University of Ottawa

Department of Mathematics and Statistics

MAT 1302D: Mathematical Methods II
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Second Midterm Exam – Solutions
March 18, 2010

Version A

Surname ______________ First Name ______________

Student # ______________ DGD (1–4) ______________

Instructions:

(a) You have 80 minutes to complete this exam.
(b) The number of points available for each question is indicated in square brackets.
(c) Unless otherwise indicated, you must justify your answers to receive full marks.
(d) All work to be considered for grading should be written in the space provided. The reverse side of pages is for scrap work. If you find that you need extra space in order to answer a particular question, you should continue on the reverse side of the page and indicate this **clearly**. Otherwise, the work written on the reverse side of pages will **not** be considered for marks.
(e) Write your student number at the top of **each page** in the space provided.
(f) No notes, books, scrap paper, calculators or other electronic devices are allowed.
(g) You are strongly advised to write in **pen**, not pencil.
(h) You may use the last page of the exam as scrap paper.

Good luck!

Please do not write in the table below.

Question	1	2	3	4	5	6	7	Total
Maximum	5	5	3	5	3	4	5	30
Grade								

1. [**5 points**] Calculate the determinant of the matrix $A = \begin{bmatrix} 1 & 0 & 4 & -2 \\ 2 & 5 & -3 & 0 \\ 3 & 0 & -1 & 2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

Solution: Since the second column contains two zeros, it is easier to use the cofactor expansion down the second column.

$$\begin{vmatrix} 1 & 0 & 4 & -2 \\ 2 & 5 & -3 & 0 \\ 3 & 0 & -1 & 2 \\ 1 & 1 & 1 & -1 \end{vmatrix} = -0\begin{vmatrix} 2 & -3 & 0 \\ 3 & -1 & 2 \\ 1 & 1 & -1 \end{vmatrix} + 5\begin{vmatrix} 1 & 4 & -2 \\ 3 & -1 & 2 \\ 1 & 1 & -1 \end{vmatrix} - 0\begin{vmatrix} 1 & 4 & -2 \\ 2 & -3 & 0 \\ 1 & 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 4 & -2 \\ 2 & -3 & 0 \\ 3 & -1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 4 & -2 \\ 3 & -1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} - 3\begin{vmatrix} 4 & -2 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 4 & -2 \\ -1 & 2 \end{vmatrix}$$

$$= 1(1-2) - 3(-4+2) + 1(8-2) = -1 + 6 + 6 = 11$$

$$\begin{vmatrix} 1 & 4 & -2 \\ 2 & -3 & 0 \\ 3 & -1 & 2 \end{vmatrix} = 1\begin{vmatrix} -3 & 0 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & -2 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & -2 \\ -3 & 0 \end{vmatrix}$$

$$= 1(-6) - 2(8-2) + 3(-6) = -6 - 12 - 18 = -36$$

$$\begin{vmatrix} 1 & 0 & 4 & -2 \\ 2 & 5 & -3 & 0 \\ 3 & 0 & -1 & 2 \\ 1 & 1 & 1 & -1 \end{vmatrix} = 5(11) - 36 = 55 - 36 = 19$$

2. [**5 points**] Is the matrix $B = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ -1 & 1 & 1 & -2 \\ 1 & -1 & 2 & 2 \end{bmatrix}$ invertible? If so, then find B^{-1}.

Solution:

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & -2 & 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_4 \to R_4 - R_1]{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1}} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to \frac{1}{3}R_3} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + \frac{1}{3}R_4]{R_1 \to R_1 - R_4} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_3} \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 2 & 0 & -\frac{2}{3} & -\frac{5}{3} \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & -\frac{2}{3} & -\frac{5}{3} \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

it follows that B is invertible and $B^{-1} = \begin{bmatrix} 1 & 1 & -\frac{2}{3} & -\frac{5}{3} \\ -1 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ -1 & 0 & 0 & 1 \end{bmatrix}$

3. (a) [**1 point**] Let A and B be 4×4 matrices. You don't know what A and B are but you know that $\det A = -2$ and $\det B = 3$. Calculate $\det(A^T B^{-2} A B^T)$.

Solution: We have $\det A^T = \det A = -2$, $\det B^T = \det B = 3$, and $\det B^{-1} = \dfrac{1}{\det B} = \dfrac{1}{3}$. Therefore :

$$\det(A^T B^{-2} A B^T) = \det A^T \cdot \det B^{-1} \cdot \det B^{-1} \cdot \det A \cdot \det B^T = (-2)(\frac{1}{3})(\frac{1}{3})(-2)(3) = \frac{4}{3}$$

(b) [**2 points**] Let C and D be 5×5 matrices. You don't know what C and D are, but you know that

$$C^T D C^{-1} = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 \\ 2 & -5 & 3 & -1 & 0 \\ 1 & 3 & 0 & -4 & -2 \end{bmatrix}.$$

Calculate $\det D$.

Solution: We calculate the determinant of both sides. For the left hand side we have :

$$\det(C^T \det D \det C^{-1}) = \det C^T \cdot \det D \cdot \det C^{-1} = \det C \cdot \det D \cdot \det C^{-1} = \det D$$

For the right hand side, the matrix is upper triangular and hence its determinant is the product of diagonal entries, which is 16.

Therefore $\det D = 16$.

4. Let $A = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & -1 \\ 2 & -4 & 1 & 0 & 5 & 3 \\ -1 & 2 & 1 & 2 & -2 & 4 \end{bmatrix}$.

(a) [**3 points**] Find a basis for Col A.

