~LA: The Basics~

Vectors: Points from the origin in $\mathbb{R}^n$ [collumn matrix of 'n' entries]

↳ Adding (head to tail) => gives parallelogram, creats new vector  ... = A+B

↳ Multiplying by a scalar: gives a colinear vector 

Linear Combination: A matrix 'A' whose collumns are vectors w/ matrix 'b' solutions.
Solving gives the intersection point of the vectors. (vector 'x')
'b' is a combination of v's depending on weights 'c'

♪ general solution = free + b

↳ using scalers, the vectors can be added to give 'b' (solve for scalars).

Matrix Operations: Notation: A(1,1) is the 1st row, 1st collumn => ♪ Always RC ♫

↳ Multiplication → ↓ row by collumn, then add   • $(A^T)^T = A$, • $(rA)^T = rA^T$ •

↳ Transpose: rows of 1 become collumns of other.   • $(AB)^T = B^T A^T$ • $(A+B)^T = A^T + B^T$ •

Inner Product: $u \cdot v = u^T v$ or Multiply coursponding elements.

Length: $\|v\| = \sqrt{v \cdot v}$ $\|v\|^2 = v \cdot v$

↳ normalizing $v \Rightarrow \frac{1}{\|v\|} v$ creates a unit vector 'u' (length 1) in the direction of v

Distance: Btw u & v is $dist(u,v) = \|u - v\|$

Angle Btw Vectors $\cos\theta = \frac{x \cdot y}{\|x\|\|y\|}$ solve for θ

Rotation Matrix rotates a vector about the origin by angle θ (counterclockwise)

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ⇐ vector   Multiply A by vector to get rotated vector.

Cramer's Rule: A way to solve $Ax = b$ for vector x

$x_i = \frac{\det A_i(b)}{\det A}$ where $A_i(b)$ is matrix 'A' w/ 'b' replacing collumn 'i'.

Elementary Matrix: Performing one row operation on an I matrix gives E
$E E^{-1} = I$ (elementary Matrices are invertable)

Homogeneous System: $Ax = 0$ (there will always be a solution)

↳ trivial solution: $x = 0$ (all x's) ↳ Non-trivial: At least one free variable ($x \neq 0$)

Linearly Independant (L.I): if the only solution is $x=0$ (for all x) => The trivial solution

Linearly Dependant (L.D): if the free variable determines the values of all others (non-trivial)

Area: given A (a 2x2) **Area = |det A|** gives area of parallelogram (created by vectors)

Volume: given A (a 3x3) **Volume = |det A|** gives volume of parallelepid (created by vectors)

Coordinate Vector $[x]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is coordinate vector of x relative to basis "B" $x = c_1v_1 + c_2v_2 \cdots c_nv_n$

$[v_1 \; v_2 \; \vdots \; x] \Rightarrow$ gives $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

↳ given $B = [v_1], [v_2]$ & $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ or $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ } solve for other

~Determinents~

2x2 matrix: det A = ad − bc

3 x 3 matrix:

Multiply Multiply

$$\left[\begin{array}{ccc:cc} a & d & g & a & d \\ b & e & h & b & e \\ c & f & i & c & f \end{array}\right]$$

j k L m n o

$(m+n+o)-(j+k+L)$

Triangular if all entries above or below main diagonal are zeros:
det = main diagonal entries multiplied together

Row Operations: Multiplying a row by a scalar (Multiply determinent by inverse) $\begin{bmatrix} a \cdot R_i \\ a \cdot \det \end{bmatrix}$
Swapping two rows (Multiply determinent by −1) $[R_1 \leftrightarrow R_2 \quad -\det]$

Square Matrix: $\sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$ cross out row i, collumn j $\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}$

foiled out (a 2x2 example)

$\det A = (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12}) + (-1)^{1+1} a_{21} (\det A_{21}) + (-1)^{1+2} a_{22} (\det A_{22})$

Properties: det A = 0 if A has 2 identical rows or collumns
det A = det B if A & B are different by row placement *??

