^{-3} \ln(1) + C_2 (1)^{-3}$

so $-2C_1 + C_2 = 0 \Rightarrow C_2 = 4$

$\therefore$ the unique solution is $\boxed{y(x) = 2x^{-2} + 4x^{-2} \ln x}$

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Question 4. Solve the initial value problem:

$$y''' - 7y'' + 15y' - 9y = 0, \quad y(0) = 4, \quad y'(0) = 13, \quad y''(0) = 46.$$

the char. eq. is $\lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$

by inspection, $\lambda = 1$ is a root, so

$$\lambda^3 - 7\lambda^2 + 15\lambda - 9 = (\lambda - 1)(\lambda^2 - 6\lambda + 9) = (\lambda - 1)(\lambda - 3)^2$$

ie $\lambda_1 = 1$, $\lambda_{2,3} = 3$ and the general solution

is $y(x) = c_1 e^x + c_2 e^{3x} + c_3 x e^{3x}$

$y(0) = 4 \Rightarrow 4 = c_1 e^0 + c_2 e^0 + c_3(0)e^0 \Rightarrow c_1 + c_2 = 4$ (1)

$y'(x) = c_1 e^x + 3c_2 e^{3x} + c_3 e^{3x} + 3c_3 x e^{3x}$

$y'(0) = 13 \Rightarrow 13 = c_1 e^0 + 3c_2 e^0 + c_3 e^0 + 3c_3(0)e^0$

$\Rightarrow c_1 + 3c_2 + c_3 = 13$ (2)

$y''(x) = c_1 e^x + 9c_2 e^{3x} + 6c_3 e^{3x} + 9c_3 x e^{3x}$

$y''(0) = 46 \Rightarrow 46 = c_1 e^0 + 9c_2 e^0 + 6c_3 e^0 + 9c_3(0)e^0$

$\Rightarrow c_1 + 9c_2 + 6c_3 = 46$ (3)

(2) − (1) $2c_2 + c_3 = 9$ (4)

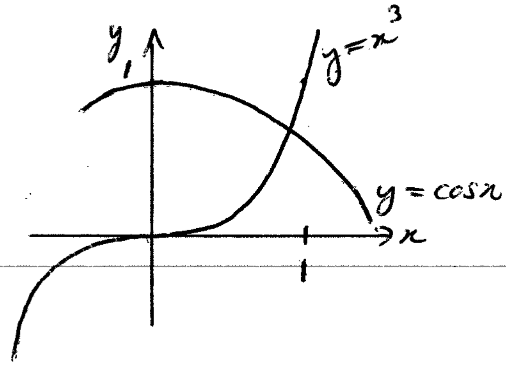
(3) − (1) $8c_2 + 6c_3 = 42$ (5)

(5) − 4×(4) $2c_3 = 6 \Rightarrow c_3 = 3$

then $c_2 = 3$ and $c_1 = 1$

∴ the unique solution is $\boxed{y(x) = e^x + 3e^{3x} + 3xe^{3x}}$

Question 5.
Use Newton's Method to find the point of intersection of the curves $y = x^3$ and $y = \cos x$ to six decimal places. Start with $x_0 = 1$. Verify your answer by checking that the solution is correct.



let $f(x) = x^3 - \cos x$
then $f'(x) = 3x^2 + \sin x$

Newton's Method $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n^3 - \cos x_n}{3x_n^2 + \sin x_n}$$

$$x_1 = 1 - \frac{(1)^3 - \cos(1)}{3(1)^2 + \sin(1)} = 0.880\,333$$

$$x_2 = 0.865\,648$$

$$x_3 = 0.865\,474 = x_4 \quad \therefore \text{ stop}$$

check:
$$\left.\begin{aligned}(0.865474)^3 &= 0.648\,279\\ \cos(0.865474) &= 0.648\,279\end{aligned}\right\} \text{ okay!}$$

So the point of intersection is

$$\boxed{(0.865474,\ 0.648\,279)}$$

(B)

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Question 1. Solve the initial value problems:

(a) $y' = \dfrac{3x^2 \cos^2 y}{x^3+1}$, $y(0)=0$

The DE is separable $\sec^2 y \, dy = \dfrac{3x^2}{x^3+1} dx$

integrate on both sides $\int \sec^2 y \, dy = \int \dfrac{3x^2}{x^3+1} dx + C$

we get $\tan y = \ln|x^3+1| + C$

or $y = \arctan(\ln|x^3+1| + C)$ (general solution)

