MAT 1348 Assignment 4 Solutions

1. * To prove that R is reflexive:
Take any $f \in \mathcal{F}$. Then $f(x) = 1 \cdot f(x)$ for all $x \in \mathbb{R}$, so $f = cf$ for the constant $c = 1$.
Hence $(f, f) \in R$.

* To prove that R is symmetric:
Take any $f, g \in \mathcal{F}$. Suppose $(f, g) \in R$. Then there exists a constant $c \in \mathbb{R} - \{0\}$ such that $f(x) = c\,g(x)$ for all $x \in \mathbb{R}$. But then, since $c \neq 0$, $g(x) = \frac{1}{c} f(x)$, where $\frac{1}{c} \in \mathbb{R} - \{0\}$, for all $x \in \mathbb{R}$.
Hence $(g, f) \in R$.

* To prove that R is transitive:
Take any $f, g, h \in \mathcal{F}$, and suppose $(f, g), (g, h) \in R$. Then there exist constants $c, d \in \mathbb{R} - \{0\}$ s.t. for all $x \in \mathbb{R}$, $f(x) = c\,g(x)$ and $g(x) = d\,h(x)$.
Therefore, for all $x \in \mathbb{R}$, $f(x) = (cd)\,h(x)$, where $cd \in \mathbb{R} - \{0\}$. Hence $(f, h) \in R$.

Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

1b) Equivalence classes of R on the set $A = \{f_1, \ldots, f_9\}$:

$[f_1]_R = \{f_1, f_8, f_9\}$

$[f_2]_R = \{f_2\}$

$[f_3]_R = \{f_3, f_5\}$

$[f_4]_R = \{f_4\}$

$[f_6]_R = \{f_6, f_7\}$

(2a) The equivalence class of $(1.2, -2.8)$ with respect to R contains:

$(1.3, -2.1)$

$(1, -3)$

(observe that $\lfloor 1.2 \rfloor = 1$ and $\lfloor -2.8 \rfloor = -3$)

(2b) $[(1.2, 2.3)]_R = \{(x,y) : (x,y) R (1.2, 2.3)\}$

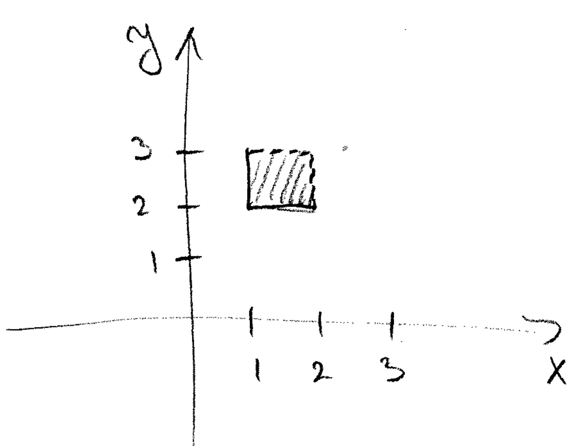
$= \{(x,y) : \lfloor x \rfloor = \lfloor 1.2 \rfloor \text{ and } \lfloor y \rfloor = \lfloor 2.3 \rfloor\}$

$= \{(x,y) : \lfloor x \rfloor = 1 \text{ and } \lfloor y \rfloor = 2\}$

$= \{(x,y) : 1 \leq x < 2 \text{ and } 2 \leq y < 3\}$

$= [1,2) \times [2,3)$

Thus, the equiv. class of $(1.2, 2.3)$ is a unit square in $\mathbb{R}^2$ with vertices $(1,2), (1,3), (2,2), (2,3)$ and two open sides (dashed)



(3a) This means arranging 6 out of 9 people (not including the bride and groom) in a row:

Ans: $P(9,6) = 9\cdot8\cdot7\cdot6\cdot5\cdot4 = \underline{\underline{60{,}480}}$

(3b)

