**The University of British Columbia**

Midterm 1 Solutions - February 3, 2012

**Mathematics 105, 2011W T2**

**Sections 204, 205, 206, 211**

1. (a) Let

$$f(x, y) = x \sin(y).$$

If the value of $(x, y)$ changes from $(0, \frac{\pi}{2})$ to $(0.1, \frac{\pi}{2} - 0.2)$, estimate the corresponding change in the value of $f$.

(8 points)

**Solution:** We have $\frac{\partial f}{\partial x} = \sin(y)$ and $\frac{\partial f}{\partial y} = x\cos(y)$. From the definition of the derivative,

$$f(x + \Delta x, y + \Delta y) - f(x, y) \approx \frac{\partial f}{\partial x}(x, y) \cdot \Delta x + \frac{\partial f}{\partial y}(x, y) \cdot \Delta y.$$

In our case, $x = 0$, $y = \pi/2$, $\Delta x = 0.1$ and $\Delta y = -0.2$. Hence,

$$\begin{aligned} f(0.1, \frac{\pi}{2} - 0.2) - f(0, \frac{\pi}{2}) &\approx \sin(\frac{\pi}{2})(0.1) + (0 \cdot \cos(\frac{\pi}{2}))(-0.2) \\ &= 1 \cdot 0.1 + 0 \cdot (-0.2) = 0.1. \end{aligned}$$

So the change in $f$ is about 0.1

(b) Let $\mathbf{v} = \langle \frac{4}{3}, -\frac{1}{2} \rangle$ and $\mathbf{w} = \langle 16, -6 \rangle$. Are the two vectors $\mathbf{v}$ and $\mathbf{w}$ parallel, perpendicular or neither? Justify your answer.

(8 points)

**Solution:** The two vectors are *not perpendicular*, since

$$\frac{4}{3} \cdot 16 + \left(-\frac{1}{2}\right) \cdot (-6) = \frac{64}{3} + 3 = \frac{73}{3} \neq 0.$$

The two vectors *are parallel*, since $12\mathbf{v} = \langle 12 \cdot \frac{4}{3}, 12 \cdot -\frac{1}{2} \rangle = \langle 16, -6 \rangle = \mathbf{w}$.

(c) A plane $\mathcal{P}$ is parallel to the plane $2x - y + z = 0$ and passes through the point $P_0(1, -1, 3)$. Find the equation of the plane $\mathcal{P}$.

(8 points)

**Solution:** Since $\mathcal{P}$ is parallel to the plane $2x - y + z = 0$, the vector $\langle 2, -1, 1\rangle$ is a normal vector for $\mathcal{P}$. Since $\mathcal{P}$ passes through the point $P_0(1, -1, 3)$, an equation for $\mathcal{P}$ is given by

$$2(x-1) + (-1)(y-(-1)) + 1(z-3) = 0.$$

Rearranging gives

$$2x - y + z = 6.$$

(d) Compute the right Riemann sum with four equal subintervals for $f(x) = 16x^2$ in the interval $[0, 1]$.

(8 points)

**Solution:** The general form of a Riemann sum is $\sum_{k=1}^{n} f(\overline{x}_k)\Delta x$. In our case,

$$f(x) = 16x^2, \quad n = 4, \quad \Delta x = \frac{1-0}{4} = \frac{1}{4}, \quad \text{and} \quad \overline{x}_k = a + k\Delta x = 0 + k\frac{1}{4} = \frac{k}{4}.$$

(To obtain the expression for $\overline{x}_k$, we recall that we are dealing with a *right* Riemann sum.) So our Riemann sum is

$$\sum_{k=1}^{4} 16\left(\frac{k}{4}\right)^2 \frac{1}{4} = \sum_{k=1}^{4} \frac{1}{4}k^2 = \frac{1}{4}\sum_{k=1}^{4} k^2$$
$$= \frac{1}{4}\left(1^2 + 2^2 + 3^2 + 4^2\right) = \frac{30}{4} = \frac{15}{2}.$$

(e) Given functions

$$F(x, y) = x + e^y, \quad \text{and} \quad G(x, y) = y + e^x,$$

does there exist a function $f(x, y)$ such that $\nabla f(x, y) = \langle F, G \rangle$? Justify your answer in detail, citing any result that you use.

