1. If $f(x,y)$ is a differentiable function such that $\vec{\nabla} f(1,2) = \vec{\imath}$, only one of the following curves can be the level curve for f through the point $(1,2)$. Which one?

A. $y = 1 + x$

B. $y = \dfrac{2}{x}$

C. $y = 1 + e^{x-1}$

D. $x = 1$

E. $y = 2e^{x-1}$

F. $y = 2 + (x-1)^2$

The level curve has to be $\perp$ to the vector $\vec{\imath}$. The only curve which satisfies this condition (of those listed) is the vertical line $x=1$.

2. If $f(x,y) = x^2 - y^2$, then which of the following numbers corresponds to the global maximum value of f subject to the constraint $x^2 + 4y^2 = 1$?

A. $\dfrac{5}{4}$

B. $\dfrac{-1}{2}$

C. That global maximum value does not exist

D. $\dfrac{1}{4}$

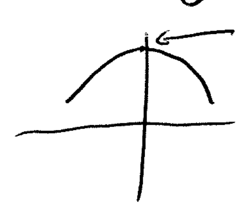
E. 1

F. 0

$\Rightarrow x^2 = 1 - 4y^2$

$\Rightarrow f = 1 - 4y^2 - y^2$

$= 1 - 5y^2$



max value is 1.

3. If $z = f(x, y)$ and $x = x(u, v)$, $y = y(u, v)$, which of the following formulas corresponds to the chain rule for the partial derivative $\dfrac{\partial z}{\partial u}$?

A. $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$

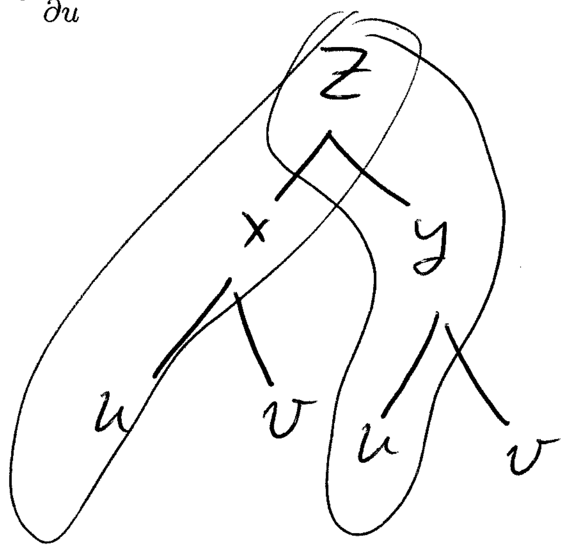
B. $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$

C. $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial v} + \dfrac{\partial z}{\partial v}$

D. $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$

E. $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u}$

F. $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v}\dfrac{\partial v}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$



4. Find and classify the critical points of the function $f(x, y) = xy(12 - 4x - 3y)$.

$\frac{1}{2}$ $f_x = 12y - 8xy - 3y^2 = y(12 - 8x - 3y)$ $\quad f_y = 12x - 4x^2 - 6xy = x(12 - 4x - 6y)$ $\frac{1}{2}$ $\quad = 12xy - 4x^2y - 3xy^2$

$f_x = 0 \Rightarrow y = 0$ or $12 - 8x - 3y = 0$

i) if $y = 0$, 2ND eqn becomes $x(12 - 4x - 6 \cdot 0) = 0$

$\Rightarrow x(12 - 4x) = 0 \Rightarrow x = 0$ or $x = 3$

$\Rightarrow$ two critical points: $(0,0)$ $\frac{1}{2}$ and $(3,0)$ $\frac{1}{2}$

ii) if $12 - 8x - 3y = 0 \Leftrightarrow \underline{y = 4 - \frac{8}{3}x}$, 2ND eqn becomes

$x(12 - 4x - 6(4 - \frac{8}{3}x)) = 0 \Rightarrow x(12 - 4x - 24 + 16x) = 0 \Rightarrow x(12x - 12) = 0$

$\frac{1}{2}$ $\quad \frac{1}{2}$ $\Rightarrow \underline{x = 0}$ or $\underline{x = 1}$

$\Rightarrow$ two more critical points: $(0,4)$ and $(1, \frac{4}{3})$ $\quad y = 4 - \frac{8}{3} \cdot 0 = 4$ $\quad y = 4 - \frac{8}{3} \cdot 1 = \frac{4}{3}$.

