Math 267 Midterm Exam I
9:00am, Monday, Feb. 6, 2012
at Buchanan A104
Duration: **70 minutes**

Full Name: ______________________ Student Number: ______________________

Do not open this test until instructed to do so! This exam should have **9** pages (including this cover) plus the formula sheet. **The formula sheet is attached in the end.**
No textbooks, calculators, or other aids are allowed. Turn off any cell phones, pagers, etc. You must remain in this room until you have finished the exam. **Show all your work.** Use the back of the page if necessary.

Read these UBC rules governing examinations:

(i) Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.

(ii) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

(iii) No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

(iv) Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

- Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
- Speaking or communicating with other candidates.
- Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

(v) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Problem	1	2	3	4	**Total**
Out of	10	10	10	10	40
Score					

Problem 1. [10 points]
Recall that i is the imaginary number $i = \sqrt{-1}$.

(a) (2 points) Let $z = -1 + \sqrt{3}\, i$ and $w = 2i$. Compute $z^3 + (\overline{w})^{-3}$. (Here $\overline{w}$ is the complex conjugate of w.)

(b) Let

$$f(t) = \sum_{k=0}^{100} \Big[-\frac{1}{2} + \frac{\sqrt{3}}{2} i \Big]^k e^{ik\pi t}.$$

(i) (4 points) Calculate $\int_{-1}^{1} f(t) e^{i30\pi t} dt$.

(ii) (4 points) Calculate $\int_{-1}^{1} |f(t)|^2 dt$.

Solution:

(a): $z = -1 + \sqrt{3}i = 2[-\frac{1}{2} + \frac{\sqrt{3}}{2} i] = 2e^{i2\pi/3}$ and $\overline{w} = -2i$. Thus,

$$z^3 + (\overline{w})^{-3} = 2^3 e^{i2\pi} + \Big(\frac{1}{-2i}\Big)^3 = 2^3 - \frac{i}{8} = \underline{8 - \frac{i}{8}}$$

(b):
(i) :

$$\int_{-1}^{1} \sum_{k=0}^{100} \Big[-\frac{1}{2} + \frac{\sqrt{3}}{2} i \Big]^k e^{ik\pi t} e^{i30\pi t} dt$$
$$= \sum_{k=0}^{100} \Big[-\frac{1}{2} + \frac{\sqrt{3}}{2} i \Big]^k \int_{-1}^{1} e^{ik\pi t} e^{i30\pi t} dt$$
$$= 0 \qquad \text{(by orthogonality since } k \geq 0\text{: no case of } k = -30.\text{).}$$

(ii) : We use Parseval's relation

$$\frac{1}{2} \int_{-1}^{1} |f(t)|^2 dt = \sum_{k=0}^{100} \Big| -\frac{1}{2} + \frac{\sqrt{3}}{2} i \Big|^{2k} = \sum_{k=0}^{100} |e^{i2\pi/3}|^{2k} = \sum_{k=0}^{100} 1 = 101.$$

Thus, the answer is <u>202.</u>

Problem 2. [10 points]: Consider the following initial boundary value problem for the **wave equation**:

$$\begin{aligned} &u_{tt}(x,t) = u_{xx}(x,t), \qquad \text{for } 0 < x < \pi,\ t > 0; \\ &u(0,t) = 0, \quad u_x(\pi,t) = 0, \qquad \text{for } t > 0. \end{aligned}$$

(Note the derivative boundary condition $u_x(\pi,t) = 0$ at $x = \pi$.)
Find **all nontrivial** solutions $u(x,t)$ of the following type:

$$u(x,t) = X(x)T(t)$$

where $X(x)$ and $T(t)$ are functions of the variables x, t, respectively.

Solution: For $u(x,t) = X(x)T(t)$. From the PDE we get $X''(x)T(t) = X(x)T''(t)$. thus,

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \sigma$$

We get (considering also the boundary condition $X(0)T(t) = 0$ and $X'(\pi)T(t) = 0$.

$$\begin{aligned} &X'' = \sigma X, \\ &X(0) = 0, \quad X'(\pi) = 0 \end{aligned}$$

and $T''(t) = \sigma T(t)$.
For the X problem:

Case: $\sigma = 0$: Then, $X'' = 0$, thus, $X(x) = Ax + B$. But, $X(0) = 0$, so $B = 0$. And $X'(\pi) = 0$, so $A = 0$. So, we get only **trivial solution**. Thus, we exclude this case.