- # all arrangements of 6 out of 11 people in a row: $P(11,6)$
- # of arrangements of 6 out of 11 people in a row where the 6 include neither the bride nor the groom (from (a)): $P(9,6)$
- # of arrangements of 6 out of 11 people in a row where the 6 include at least one of the bride and groom:

$$P(11,6) - P(9,6) = 11\cdot10\cdot9\cdot8\cdot7\cdot6 - 9\cdot8\cdot7\cdot6\cdot5\cdot4$$

$$= 372{,}240 - 60{,}480$$

$$= \underline{\underline{311{,}760}}$$

(3c) Task T_1: choose 4 out of 9 people (not including the bride and groom) .... $\binom{9}{4}$ ways

Task T_2: arrange these 4 people together with the bride and groom in a row ... $6!$ ways

By the product rule: $\binom{9}{4}6! = \frac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2\cdot1}\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$

$$= \underline{\underline{90{,}720}}$$

(4) Let

$A = \{n \in \mathbb{Z} : 1 \le n \le 500,\ 3 | n\}$

$B = \{n \in \mathbb{Z} : 1 \le n \le 500,\ 4 | n\}$

We want to find $|A \cup B|$.

By PIE, $|A \cup B| = |A| + |B| - |A \cap B|$

Now $A \cap B = \{n \in \mathbb{Z} : 1 \le n \le 500,\ 12 | n\}$.

Hence: $|A| = \lfloor \frac{500}{3} \rfloor = 166$

$|B| = \lfloor \frac{500}{4} \rfloor = 125$

$|A \cap B| = \lfloor \frac{500}{12} \rfloor = 41$

and $|A \cup B| = 166 + 125 - 41 = \underline{250}$

(5)
- # 4-element subsets of S: $|\mathcal{A}| = \binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$
- smallest possible sum of elements of a 4-element subset of S: $1+2+3+4 = 10$
- largest possible sum of elements of a 4-element subset of S: $7+8+9+10 = 34$
- # possible sums of elements of sets in $\mathcal{A}$: 25 (i.e. possible sums are $10, 11, \ldots, 34$)

Using the Pigeonhole Principle:

boxes: 25 possible sums

objects: 210 sets (elements of $\mathcal{A}$)

Since $210 = 8 \cdot 25 + 10 > 8 \cdot 25 + 1$, we know by PP that at least 1 box will contain at least 9 objects (note $n = 25$, $k = 8$).
That is, at least 9 4-element subsets will have equal sums of elements.

(6) Take any bit string s of length 13, and let o and z be the number of ones and zeroes in s.
Then $o + z = 13$. If $o \leq 10$, then $z \geq 3$.
If, in addition, $z \leq 5$, then $z \in \{3, 4, 5\}$.
Hence the number of such bit strings is:

$$\binom{13}{3} + \binom{13}{4} + \binom{13}{5} = \frac{13 \cdot 12 \cdot 11}{3 \cdot 2 \cdot 1} + \frac{13 \cdot 12 \cdot 11 \cdot 10}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 286 + 715 + 1287 = \underline{2288}$$

(7a) T_1: to choose the GP winner ... 75 ways
T_2: to choose the SP winner (after the GP winner has been chosen) ... 74 ways
T_3: to choose the winners of 3 more prizes : ... $\binom{73}{3}$ ways

By Product Rule: $75 \cdot 74 \cdot \binom{73}{3} = 75 \cdot 74 \cdot \frac{73 \cdot 72 \cdot 71}{3 \cdot 2 \cdot 1}$

$$= \underline{345{,}187{,}800}$$

(7b) T_1: to award the GP ... 50 ways

T_2: to award the SP (after GP has been awarded) ... 49 ways

T_3: to award the 3^{rd} prize (after GP and SP have been awarded) ... 73 ways

T_4: .... 4^{th} prize .. 72 ways

T_5: .... 5^{th} prize .. 71 ways

By Product Rule: $50 \cdot 49 \cdot 73 \cdot 72 \cdot 71$

$= \underline{\underline{914,281,200}}$