Four critical points in total: $\boxed{(0,0),\ (3,0),\ (0,4) \text{ and } (1, 4/3)}$

$\underline{\text{Classification}}$: $f_{xx} = -8y$, $f_{xy} = f_{yx} = 12 - 8x - 6y$, $f_{yy} = -6x$

$f_{xx} f_{yy} - (f_{xy})^2 = 48xy - (12 - 8x - 6y)^2 = D$ $\quad \frac{1}{2}$

$\underline{(0,0)}$: $D = -144 < 0 \Rightarrow (0,0)$ is a $\underline{\text{saddle}}$

$\underline{(3,0)}$: $D = -144 < 0 \Rightarrow (3,0)$ is a $\underline{\text{saddle}}$ $\quad 1$

$\underline{(0,4)}$: $D = -144 < 0 \Rightarrow (0,4)$ is a $\underline{\text{saddle}}$

$(1, \frac{4}{3})$: $D = 64 - 16 = 48 > 0$, $f_{xx} = -\frac{32}{3} < 0 \Rightarrow$ $\quad \frac{1}{2}$

$(1, \frac{4}{3})$ is a $\underline{\text{local max}}$.

4. Find and classify the critical points of the function $f(x,y) = xy(12-3x-4y)$.

$= 12xy - 3x^2y - 4xy^2$

$f_x = 12y - 6xy - 4y^2$ $\quad$ $f_y = 12x - 3x^2 - 8xy$ $\quad \frac{1}{2}$

$\frac{1}{2}$ $= y(12-6x-4y)$ $\quad$ $= x(12-3x-8y)$

$f_x = 0 \Rightarrow y = 0$ or $12-6x-4y=0$

i) if $y=0$, 2ND eqn becomes $x(12-3x-8\cdot 0)=0 \Rightarrow x(12-3x)=0 \Rightarrow$
$x=0$ or $x=4$

$\frac{1}{2}$ $\quad \frac{1}{2}$

$\Rightarrow$ two critical points: $(0,0)$ and $(4,0)$

ii) if $12-6x-4y=0 \Leftrightarrow y = 3-\frac{3}{2}x$, 2ND eqn becomes

$x(12-3x-8(3-\frac{3}{2}x))=0 \Rightarrow x(12-3x-24+12x)=0 \Rightarrow x(9x-12)=0$

$\Rightarrow \underbrace{x=0}_{y=3-\frac{3}{2}\cdot 0 = 3}$ or $\underbrace{x=4/3}_{y=3-\frac{3}{2}\cdot\frac{4}{3} = 1}$

$\frac{1}{2}$ $\quad \frac{1}{2}$

$\Rightarrow$ two more critical points: $(0,3)$ and $(\frac{4}{3},1)$

Four critical points in total: $(0,0)$, $(4,0)$, $(0,3)$ and $(\frac{4}{3},1)$.

Classification: $f_{xx} = -6y$ $\quad f_{xy} = f_{yx} = 12-6x-8y$, $\quad f_{yy} = -8x$ $\quad \frac{1}{2}$

$D = f_{xx}f_{yy} - (f_{xy})^2 = 48xy - (12-6x-8y)^2$ $\quad 1$

$(0,0)$: $D = -144 < 0 \Rightarrow (0,0)$ is a saddle

$(4,0)$: $D = -144 < 0 \Rightarrow (4,0)$ is a saddle $\quad 1$

$(0,3)$: $D = -144 < 0 \Rightarrow (0,3)$ is a saddle

$(\frac{4}{3},1)$: $D = 64-16 = 48 > 0$, $f_{xx} = -6 < 0 \Rightarrow$ $\quad \frac{1}{2}$

$(\frac{4}{3},1)$ is a local max.

Lagrange multipliers approach.

5. Find the global maximum and the global minimum of the function $f(x,y) = xy$ on the region

$$A = \{ (x,y) \in \mathbb{R}^2 \,|\, \underbrace{x^2 + y^2}_{g(x,y)} \leq 9 \}.$$

constraint function.

(1)

$\boxed{f_x = y, \; f_y = x}$ $\quad g_x = 2x \quad g_y = 2y$

Critical point of f: $f_x = 0$, $f_y = 0 \Rightarrow \boxed{(0,0) \in A}$ (1)

Boundary of A: (1) $\boxed{\left.\begin{array}{l}\vec{\nabla} f = \lambda \vec{\nabla} g \\ g = 9\end{array}\right\}}$ $\boxed{\begin{array}{l} y = 2\lambda x \\ x = 2\lambda y \\ x^2+y^2 = 9\end{array}}$ (1)

$y = 2\lambda x \xRightarrow{2^{ND}\text{ eq}} x = 2\lambda(2\lambda x) \Rightarrow x = 4\lambda^2 x \Rightarrow x(1-4\lambda^2) = 0$

$\Rightarrow x = 0, \; \lambda = \frac{1}{2}$ or $\lambda = -\frac{1}{2}$

if $x=0$ then $y=0$, but then 3^{RD} eqn is not satisfied. $0^2+0^2 \neq 9$.