Case: $\sigma \neq 0$: Then, $X'' = \sigma X$. Characteristic equation is $r^2 = \sigma$, and $r = \pm\sqrt{\sigma}$. Thus, the general solution is $X(x) = C_1 e^{\sqrt{\sigma}x} + C_2 e^{-\sqrt{\sigma}x}$ Now, use the boundary condition, $X(0) = 0$, so, $C_1 + C_2 = 0$. Thus, $X(x) = C_1[e^{\sqrt{\sigma}x} - e^{-\sqrt{\sigma}x}]$. $X'(x) = C_1\big[\sqrt{\sigma}e^{\sqrt{\sigma}x} + \sqrt{\sigma}e^{-\sqrt{\sigma}x}\big] = C_1\sqrt{\sigma}\big[e^{\sqrt{\sigma}x} + e^{-\sqrt{\sigma}x}\big]$. Use the boundary condition $X'(\pi) = 0$, to get $C_1\sqrt{\sigma}\big[e^{\sqrt{\sigma}\pi} + e^{-\sqrt{\sigma}\pi}\big] = 0$. Since $C_1 \neq 0$ (otherwise we have trivial solution) and $\sigma \neq 0$ (in the present case), we see $e^{\sqrt{\sigma}\pi} + e^{-\sqrt{\sigma}\pi} = 0$, i.e. $e^{2\sqrt{\sigma}\pi} = -1$. Since $-1 = e^{ik\pi}$, for odd k .We see that $2\sqrt{\sigma}\pi = ik\pi$, for odd k Thus,

$$\sigma = -\frac{k^2}{4} \qquad \text{odd } k$$

Therefore, $X(x) = C_1\big[e^{i(kx/2} - e^{-ikx/2}\big] = D\sin(kx/2)$, for odd $k = 1, 3, 5, \cdots$.
Now, from $T''(t) = \sigma T(t)$, with $\sigma = -k^2/4$, odd k , we get

$$T(t) = A_k\cos(kt/2) + B_k\sin(kt/2), \qquad k = 1, 3, 5, \cdots.$$

Therefore, we have

$$\underline{u(x,t) = X(x)T(t) = \sin(kx/2)\Big[\alpha_k\cos(kt/2) + \beta_k\sin(kt/2)\Big], \qquad k = 1, 3, 5, \cdots.}$$

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Problem 3. [10 points]: Consider the following **heat** conduction problem on a rod of length 1:

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \qquad \text{for } 0 < x < 1,\ t > 0;\\ &u(0,t) = u(1,t) = 0, \qquad \text{for } t > 0;\\ &u(x,0) = f(x), \qquad \text{for } 0 \le x \le 1. \end{aligned}$$

(a) (3 points) Find the solution $u(x,t)$ for $f(x) = 3\sin(\pi x) + 2\sin(5\pi x)$.

(You **do not** have to explain the method of separation of variables.)

(b) (5 points) Find the solution $u(x,t)$ for $f(x) = 1$.

(You **do not** have to explain the method of separation of variables.)

(c) (2 points) What is the limit of $u(x,t)$ in part (b) as $t \to \infty$?

Solution:

The solution is of the form $u(x,t) = \sum_{k=1}^{\infty} b_k e^{-k^2\pi^2 t}\sin(k\pi x)$.

(a): Use the initial condition

$$3\sin(\pi x) + 2\sin(5\pi x) = u(x,0) = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$$

Comparing both sides, we see $b_1 = 3, b_5 = 2, \quad b_k = 0$ for $k \neq 1, 5$. Thus,

$$\underline{u(x,t) = 3e^{-\pi^2 t}\sin(\pi x) + 2e^{-5^2\pi^2 t}\sin(5\pi x)}$$

(b): Use the initial condition $1 = u(x,0) = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$.