(8 points)

**Solution:** No, there is no such function.
Clairaut's theorem says that if $f$ is defined on an open set $D$ in $\mathbb{R}^2$ and the mixed partials $f_{xy}$ and $f_{yx}$ are continuous there, then $f_{xy} = f_{yx}$ on all of $D$.
If $\nabla f(x, y) = \langle F, G \rangle$, then $f_{xy} = F_y = e^y$ and $f_{yx} = G_x = e^x$. So $f_{xy}$ and $f_{yx}$ are defined and continuous on all of $\mathbb{R}^2$, but it is not the case that $f_{xy} = f_{yx}$ on all of $\mathbb{R}^2$. This contradicts Clairaut's theorem and shows that no such function $f$ exists.

(f) Let $R$ be the semicircular region $\{x^2 + y^2 \leq 4,\ y \geq 0\}$. Find the maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 - 3y$$

on the *boundary of the region* $R$.

(10 points)

**Solution:** The boundary of the region has two pieces: (1) the semicircular arc consisting of points $(x, y)$ with $x^2 + y^2 = 4$ and $y \geq 0$, and (2) the horizontal line segment consisting of points $(x, 0)$ with $-2 \leq x \leq 2$. We treat the two pieces separately.

*Semicircular arc:*
Approach 1: We use Lagrange multipliers. Any point interior to the arc corresponding to a maximum or minimum value must satisfy

$$\nabla f = \lambda \nabla g,$$

where $f(x, y) = x^2 + y^2 - 3y$ and $g(x, y) = x^2 + y^2 - 4$. We have $\nabla f = \langle 2x, 2y - 3\rangle$ and $\nabla g = \langle 2x, 2y\rangle$. So

$$2x = \lambda(2x)$$
$$2y = \lambda(2y - 3).$$

The first equation shows that either $x = 0$ or $\lambda = 1$. From the second equation, we see that $\lambda \neq 1$. Hence $x = 0$. Since $x^2 + y^2 = 4$ and $y \geq 0$, we have $y = 2$. So we have to test the point $(0, 2)$. Since we are only looking at the upper half of $x^2 + y^2 = 4$ and not the entire circle, we must also test the endpoints of the arc, namely $(-2, 0)$ and $(2, 0)$. We find that

$$f(0, 2) = -2, \quad f(-2, 0) = 4, \quad f(2, 0) = 4.$$

So on the semicircular arc, the minimum is $-2$ and the maximum is 4.

Approach 2: The points on the semicircular arc have the form $(2\cos(\theta), 2\sin(\theta))$ for $0 \leq \theta \leq \pi$. So finding the maximum and minimum of $f$ on this arc amounts to finding the extrema of the one-variable function $F(\theta) = f(2\cos(\theta), 2\sin(\theta))$ on $[0, \pi]$. We have

$$F(\theta) = 4\cos^2(\theta) + 4\sin^2(\theta) - 6\sin(\theta) = 4 - 6\sin(\theta).$$

Thus, $F'(\theta) = -6\cos(\theta)$. So $F$ has a unique critical point on $[0, \pi]$ at $\theta = \pi/2$, where $F(\pi/2) = 4 - 6\sin(\pi/2) = -2$. Checking the endpoints of the interval, $F(0) = 4$ and $F(\pi) = 4$. So on the semicircular arc, the minimum is $-2$ and the maximum is 4.

*Horizontal segment:*
Finding the extrema of $f$ on this segment amounts to finding the extrema of the one-variable function $F(x) = f(x, 0) = x^2$ on the closed interval $[-2, 2]$. We have $F'(x) = 2x$. Thus, the only critical point of $F$ is $x = 0$, where $F(x) = 0$. Next, we test the endpoints of $[-2, 2]$, finding that $F(-2) = 4$ and $F(2) = 4$. (This is not a surprise; these endpoints correspond to the endpoints of the semicircular arc, so we found these values above also.) So the minimum value of $f$ on the horizontal segment is 0 and the maximum value is 4.

