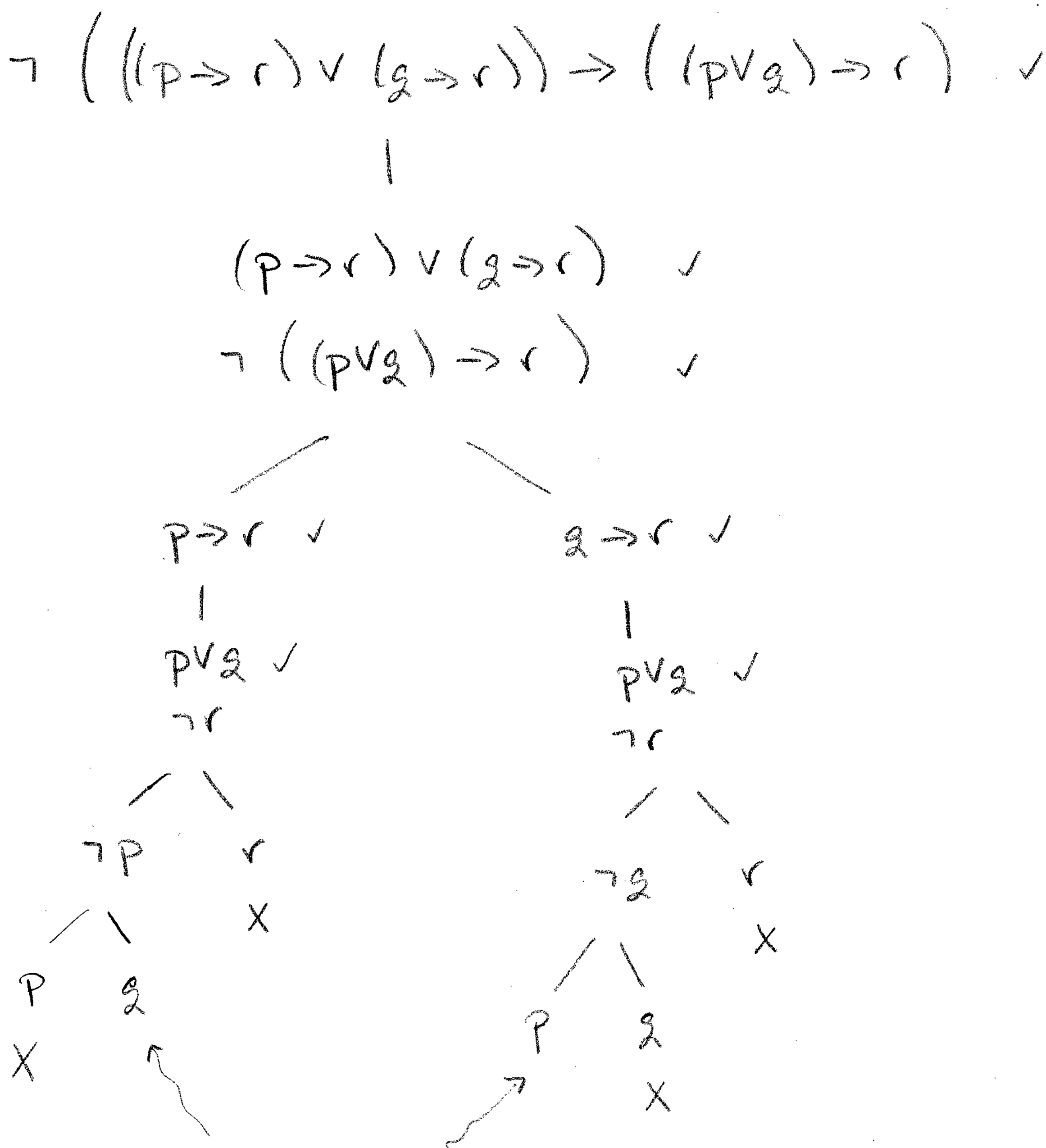
MAT 1348 - Assignment 2 Solutions

(1) Let P be the given proposition. We build a truth tree for $\neg P$:



We have 2 complete active paths; hence $\neg P$ is not a contradiction and P is not a tautology.

Counter examples: (1) P is F, q is T, r is F

(2) p is T, q is F, r is F

(2) (a)

$$\begin{array}{l} b \to (a \vee d) \\ m \vee \neg a \\ \neg d \\ \hline \therefore\ b \to m \end{array}$$

(b)

(1)	$b \to (a \vee d)$	(hypothesis)
(2)	$\neg b \vee (a \vee d)$	(equivalence, from (1))
(3)	$a \vee (\neg b \vee d)$	(comm. & assoc. law of $\vee$)
(4)	$m \vee \neg a$	(hypothesis)
(5)	$\neg a \vee m$	(comm. law of $\vee$)
(6)	$(\neg b \vee d) \vee m$	(resolution, from (3) and (5))
(7)	$d \vee (\neg b \vee m)$	(comm. & assoc. law of $\vee$)
(8)	$\neg d$	(hypothesis)
(9)	$\neg b \vee m$	(disjunctive syllogism, from (7) & (8))
(10)	$b \to m$	(equivalence, from (9))