if $\lambda = \frac{1}{2}$, then $x = y \Rightarrow x^2 + x^2 = 9 \Rightarrow 2x^2 = 9 \Rightarrow x^2 = \frac{9}{2}$

$\Rightarrow x = \frac{\pm 3}{\sqrt{2}}$

$\frac{1}{2}$ $\Rightarrow \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(\frac{-3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}, \frac{1}{2}\right)$ $\frac{1}{2}$

if $\lambda = -\frac{1}{2}$, then $x = -y \Rightarrow x^2 + (-x)^2 = 9 \Rightarrow 2x^2 = 9 \Rightarrow x^2 = \frac{9}{2}$

$\Rightarrow x = \frac{\pm 3}{\sqrt{2}}$

$\frac{1}{2}$ $\frac{1}{2}$

$\Rightarrow \left(\frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}, -\frac{1}{2}\right)$ and $\left(\frac{-3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, -\frac{1}{2}\right)$.

$f(0,0) = 0$, $\underbrace{f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = \frac{9}{2}, \; f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \frac{9}{2}}_{\text{global max } \frac{1}{2}}$, $\underbrace{f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = -\frac{9}{2}, \; f\left(\frac{-3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = -\frac{9}{2}}_{\text{global min } \frac{1}{2}}$

Parametrizing boundary approach

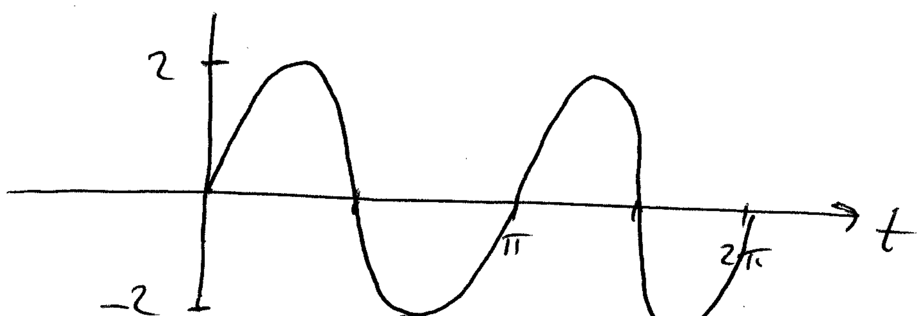
5. Find the global maximum and the global minimum of the function $f(x, y) = xy$ on the region

$$A = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4 \}.$$

1 $\boxed{f_x = y, \ f_y = x} \Rightarrow$ Critical point of f is $\boxed{(0,0) \in A}$ 1

Parametrize boundary of A: $\boxed{\begin{array}{l} x = 2\cos t \\ y = 2\sin t \end{array} \quad 0 \leq t \leq 2\pi}$ 1

$$f\Big|_{\text{boundary of } A} = 2\cos t \cdot 2\sin t = 4\sin t\cos t = 2\sin 2t$$ 1



f has a max value of 2 at $t = \frac{\pi}{4}$ and $\frac{5\pi}{4} \Rightarrow (x,y) = (\sqrt{2}, \sqrt{2})$ $\frac{1}{2}$
and $(x,y) = (-\sqrt{2}, -\sqrt{2})$ $\frac{1}{2}$

f has a min value of -2 at $t = \frac{3\pi}{4}$ and $\frac{7\pi}{4} \Rightarrow (x,y) = (-\sqrt{2}, \sqrt{2})$ $\frac{1}{2}$
and $(x,y) = (\sqrt{2}, -\sqrt{2})$ $\frac{1}{2}$

$f(0,0) = 0$, $\underbrace{f(\sqrt{2},\sqrt{2}) = 2, \ f(-\sqrt{2},-\sqrt{2}) = 2}_{\text{global max}}$, $\underbrace{f(\sqrt{2},-\sqrt{2}) = -2, \ f(-\sqrt{2},\sqrt{2}) = -}_{\text{global min}}$

$\frac{1}{2}$ $\frac{1}{2}$

Type I approach

6. Compute the following double integrals $\iint_R (y^3 + 3yx^2)\, dA$, for each of the regions R described below:

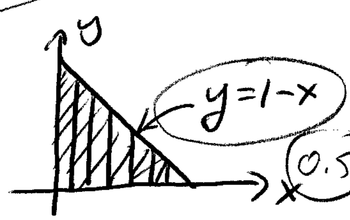
(a) R is the rectangle $0 \le x \le 1$, $0 \le y \le 2$.