*Conclusion:* On the boundary of $R$, $f$ has minimum value $-2$ and maximum value 4.

2. Find *all* critical points of the function

$$f(x,y) = 3x^2 + 6xy - 2y^3$$

Classify each point as a local maximum, local minimum, or saddle point.

( 10 + 10 = 20 points)

**Solution:** We have $f_x = 6x+6y$ and $f_y = 6x-6y^2$. Thus, $f_x = 0$ precisely when $x = -y$. If we also require that $f_y = 0$, then we must have

$$0 = 6x - 6y^2 = -6y - 6y^2 = -6y(1+y),$$

so either $y = 0$ or $y = -1$. Since $x = -y$, the critical points of $f$ are $(0,0)$ and $(1,-1)$.

We now classify these points using the second derivative test. We have

$$f_{xx} = 6, \quad f_{yy} = -12y, \quad \text{and} \quad f_{xy} = 6.$$

So the $D$-function (the discriminant) is given by

$$\begin{aligned} D(x,y) &= f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 \\ &= 6(-12y) - 6^2 \\ &= -72y - 36. \end{aligned}$$

Now we plug in our critical points:

- $D(0,0) = -36 < 0$, so $(0,0)$ is a *saddle point*.
- $D(1,-1) = 72 - 36 = 36 > 0$, while $f_{xx}(1,-1) = 6 > 0$. Hence, $(1,-1)$ is a *local minimum*.

3. A candy company produces boxes of bubblegum and gummy bears. It costs the company \$1 to produce a box of either type of candy. On the other hand, bubblegum sells for \$3 per box and gummy bears for \$5 per box. Due to limitations of sugar supply, the production scheme has to satisfy the production possibilities curve

$$\sqrt{x} + 2\sqrt{y} = 300,$$

where $x$ and $y$ denote the number of boxes of bubblegum and gummy bears that the company produces weekly. Assuming that the company manages to sell every unit produced, use the method of Lagrange multipliers to answer the following question: how many boxes of each type of candy should the company aim to produce if it is to maximize profit?

Clearly state the objective function and the constraint. There is no need to justify that the solution you obtained is the absolute max or min. **A solution that does not use the method of Lagrange multipliers will receive no credit, even if it is correct.**

(20 points)

**Solution:** Recall that profit is given by revenue minus cost. If the company manufactures $x$ boxes of bubblegum and $y$ boxes of gummy bears, then the total revenue is $3x+5y$ and the total cost is $x+y$. So the profit function $f$, which is our objective function, is given by

$$f(x, y) = 2x + 4y.$$

Our constraint is specified by the production possibilities curve; in the method of Lagrange multipliers, this corresponds to the constraint $g = 0$, where

$$g(x, y) = \sqrt{x} + 2\sqrt{y} - 300.$$

Since $\nabla f = \langle 2, 4\rangle$ while $\nabla g = \langle \frac{1}{2}x^{-1/2}, y^{-1/2}\rangle$, setting $\nabla f = \lambda \nabla g$ gives us the two equations

$$2 = \frac{\lambda}{2}x^{-1/2},$$
$$4 = \lambda y^{-1/2}$$

Dividing the first equation by $\lambda/2$ and the second by $\lambda$, we find that both $x^{-1/2}$ and $y^{-1/2}$ are equal to $4/\lambda$. So $x^{-1/2} = y^{-1/2}$ and hence $x = y$.

Since $\sqrt{x} + 2\sqrt{y} = 300$ and $x = y$, we have $3\sqrt{x} = 300$. Hence $\sqrt{x} = 100$ and $x = 10^4$. Since $y = x$, we also have $y = 10^4$. So the optimum configuration is $10^4$ boxes of bubblegum and $10^4$ boxes of gummy bears.