Compute,

$$\begin{aligned} b_k &= 2\int_0^1 1\cdot\sin(k\pi x)dx\\ &= 2\Big[\frac{-\cos(k\pi x)}{k\pi}\Big]_0^1\\ &= \frac{2}{k\pi}\Big[-(-1)^k + 1\Big]\\ &= \begin{cases} 0, & k \text{ even},\\ \frac{4}{k\pi}, & k \text{ odd}. \end{cases} \end{aligned}$$

Thus, we have

$$\underline{u(x,t) = \sum_{k=1,3,5,\cdots} \frac{4}{k\pi} e^{-k^2\pi^2 t}\sin(k\pi x)}$$

(c): As $t \to \infty$, $e^{-k^2\pi^2 t} \to 0$ for $k = 1, 2, 3, \cdots$. Thus, $\underline{u(x,t) \to 0.}$

Problem 4 [10 points]. Let $f(t)$ be the **4-periodic** (i.e. the period = 4) function defined by

$$f(t) = \begin{cases} 1, & \text{if } -2 \le t < 0; \\ 2, & \text{if } 0 \le t < 2. \end{cases}$$

(a) (2 points) Sketch the graph of $f(t)$ (show at least three periods). Is $f(t)$ odd or even?

(b) (6 points) Find the **complex Fourier series** of $f(t)$.

(c) (2 point) What is the value of **the complex Fourier series** in part (b) at $t = 4$? (Give the numerical value!)

Solution:

(a). The graph is omitted. The function is neither even nor odd.

(b): Compute

$$\begin{aligned} c_k &= \frac{1}{4}\int_{-2}^{2} f(t)e^{-ik\pi t/2}dt \\ &= \frac{1}{4}\Big[\int_{-2}^{0} f(t)e^{-ik\pi t/2}dt + \int_{0}^{2} f(t)e^{-ik\pi t/2}dt\Big] \\ &= \frac{1}{4}\Big[\int_{-2}^{0} 1e^{-ik\pi t/2}dt + \int_{0}^{2} 2e^{-ik\pi t/2}dt\Big] \\ &= \frac{1}{4}\Big\{\Big[\frac{e^{-ik\pi t/2}}{-ik\pi/2}\Big]_{-2}^{0} + 2\Big[\frac{e^{-ik\pi t/2}}{-ik\pi/2}\Big]_{0}^{2}\Big\} \end{aligned}$$

Case $k = 0$: Then, $c_0 = \frac{1}{4}\Big[\int_{-2}^{0} 1dt + \int_{0}^{2} 2dt\Big] = 3/2$.

Case $k \neq 0$. Then,

$$\begin{aligned} c_k &= \frac{1}{4}\Big\{\Big[\frac{e^{-ik\pi t/2}}{-ik\pi/2}\Big]_{-2}^{0} + 2\Big[\frac{e^{-ik\pi t/2}}{-ik\pi/2}\Big]_{0}^{2}\Big\} \\ &= \frac{1}{4}\Big\{\Big[\frac{1-e^{ik\pi}}{-ik\pi/2}\Big] + 2\Big[\frac{e^{-ik\pi}-1}{-ik\pi/2}\Big]\Big\} \\ &= \frac{1}{4}\Big\{\Big[\frac{1-(-1)^k}{-ik\pi/2}\Big] + 2\Big[\frac{(-1)^k-1}{-ik\pi/2}\Big]\Big\} \\ &= \begin{cases} \frac{1}{ik\pi} & k \text{ odd}, \\ 0 & k \text{ even}. \end{cases} \end{aligned}$$

Thus, the complex Fourier series is

$$\underline{f(t) = 3/2 + \sum_{k=\pm1,\pm3,\cdots} \frac{1}{ik\pi}e^{ik\pi t/2}}$$

(c): Notice that at $t = 4$, the function has jump discontinuity. Thus, the value of the Fourier series is $\underline{\frac{f(4-)+f(4+)}{2} = (1+2)/2 = 3/2.}$

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