- $\det(A) = \det(A^T)$
- $\det(A^{-1}) = \dfrac{1}{\det A}$ (if det $A \neq 0$)
- $\det(AB) = \det(A) \cdot \det(B)$
- $\det(aA) = a^n \det(A)$ (where a = scalar, n = size)

~Inverses~

Identity Matrix: $I = A^{-1}A = AA^{-1}$ ⇒ A matrix of 0's & 1's (reduced echelon form)

2x2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3x3: $\begin{bmatrix} A & \vdots & I \end{bmatrix}$ with $A \downarrow I$ and $I \downarrow A^{-1}$: if you reduce (A) to (I), the same operations reduce (I) to (A^{-1}).

Solving Systems $Ax = b$, $x = A^{-1}b$ (multiply both sides by A^{-1})

Properties • $(A^{-1})^{-1} = A$ • $(AB)^{-1} = B^{-1}A^{-1}$ •

• $(A^T)^{-1} = (A^{-1})^T$ • $(kA)^{-1} = \frac{1}{k}A^{-1}$ •

coefactors a coefactor of an entry is $C_{ij} = (-1)^{i+j} \det(A_{ij})$

↳ Take this of each entry to make coefactor matrix

↳ $\operatorname{adj}(A) = [\text{coefactormatrix}]^T$ $\quad A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$ ♪ Finds inverse ♪

Theorem: If A is invertible:

- Reduced A has pivot in every row & collumn
- $Ax = 0$ has only trivial solution (A is linearly independant)
- $T(x) = Ax$ is one-to-one & onto
- collumns of A span $\mathbb{R}^n$

~ Complex Numbers ~

- a number of form $(a + ib = z)$ $Re(z) = a$, $Im(z) = b$
- $i = \sqrt{-1}$, $i^2 = -1$

conjugate $z = a + ib$, $\bar{z} = a - ib$; $z + \bar{z} = 2a$, $z - \bar{z} = 2ib$; $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$; $\overline{z_1 / z_2} = \bar{z}_1 / \bar{z}_2$, $z \cdot \bar{z} = a^2 + b^2$; $\bar{\bar{z}} = z$

+ / − just add & subtract like terms (a's & ib's)

***** treat a, i & b like variables, multiply out & $i^2 = -1$

÷ Multiply & devide by $\bar{z}$ of denom. → expand & simplify & $i^2 = -1$

|z| $|z| = \sqrt{a^2 + b^2}$ ∴ $= \sqrt{z \cdot \bar{z}}$ & $|zw| = |z||w|$

Polar Form $a + ib = |z|(\cos\theta + i \sin\theta)$

↳ Find θ ⇒ $\theta = \cos^{-1}(a/|z|)$, $a = |z|\cos\theta$

⇒ $\theta = \sin^{-1}(b/|z|)$, $b = |z|\sin\theta$

De Moiver's Theorem

$$z^n = |z|^n(\cos(n\theta) + i \sin(n\theta))$$

~Eigen~

<u>Eigenvector</u> A is nxn matrix & $Ax = \lambda x$ for some scalar λ
x is a non-zero eigenvector.

<u>Eigenvalue</u> The scalar λ coorsponding to 'x' (for every λ there's an x)

$Ax = \lambda x$ } $Ax - \lambda x = 0$ } $(A - \lambda I)x = 0$ } $\det(A - \lambda I) = 0$ (characteristic eq.)

<u>Finding Eigenvalues</u>: solve $\det(A - \lambda I) = 0$ } gives polynomial, find roots ($\lambda_1, \lambda_2, \lambda_3 \ldots$)

<u>Finding Eigenvectors</u>: use λ's one at a time in $(A - \lambda I)x = 0$ => solve __ $x = 0$ for vector 'x'
↳ eigenspace: the set of all solutions 'x' => $\text{Null}(A - \lambda I)$ = eigenspace A.