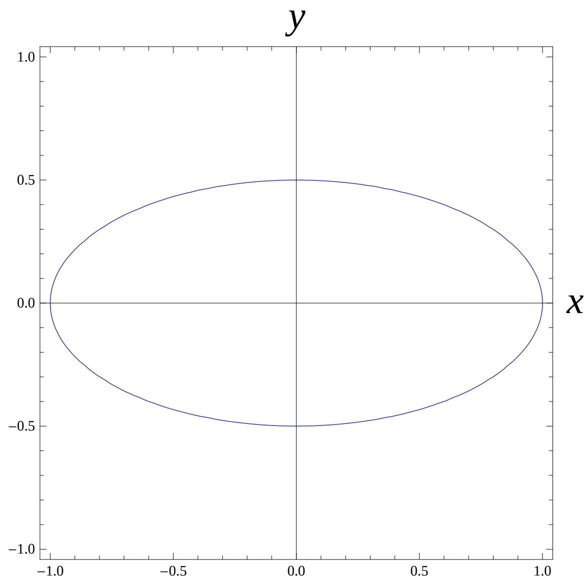
4. Consider the surface $S$ given by

$$z = 3 + x^2 + 4y^2.$$

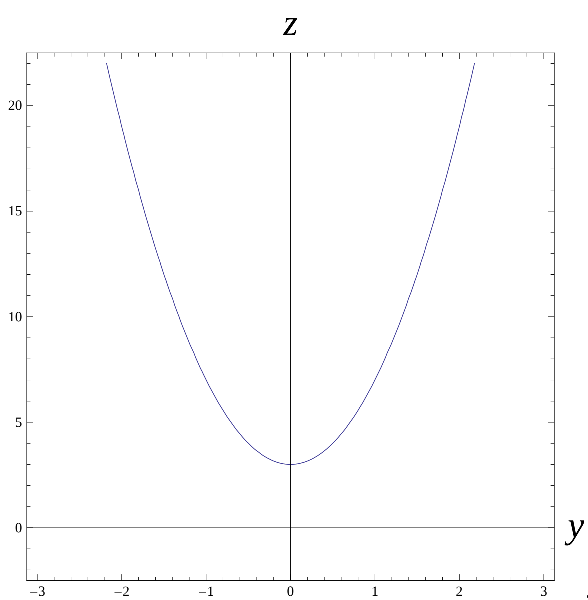
(a) Sketch the traces of $S$ in the $z = 4$ and $x = 0$ planes, labeling the axes carefully.

(3 + 3 = 6 points)

