The University of British Columbia

Second Midterm Examination

Mathematics 221

Matrix Algebra

Closed book examination Time: 50 minutes

Last Name Solutions **First** ______ **Signature** ______

Student Number ______

Special Instructions:

No calculators, open books, or notes are allowed.

There are 5 problems in this exam. Each problem is worth 10 marks.

Rules governing examinations

• Each candidate must be prepared to produce, upon request, a UBC-card for identification.

• Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

• No candidate shall be permitted to leave during the last ten minutes of the examination.

• Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.

• Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

• Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		10
3		10
4		10
5		10
Total		~~60~~ 50

PROBLEM 1. Answer the following. You do not need to justify your answers.

a) Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$. What are the possible values for the dimension of the range of T?

0 - 3

b) Suppose A is a 5×3 matrix and B is a 3×5 matrix. Suppose that $\det(BA) = 3$. What is $\det(AB)$?

0

c) Suppose that A is a 5×2 and B is a 2×4 matrix. What are the possible values for the rank and the nullity of A?

rank : 0 - 2

nullity : 2 - 4

d) Suppose that A and B are 3×3 matrices with $\det(A) = 3$ and $\det(B) = -2$. Compute each of the following:

$\det(-2A) =$ -24

$\det(A^{-1}B) =$ $-2/3$

e) Let v_1, v_2, v_3, and v_4 be vectors in $\mathbb{R}^4$. Let $A = \begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \\ v_4^T \end{pmatrix}$ (recalling that v^T denotes the *transpose* of a vector v) and suppose that $\det(A) = -3$. Compute the following determinant:

$$\det \begin{pmatrix} v_1^T + 7v_3^T \\ 2v_3^T \\ -3v_2^T \\ v_4^T + 12v_1^T - 13v_2^T \end{pmatrix}$$

$(-1)(2)(-3)(-3) = -18$

PROBLEM 2. In (a) and (b), determine which of the following sets are linear subspaces. In (c-e), determine which of the transformations are linear. In each case, circle the appropriate answer. Do *not* justify your answer.

(a) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

linear subspace   not a linear subspace

(b) $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ where A is a fixed 3×4 matrix.

linear subspace   not a linear subspace

(c) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ x+1 \\ 0 \end{pmatrix}$$

linear transformation   not a linear transformation

(d) $T : \mathbb{R} \to \mathbb{R}^3$ defined by

$$T(x) = \begin{pmatrix} x \\ -2x \\ 0 \end{pmatrix}$$

linear transformation   not a linear transformation

(e) $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = \mathbf{0}$.

linear transformation   not a linear transformation

PROBLEM 3. Consider the following matrix A and its reduced echelon matrix $REF(A)$:

$$A = \begin{pmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{pmatrix} \qquad REF(A) = \begin{pmatrix} 1 & -2 & 0 & 9 & -16 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Find a basis for the nullspace and a basis for the column space of A. What are the rank and nullity of the matrix A?

solution set for $A\vec{x} = \vec{0}$:

$$\left\{ x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix} : x_2, x_4, x_3 \in \mathbb{R} \right\}$$

basis for nullspace: $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\}$

rank $A = 2$

nullity $A = 3$

basis for column space: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix} \right\}$

PROBLEM 4. There are two linear transformations of $\mathbb{R}^2$ that map the square with vertices at $(0,0),(1,0),(0,1)$, and $(1,1)$ to the rectangle with vertices at $(0,0),(2,2),(-1,1)$, and $(1,3)$. Call them T_1 and T_2. Determine the standard matrices of T_1 and T_2.

$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}-1\\1\end{pmatrix}$ or $\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}-1\\1\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}2\\2\end{pmatrix}$

T_1: $\begin{pmatrix}2 & -1\\2 & 1\end{pmatrix}$ T_2: $\begin{pmatrix}-1 & 2\\1 & 2\end{pmatrix}$

PROBLEM 5. Let $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$, and let B be a 3×3 matrix so that:

$$(AB)^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 3 & -3 & 7/2 \\ 2 & 2 & -3/2 \end{pmatrix}$$

Find the matrix B.

$$\begin{pmatrix} 1 & -2 & 1 \\ -3 & 7 & -6 \\ 2 & 4 & 0 \end{pmatrix}$$

$(AB)^{-1} = B^{-1}A^{-1}$ $\quad B^{-1} = (AB)^{-1}A$

$$\begin{pmatrix} 0 & 1 & -1 \\ 3 & -3 & 7/2 \\ 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ -3 & 7 & -6 \\ 2 & -2 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -3 & 7 & -6 & 0 & 1 & 0 \\ 2 & -2 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 2 & -2 & -2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1/2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 4 & 1 & -1/2 \\ 0 & 1 & -1 & -1 & 0 & 1/2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1/2 \\ 0 & 0 & 1 & -2 & -1/2 & 1/4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 3 & 1/2 & -1/4 \\ 0 & 1 & 0 & -3 & -1/2 & 3/4 \\ 0 & 0 & 1 & -2 & -1/2 & 1/4 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1/2 & 5/4 \\ 0 & 1 & 0 & -3 & -1/2 & 3/4 \\ 0 & 0 & 1 & -2 & -1/2 & 1/4 \end{array}\right)$$

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