P is a matrix of eigenvectors $P = [x_1 \; x_2 \; x \ldots]$ it is linearly independant

D is a triangular matrix of main diagonal λ $D = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

<u>Diagonalization</u>: A is diagonalizable if it has 'n' L.I. eigenvectors.
↳ to show this, show $AP = PD$ .... $(A = PDP^{-1})$

if $\det(A - \lambda I) = 0$ gives <u>complex eigenvalues</u> that's ok!
↳ x is then a complex eigenvector &...
$\bar{\lambda}$ & $\bar{x}$ is anotter eigenvalue/vector set in A

=> given $\lambda = 1$ $\lambda = -2$ $\lambda = -2$, eigenspace = 2 (2 of same λ)

=> $D = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, $A^n = P \begin{bmatrix} \lambda^n & 0 \\ 0 & \lambda^n \end{bmatrix} P^{-1}$

LifeIsGood~ Exam 6

~ Orthogonal ~

~ orthogonal means perpendicular ~ $\perp$

$u \cdot v = 0$ if this equation holds true, u & v are orthogonal vectors $[u|v]$ $(u \perp v)$
↳ also if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ (pythagoras)

orthogonal compliments W is a subspace in $\mathbb{R}^n$, z is a vector
↳ if z is orthogonal to every vector in W, z is the OC of W (called $W^\perp$)
∫ given $W = \text{span}\{v_1, v_2\}$, $z \cdot v_1 = 0$ & $z \cdot v_2 = 0$, create $[z_1 z_2 | 0]$ & solve for vector z, $W^\perp = z$ ∫


[z is ⊥ to W]

Orthogonal set $S = [v_1, v_2, v_3]$, if $v_1 \cdot v_2 = 0$, $v_2 \cdot v_3 = 0$ & $v_1 \cdot v_3 = 0$, 'S' is orthogonal set
↳ These are always linearly independant
↳ if v_1, v_2, v_3 are all unit vectors, 'S' is an orthonormal set

icky { orthogonal Basis: Basis of orthog. vectors. (a subspace spanned by Orthog. set has an orthog. Basis)
↳ orthonormal Basis: a Basis consisting of orthonormal vectors.

orthogonal Matrix: square, invertible, $U^T U = I$ (has orthonormal vectors)
↳ $\|Ux\| = \|x\|$ } $(Ux)\cdot(Uy) = x \cdot y$ } $(Ux)\cdot(Uy) = 0 \iff x \cdot y = 0$ } all cuz $\|U\| = 1$

Solving system $c_j = \dfrac{x \cdot u_j}{u_j \cdot u_j}$ [if $\{u_1, u_2 \ldots\}$ is an orthogonal basis.] gives constants $b = c_1 v_1 + c_2 v_2 + \ldots$ for linear combo.

$\text{Proj}_u y$: $\hat{y} = \dfrac{y \cdot u}{u \cdot u} u$ } $\hat{y}$ is the orthogonal projection of 'y' onto 'u'

decomposition $z = y - \hat{y}$, the decomp is $y = \hat{y} + z$

$\text{Proj}_W x$: 'W' is a subspace with an orthogonal basis $\{u_1, u_2 \cdots\}$, 'x' is a vector

↳ $\hat{x} = \dfrac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \dfrac{x \cdot u_2}{u_2 \cdot u_2} u_2 \cdots$ ∫ $\hat{x}$ is in W, z is in $W^\perp$ (z is $\perp$ u) ∫

decomposition

$z = x - \hat{x}$, $x = \hat{x} + z$

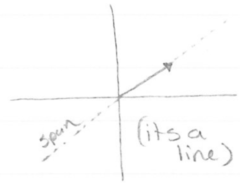
~Space~

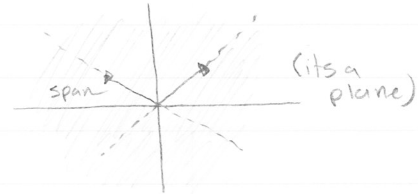
Span: The space the vectors cover

↳ its vectors $v_1, v_2 \cdots v_n$ multiplyed by scalars $c_1, c_2 \cdots c_n$

↳ its the set of all linear combinations $c_1v_1 + c_2v_2 + \cdots c_nv_n$

It contains every scalar multiple of v (The zero vector is always in the span.)