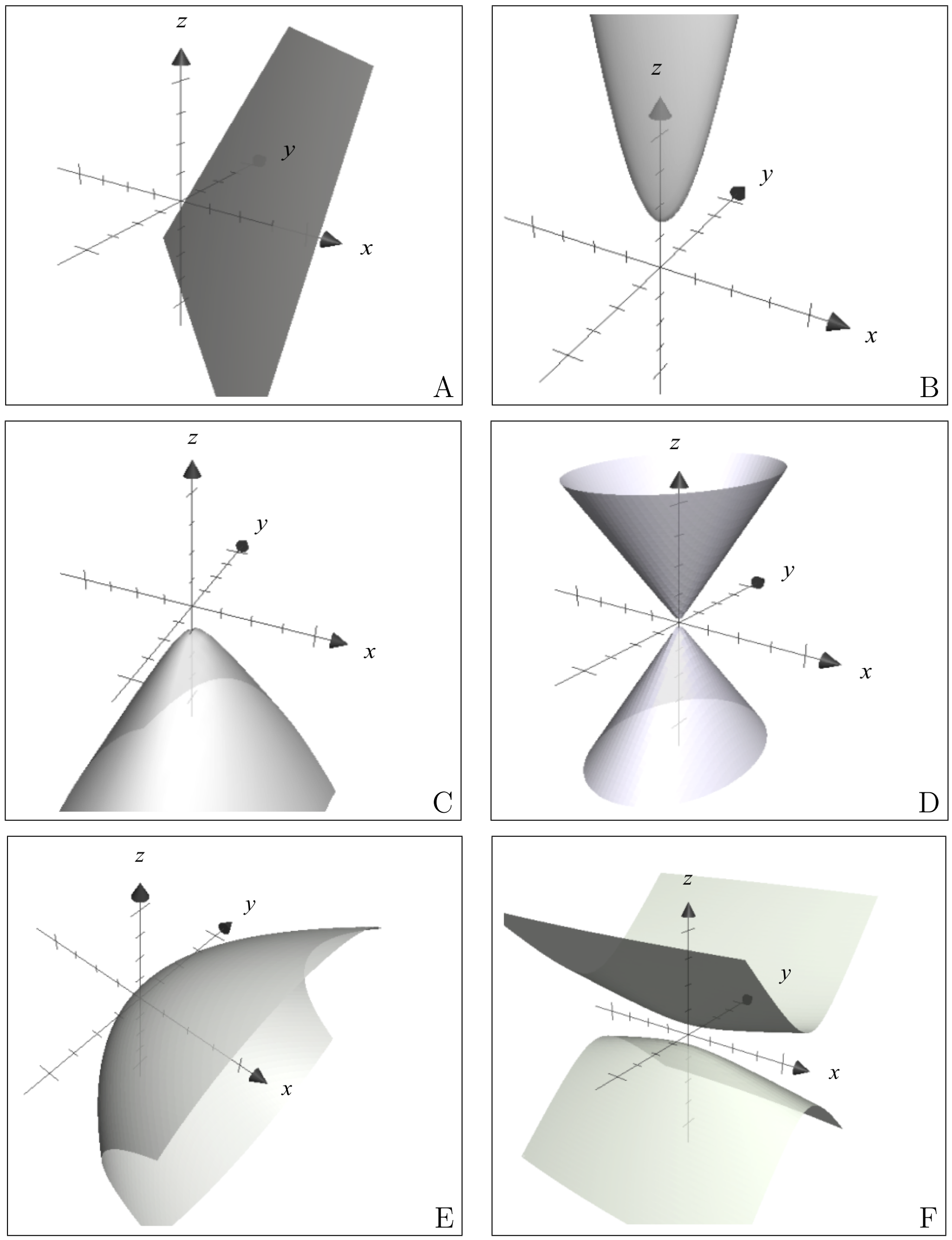
**Solution:** The trace in the $z = 4$ plane is specified by the equation $4 = 3 + x^2 + 4y^2$, i.e., $x^2 + 4y^2 = 1$. This is an ellipse:




The trace in the $x = 0$ plane is specified by the equation $z = 3 + 4y^2$, which is a parabola:



(b) Based on the traces you sketched in part (a), which of the following renderings represents the graph of the surface?




**Solution:** B.

5. (Extra credit) Transform the limit of the following Riemann sum to a definite integral:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{n}\left(1+\frac{2k}{n}\right)^5. \tag{1}$$

(10 points)

**Solution:** Recall that the general form of a Riemann sum is

$$\sum_{k=1}^{n} f(\overline{x}_k)\Delta x. \tag{2}$$

Comparing this with (1), we see that $\Delta x = \frac{2}{n}$. Since $\Delta x = \frac{b-a}{n}$, we have $b - a = 2$, and so $[a, b]$ is an interval of length 2. In order that (1) match up with (2) we also need to ensure that

$$f(\overline{x}_k) = \left(1+\frac{2k}{n}\right)^5.$$

Since this expression for $f(\overline{x}_k)$ involves $2k/n = k\cdot\Delta x$ (and not $(k-1)\Delta x$ or $(k-\frac{1}{2})\Delta x$) we suspect that we are dealing with a right-endpoint Riemann sum. In that case,

$$\overline{x}_k = a + k\Delta x = a + \frac{2k}{n}.$$

If we now choose $a = 1$, then

$$\left(1+\frac{2k}{n}\right)^5 = \overline{x}_k^5,$$

which matches $f(\overline{x}_k)$ if we choose $f(x) = x^5$.

Since $a = 1$ and $b - a = 2$, the interval $[a, b] = [1, 3]$. We conclude that the given limit of Riemann sums coincides with the definite integral

$$\int_1^3 x^5\,dx.$$

**Note**: Other (equivalent) answers are possible, e.g., $\int_0^2 (1+x)^5\,dx$ is also acceptable.