$y(0)=0 \Rightarrow 0 = \arctan(\ln(1)+C) = \arctan(C) \Rightarrow C=0$

$\therefore$ the unique solution is $\boxed{y = \arctan(\ln|x^3+1|)}$

(b) $y' + \dfrac{y}{x} = 3x$, $y(1)=3$

f-o-l with $f(x) = \frac{1}{x}$, $r(x) = 3x$

$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$y(x) = \dfrac{1}{x}\left[\int x(3x)\,dx + C\right]$

$= \dfrac{1}{x}\left(\int 3x^2 dx + C\right)$

$= \dfrac{1}{x}(x^3 + C) = x^2 + Cx^{-1}$ (general solution)

$y(1)=3 \Rightarrow 3 = (1)^2 + C(1)^{-1} \Rightarrow C=2$

$\therefore$ the unique solution is $\boxed{y(x) = x^2 + 2x^{-1}}$

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Question 2. Solve the initial value problem: (not separable)

$$(3x^2y - 2y^2\sin x)\,dx + (2x^3 + 6y\cos x + 2)\,dy = 0, \quad y(0) = 1\,.$$

$M(x,y) = 3x^2y - 2y^2\sin x \Rightarrow M_y = 3x^2 - 4y\sin x$

$N(x,y) = 2x^3 + 6y\cos x + 2 \Rightarrow N_x = 6x^2 - 6y\sin x$

$\Big\}$ $M_y \neq N_x$ DE not exact

$$\frac{M_y - N_x}{M} = \frac{-3x^2 + 2y\sin x}{3x^2y - 2y^2\sin x} = \frac{-1}{y} \Rightarrow \mu(y) = e^{-\int \frac{-1}{y}dy} = e^{\ln y} = y$$

DE becomes $(3x^2y^2 - 2y^3\sin x)dx + (2x^3y + 6y^2\cos x + 2y)dy = 0$

$M^*(x,y) = 3x^2y^2 - 2y^3\sin x \Rightarrow M^*_y = 6x^2y - 6y^2\sin x$

$N^*(x,y) = 2x^3y + 6y^2\cos x + 2y \Rightarrow N^*_x = 6x^2y - 6y^2\sin x$

$\Big\}$ $M^*_y = N^*_x$ DE exact

$$F(x,y) = \int N^*(x,y)\,dy + h(x) = \int(2x^3y + 6y^2\cos x + 2y)dy + h(x)$$

$$= x^3y^2 + 2y^3\cos x + y^2 + h(x)$$

then $\frac{\partial F}{\partial x} = 3x^2y^2 - 2y^3\sin x + h'(x) = M^*(x,y) = 3x^2y^2 - 2y^3\sin x$

so $h'(x) = 0$ and we take $h(x) = 0$

thus $F(x,y) = x^3y^2 + 2y^3\cos x + y^2$

and the general solution is $x^3y^2 + 2y^3\cos x + y^2 = C$

$y(0) = 1 \Rightarrow (0)^3(1)^2 + 2(1)^3\cos(0) + (1)^2 = C \Rightarrow C = 3$

$\therefore$ the unique solution is $\boxed{x^3y^2 + 2y^3\cos x + y^2 = 3}$

Question 3. Solve the initial value problems:

(a) $y'' - 2y' + 10y = 0, \quad y(0) = 3, \quad y'(0) = 6$

the char. eq. is $\lambda^2 - 2\lambda + 10 = 0 \Rightarrow \lambda_{1,2} = 1 \pm 3i$

general solution is $y(x) = C_1 e^x \cos(3x) + C_2 e^x \sin(3x)$

$y(0) = 3 \Rightarrow 3 = C_1 e^0 \cos(0) + C_2 e^0 \sin(0) \Rightarrow C_1 = 3$

$y'(x) = C_1 e^x \cos(3x) - 3C_1 e^x \sin(3x) + C_2 e^x \sin(3x) + 3C_2 e^x \cos(3x)$

$y'(0) = 6 \Rightarrow 6 = C_1 e^0 \cos(0) - 3C_1 e^0 \sin(0) + C_2 e^0 \sin(0) + 3C_2 e^0 \cos(0)$

so $C_1 + 3C_2 = 6 \Rightarrow C_2 = 1$

$\therefore$ the unique solution is $\boxed{y(x) = 3e^x \cos(3x) + e^x \sin(3x)}$