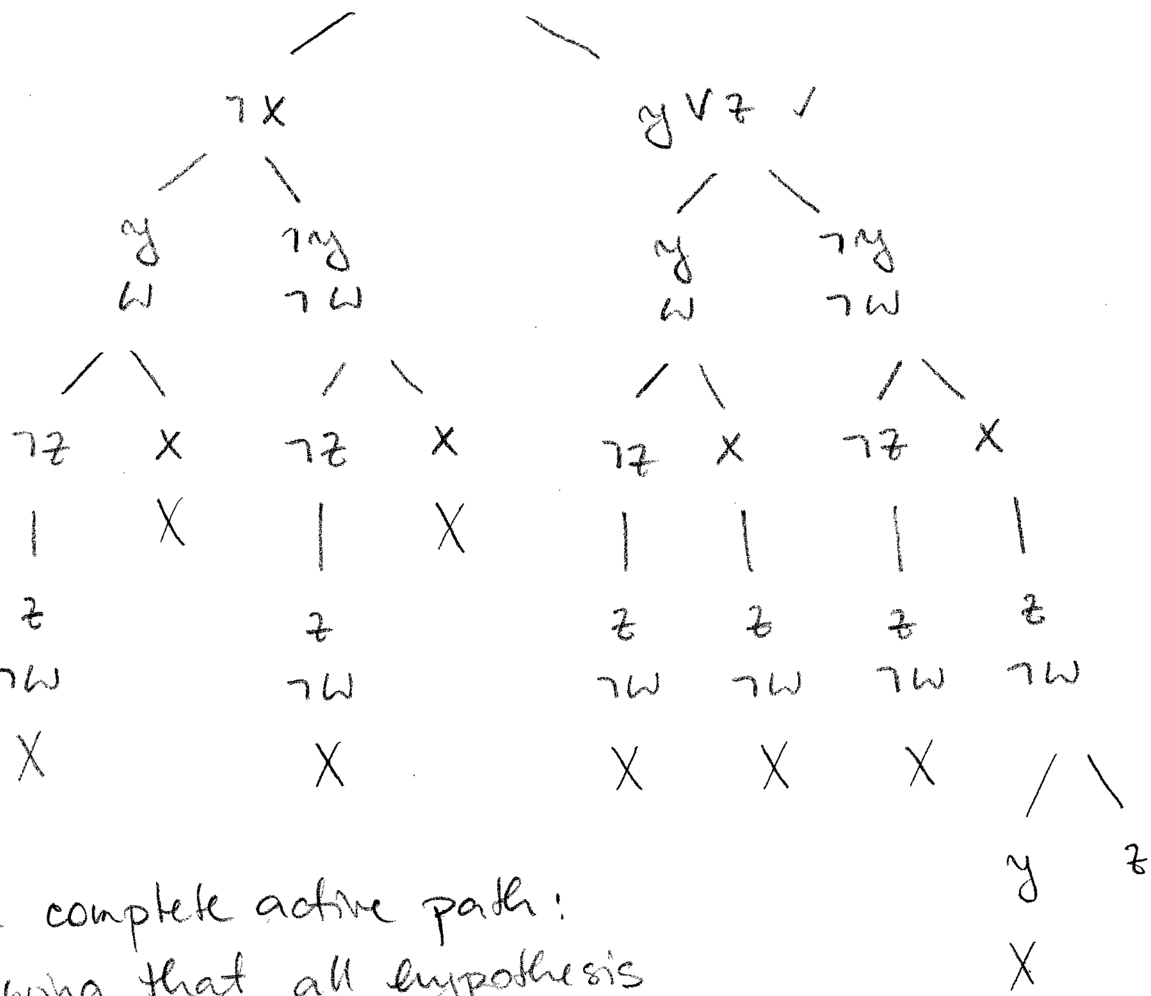
(3)

$x \to (y \lor z)$ ✓

$y \leftrightarrow w$ ✓

$z \to x$ ✓

$\neg(z \to w)$ ✓



One complete active path: showing that all hypothesis can be T while the conclusion is F. Counterexample: x is T

y is F

z is T

w is F

Argument is

<u>invalid</u>.

(4) Let p : "a and b are rational numbers."

q : "a^2+2b is a rational number."

We must prove $p \to q$.

For a direct proof, assume p and show q follows.

Assume p: a and b are rational. Hence there exist integers p, q, p', q' such that $q, q' > 0$ and $a = \frac{p}{q}$, $b = \frac{p'}{q'}$.

Then $a^2+2b = \left(\frac{p}{q}\right)^2 + 2\frac{p'}{q'} = \frac{p^2q' + 2q^2p'}{q^2q'}$.

Since p, q, p', q' are integers, so are $p^2q' + 2q^2p'$ and q^2q'. Since $q, q' > 0$, also $q^2q > 0$.

It follows that $a^2+2b = \frac{m}{n}$, where m and n are integers and $n > 0$.

Therefore, a^2+2b is rational.

⑤ Let p : "n is a perfect square"

q : "$n+4$ is not a perfect square".

We must show $p \to q$ using a proof by contradiction. To that end, assume $\neg(p \to q)$ and derive a contradiction. Note that $\neg(p \to q) \equiv p \wedge (\neg q)$.

Hence suppose that n is a perfect square (p) and $n+4$ is a perfect square $(\neg q)$.

Hence there are integers m and k s.t. $n = m^2$ and $n+4 = k^2$. Notice that we may assume $m, k > 0$ and hence $k > m$.

Now:

$$4 = (n+4) - n = k^2 - m^2 = (k-m)(k+m)$$

Since k and m are positive integers and $k > m$, both $k+m$ and $k-m$ are pos. integers, and $k+m > k-m$. The only way to write $4 = (k-m)(k+m)$ is thus for $k-m = 1$ and $k+m = 4$. But then $2k = (k-m) + (k+m) = 1+4 = 5$, a contradiction since k is an integer.

We conclude that $p \to q$ is true.