Solution: We row reduce the matrix to find its echelon form :

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & -1 \\ 2 & -4 & 1 & 0 & 5 & 3 \\ -1 & 2 & 1 & 2 & -2 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 5 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 5 \\ 0 & 0 & 0 & -1 & 2 & -2 \end{bmatrix}$$

Since the first, third and fourth columns are pivot columns, we find the following basis for Col A :

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

(b) [**1 point**] Determine the rank of A.

Solution: rank $A =$ number of elements of a basis of Col $A = 3$.

(c) [**1 point**] Determine the nullity of A. (Recall that the nullity of A is $\dim \operatorname{Nul} A$.)

Solution: By the Rank Theorem,

$$\dim \operatorname{Nul} A = \text{number of columns of } A - \operatorname{rank} A = 6 - 3 = 3.$$

5. Determine which of the following sets are subspaces of $\mathbb{R}^n$ for the given n. In each case, you should justify your answer.

(a) [**1 point**] $H = \left\{ \begin{bmatrix} x \\ -2x \\ x \end{bmatrix} \;\middle|\; x \geqslant 0 \right\}$, for $n = 3$.

Solution: Not a subspace. For example ,
$\mathbf{a} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is in H but $(-1)\mathbf{a} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ is not in H.

(b) [**1 point**] $H = \left\{ \begin{bmatrix} x-y \\ 2x-y \\ -2y \end{bmatrix} \;\middle|\; x, y \text{ are arbitrary scalars} \right\}$, for $n = 3$.

Solution: It is a subspace. Indeed we have

$$\begin{bmatrix} x-y \\ 2x-y \\ -2y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}$$

and since x and y are arbitrary, H is equal to Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} \right\}$
(Geometrically, H is a plane in $\mathbb{R}^3$ which passes through the origin.)

(c) [**1 point**] $H = \left\{ \begin{bmatrix} x-2y+3 \\ -x+2y+3z \\ -1 \\ -z+y \end{bmatrix} \;\middle|\; x, y, z \text{ are arbitrary scalars} \right\}$, for $n = 4$.

Solution: Not a subspace, because clearly the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is not in H.

6. An economy has two sectors: Manufacturing and Service. In order to produce one unit of output, Manufacturing requires .4 units from its own sector and .5 units from Service. On the other hand, Service requires .2 units from its own sector and .6 units from Manufacturing to produce one unit of output.

(a) [**1 point**] Write down the consumption matrix C for this economy.

Solution:
If Manufacturing is the first sector and Service is the second sector, then $C = \begin{bmatrix} .4 & .6 \\ .5 & .2 \end{bmatrix}$

(b) [**1 point**] Determine the intermediate demands if Service decides to produce 40 units and Manufacturing decides to produce 50 units.

Solution:
The intermediate demand will be

$$40 \begin{bmatrix} .6 \\ .2 \end{bmatrix} + 50 \begin{bmatrix} .4 \\ .5 \end{bmatrix} = \begin{bmatrix} 44 \\ 33 \end{bmatrix}$$

which means 44 units from Manufacturing and 33 units from Service.

(c) [**2 points**] Determine the production levels required to meet a final demand of 27 units from Manufacturing and 90 units from Service.

Solution: We should solve the Leontief equation $(I - C)\mathbf{x} = \begin{bmatrix} 27 \\ 90 \end{bmatrix}$. We have

$$\mathrm{I} - C = \begin{bmatrix} .6 & -.6 \\ -.5 & .8 \end{bmatrix} \quad \Longrightarrow \quad (\mathrm{I} - C)^{-1} = \frac{1}{.18} \begin{bmatrix} .8 & .6 \\ .5 & .6 \end{bmatrix} = \begin{bmatrix} \frac{80}{18} & \frac{60}{18} \\ \frac{50}{18} & \frac{60}{18} \end{bmatrix}$$

$$\Longrightarrow \mathbf{x} = (\mathrm{I} - C)^{-1} \begin{bmatrix} 27 \\ 90 \end{bmatrix} = \begin{bmatrix} \frac{80}{18} & \frac{60}{18} \\ \frac{50}{18} & \frac{60}{18} \end{bmatrix} \begin{bmatrix} 27 \\ 90 \end{bmatrix} = \begin{bmatrix} 120 + 300 \\ 75 + 300 \end{bmatrix} = \begin{bmatrix} 420 \\ 375 \end{bmatrix}$$

7. Let $A = \begin{bmatrix} 1 & -2 & 2 & 0 & -5 \\ 1 & -2 & 2 & 2 & -7 \\ -2 & 4 & -4 & 0 & 10 \\ 1 & -2 & 2 & 1 & -6 \end{bmatrix}$

(a) [**4 points**] Find a basis for Nul A.

Solution:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & 0 & -5 & 0 \\ 1 & -2 & 2 & 2 & -7 & 0 \\ -2 & 4 & -4 & 0 & 10 & 0 \\ 1 & -2 & 2 & 1 & -6 & 0 \end{array}\right] \xrightarrow[R_4 \to R_4 - R_1]{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1}} \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 0 & -5 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_4 \to R_4 - R_2} \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_4 : basic
x_2, x_3, x_5 : free

$$\implies \begin{cases} x_1 = 2x_2 - 2x_3 + 5x_5 \\ x_4 = x_5 \end{cases}$$

$$\implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - 2x_3 + 5x_5 \\ x_2 \\ x_3 \\ x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

And we find the following basis for Nul A :

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(b) [**1 point**] Determine the rank of A.

Solution: By the Rank Theorem,

$$\text{rank}\, A = \text{number of columns of } A - \dim \text{Nul}\, A = 5 - 3 = 2$$