(b) $x^2y'' + 7xy' + 9y = 0, \quad x > 0, \quad y(1) = 1, \quad y'(1) = -1$

the char eq. is $m(m-1) + 7m + 9 = m^2 + 6m + 9 = (m+3)^2 = 0$

so $m_1 = m_2 = -3$

the general solution is $y(x) = C_1 x^{-3} + C_2 x^{-3} \ln x$

$y(1) = 1 \Rightarrow 1 = C_1 (1)^{-3} + C_2 (1)^{-3} \ln(1) \Rightarrow C_1 = 1$

$y'(x) = -3C_1 x^{-4} - 3C_2 x^{-4} \ln x + C_2 x^{-4}$

$y'(1) = -1 \Rightarrow -1 = -3C_1 (1)^{-4} - 3C_2 (1)^{-4} \ln(1) + C_2 (1)^{-4}$

so $-3C_1 + C_2 = -1 \Rightarrow C_2 = 2$

$\therefore$ the unique solution is $\boxed{y(x) = x^{-3} + 2x^{-3} \ln x}$

Question 4. Solve the initial value problem:

$$y''' - 5y'' + 8y' - 4y = 0, \quad y(0) = 3, \quad y'(0) = 7, \quad y''(0) = 17.$$

the char. eq. is $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

by inspection, $\lambda = 1$ is a root, so

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda-1)(\lambda^2 - 4\lambda + 4) = (\lambda-1)(\lambda-2)^2$$

So $\lambda_1 = 1$, $\lambda_{2,3} = 2$ and the general solution

is $y(x) = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}$

$y(0) = 3 \Rightarrow 3 = c_1 e^0 + c_2 e^0 + c_3(0)e^0 \Rightarrow c_1 + c_2 = 3$ (1)

$y'(x) = c_1 e^x + 2c_2 e^{2x} + c_3 e^{2x} + 2c_3 x e^{2x}$

$y'(0) = 7 \Rightarrow 7 = c_1 e^0 + 2c_2 e^0 + c_3 e^0 + 2c_3(0)e^0$

$\Rightarrow c_1 + 2c_2 + c_3 = 7$ (2)

$y''(x) = c_1 e^x + 4c_2 e^{2x} + 4c_3 e^{2x} + 4c_3 x e^{2x}$

$y''(0) = 17 \Rightarrow 17 = c_1 e^0 + 4c_2 e^0 + 4c_3 e^0 + 4c_3(0)e^0$

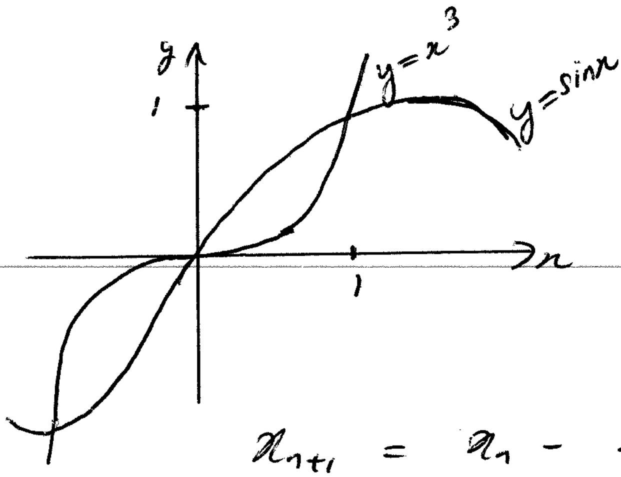
$\Rightarrow c_1 + 4c_2 + 4c_3 = 17$ (3)

(2) − (1) $c_2 + c_3 = 4$ (4) (5) − 3×(4) $c_3 = 2$

(3) − (1) $3c_2 + 4c_3 = 14$ (5) then $c_2 = 2$ and $c_1 = 1$

∴ the unique solution is $\boxed{y(x) = e^x + 2e^{2x} + 2xe^{2x}}$

Question 5.
Use Newton's Method to find the (positive) point of intersection of the curves $y = x^3$ and $y = \sin x$ to six decimal places. Start with $x_0 = 1$. Verify your answer by checking that the solution is correct.



let $f(x) = x^3 - \sin x$
then $f'(x) = 3x^2 - \cos x$

$$x_{n+1} = x_n - \frac{x_n^3 - \sin x_n}{3x_n^2 - \cos x_n}$$

$$x_1 = 1 - \frac{(1)^3 - \sin(1)}{3(1)^2 - \cos(1)} = 0.935549$$

$$x_2 = 0.928702$$

$$x_3 = 0.928626 = x_4 \quad \therefore \text{stop}$$

check: $(0.928626)^3 = 0.800797$
$\sin(0.928626) = 0.800798$ } okay!

$\therefore$ the point of intersection is

$$\boxed{(0.928626,\ 0.800797)}$$