Subspace: Almost the same as span. Given $v_1, v_2 \cdots v_n$, the set of all linear combos is a subspace of $\mathbb{R}^n$. So, span$\{v_1, v_2 \cdots v_n\}$ is actually the subspace spanned by $v_1, v_2 \cdots v_n$.

↳ 3 conditions: Contains 0 vector • closed under addition & scalar multiplication (think of span)

Collumn Space (Col A): the set of all linear combos of the collumns of A. (span$\{v_1 \cdots v_n\}$)

↳ Col A of $m \times n$ is a subspace of $\mathbb{R}^m$ (if $Ax = b$ is consistant, b is in Col A)

Null space (Null A): The set of all solutions to $Ax = 0$ (its a subspace of $\mathbb{R}^n$ in $m \times n$)

↳ explicit description, parametric form of $Ax = 0$ (if $Ax = 0$, x is in Null A)

Basis: A basis for a subspace H of $\mathbb{R}^n$ is a L.I. set in H that spans H

↳ collumns of invertible matrix are L.I. & span $\mathbb{R}^n$ (by definition) so they are a basis for $\mathbb{R}^n$

↳ $Ax = 0$ (parametric form) shows a basis for Nul A } wtf.

↳ writing all free variables in terms of pivots, pivots span Col A. }

⇒ Any set of 'p' elements of H that spans H is a basis for H (L.I. set of 'p' elements)

Dimension: of a subspace: $\dim H$ = # of vectors in any basis for H also = # pivots

: of Nul A: the number of free variables in $Ax = 0$

Rank A: dimension of Col A = the # of pivot collumns of A (or # of non-zero rows)

↳ Rank Theorem: $n = \text{rank } A + \dim \text{Nul } A$ or $n = \dim \text{Col } A + \dim \text{Nul } A$ (n = # collumns)

Basis of Col A = vectors which when reduced have pivot positions

Basis of Null A = write solution to $Ax = 0$ in parametric form

Basis = linearly independant

Subspace = can it contain 0 vector?

Rank = # pivot collumns

dim Col A = # pivots

dim Null A = # free variables

dim Subspace = # of vectors in basis

= $\mathbb{R}^n$ $(n-1)$ ⇐

~Transformations~

♪ Think of 'b' resulting from 'A' acting on 'x' (by multiplication) ♫.

↳ solving for 'x' is finding all the vectors 'x' in $\mathbb{R}^n$ that are transformed into 'b' under the 'action' of 'A'.

↳ 'A' is a transformation that assigns to each vector 'x' in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$

Domain = $\mathbb{R}^n$ $T: \mathbb{R}^n \to \mathbb{R}^m$ **codomain** = $\mathbb{R}^m$ ($m \times n$ matrix)

image: $T(x)$ is the image of 'x' **range**: The set of all images $T(x)$ (all linear combos in A)

Sheer: A transformation that when applied makes a parallelogram.