(b) R is the triangle in the plane with vertices at $(0,0)$, $(1,0)$ and $(0,1)$.

$$a)\ \iint_R (y^3+3yx^2)\, dA = \boxed{\int_0^1 \int_0^2 (y^3+3yx^2)\, dy\, dx} = \underbrace{\int_0^1 \left(\frac{y^4}{4} + \frac{3y^2x^2}{2} \Big/_0^2 \right) dx}_{\frac{1}{2}} =$$

1.5

$$\frac{1}{2}\quad \int_0^1 (4+6x^2)\, dx = 4x+2x^3 \Big/_0^1 = 4+2 = 6 \quad \frac{1}{2}$$

b) 

0.5

$$\iint_R (y^3+3yx^2)\, dA = \underbrace{\int_0^1 \int_0^{1-x} (y^3+3yx^2)\, dy\, dx}_{2}$$

$$= \int_0^1 \left(\left(\frac{y^4}{4} + \frac{3}{2} y^2 x^2 \right) \Big/_0^{1-x} \right) dx = \int_0^1 \left(\frac{(1-x)^4}{4} + \frac{3}{2}(1-x)^2 x^2 \right) dx =$$

$\frac{1}{2}$

$$\int_0^1 \left(\frac{1}{4}(1-4x+6x^2-4x^3+x^4) + \frac{3}{2}(1-2x+x^2)x^2 \right) dx =$$

$\frac{1}{2}$

$$\int_0^1 \left(\frac{1}{4} - x + \frac{3}{2}x^2 - x^3 + \frac{1}{4}x^4 + \frac{3}{2}x^2 - 3x^3 + \frac{3}{2}x^4 \right) dx = \int_0^1 \left(\frac{1}{4} - x + 3x^2 - 4x^3 + \frac{7}{4}x^4 \right) dx$$

$$= \frac{1}{4}x - \frac{x^2}{2} + x^3 - x^4 + \frac{7}{20}x^5 \Big/_0^1 = \frac{1}{4} - \frac{1}{2} + 1 - 1 + \frac{7}{20} = \frac{7}{20} - \frac{1}{4} = \frac{7}{20} - \frac{5}{20}$$

$$= \frac{2}{20} = \frac{1}{10}$$

Type II approach

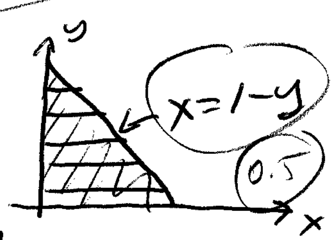
6. Compute the following double integrals $\iint_R (x^3 + 3xy^2)\, dA$, for each of the regions R described below:

(a) R is the rectangle $0 \le x \le 2$, $0 \le y \le 1$.

(b) R is the triangle in the plane ~~with vertices at $(0,0)$, $(1,0)$~~ and $(0,1)$.

a) $\iint_R (x^3+3xy^2)\,dA = \boxed{\int_0^1 \int_0^2 (x^3+3xy^2)\,dx\,dy} = \int_0^1 \left(\frac{x^4}{4} + \frac{3}{2}x^2y^2 \Big|_0^2\right) dy =$ 1.5 $\frac{1}{2}$

$\frac{1}{2}$ $\int_0^1 (4+6y^2)\,dy = 4y + 2y^3 \Big|_0^1 = 4+2 = 6$

b) 

0.5

$\iint_R (x^3+3xy^2)\,dA = \int_0^1 \int_0^{1-y} (x^3+3xy^2)\,dx\,dy$ 2

$= \int_0^1 \left(\frac{x^4}{4} + \frac{3}{2}x^2y^2 \Big|_0^{1-y}\right) dy = \int_0^1 \left(\frac{(1-y)^4}{4} + \frac{3}{2}(1-y)^2y^2\right) dy =$ $\frac{1}{2}$

$\int_0^1 \left(\frac{1}{4}(1-4y+6y^2-4y^3+y^4) + \frac{3}{2}(1-2y+y^2)y^2\right) dy =$ $\frac{1}{2}$

$\int_0^1 \left(\frac{1}{4} - y + \frac{3}{2}y^2 - y^3 + \frac{1}{4}y^4 + \frac{3}{2}y^2 - 3y^3 + \frac{3}{2}y^4\right) dy = \int_0^1 \left(\frac{1}{4} - y + 3y^2 - 4y^3 + \frac{7}{4}y^4\right) dy$

$= \frac{1}{4}y - \frac{y^2}{2} + y^3 - y^4 + \frac{7}{20}y^5 \Big|_0^1 = \frac{1}{4} - \frac{1}{2} + 1 - 1 + \frac{7}{20} = -\frac{1}{4} + \frac{7}{20} = -\frac{5}{20} + \frac{7}{20}$

$= \frac{2}{20} = \frac{1}{10}$