↳ x-sheer (stratches horizontally ▱) ↳ y-sheer (streatchs graph vertically ▱)

Linear Transformation: All matrix transformations are linear. $T(u+v) = T(u) + T(v)$

↳ They preserve vector addition & scalar multiplication. $T(cu) = cT(u)$

onto: has a pivot in every row

↳ if there is at least one solution to $T(x)=b$ & no case of no solution

one-to-one: has a pivot in every collumn

↳ if there is one unique solution for $T(x)=b$ or if there is no solution (uniqueness)

♪ To be both one-to-one & onto it must be square ♪

Identity: 'T' is completely determined by what it does to collumns of identity matrix. (e_1, e_2)

Area: 'S' = a parallelogram in $\mathbb{R}^2$ **A of T(S) = |det A| • Area of S** $(A = |\det A| \cdot |\det S|)$

↳ S is original, A is transformation

Volume: 'S' is a parallelpiped in $\mathbb{R}^3$ **V of T(S) = |det A| • Volume of S** $(V = |\det A| \cdot |\det S|)$

LinAlg Break Down

Linear Equation: $a_1x_1 + a_2x_2 + \cdots a_nx_n = b$

↳ This can be written in matrix form $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b$

when solved, $[A \vdots b]$ gives the values of $x_1, x_2, \cdots x_n$. (solution set)
→ systems of linear equations are solved this way

consistant system has 1 unique solution or infinitly many solutions
inconsistant system has no solution

Row Echelon Form: all zero rows on bottom
1st non-zero in row is to right of 1st non zero in previous row
All entries below leading entries are zeros.

Reduced Echelon Form: is in row echelon form
All leading entries are 1
Each leading entry is only nonzero in its row.

pivot a non zero in a **pivot position** in a **pivot collumn** (who's variable is called basic)

Elementary Row Operations are how to get a matrix into REF or RREF
Multiply by a constant, Add 1 row to another or to a multiple of another.
↳ At REF use back substitution or reduce to RREF to get solution set.

Questions: Solve this linear system (given 2 or more eq's)
↳ put into Augmented matrix & solve.
When there are infinite solutions
↳ give free variable new name, write everything in terms of it
When there is no solution
↳ write $0x_1 + 0x_2 + 0x_3 \cdots \neq 10$ (or whatever constant is)

This has to do w/ vectors

After all this:

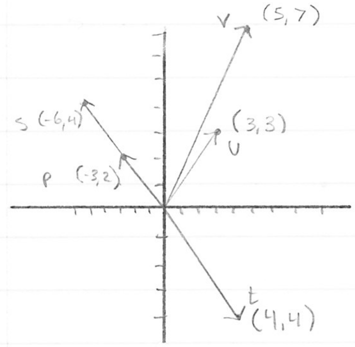
$Ax = b$

$x_1a_1 + x_2a_2 + \cdots x_na_n = b$

$[a_1 \; a_2 \; \cdots \; a_n \vdots b]$

are all exactly the same thing.

Vectors in $\mathbb{R}^2$: Points from (0,0), defined by ordered pairs: $(x,y) = \begin{bmatrix}x\\y\end{bmatrix}$ (2x1)



vectors: $\bar{v} = \begin{bmatrix}5\\7\end{bmatrix}$ $\bar{u} = \begin{bmatrix}3\\3\end{bmatrix}$ $\bar{t} = \begin{bmatrix}4\\4\end{bmatrix}$ $\bar{p} = \begin{bmatrix}-3\\2\end{bmatrix}$ $\bar{s} = \begin{bmatrix}-6\\4\end{bmatrix}$

Adding: $\bar{v} + \bar{u} = \begin{bmatrix}5\\7\end{bmatrix} + \begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}5+3\\7+3\end{bmatrix} = \begin{bmatrix}8\\10\end{bmatrix}$ can add vectors head to tail (parallelogram)

Multiply by constant: $2\bar{p} = 2\begin{bmatrix}-3\\2\end{bmatrix} = \begin{bmatrix}2(-3)\\2(2)\end{bmatrix} = \begin{bmatrix}-6\\4\end{bmatrix} = \bar{s}$

Vectors in $\mathbb{R}^3$: points from (0,0,0) defined in the 3D plane $(x,y,z) = \begin{bmatrix}x\\y\\z\end{bmatrix}$ (3x1) ordered triples

vectors in $\mathbb{R}^n$: $\bar{v} = \begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$ 'n' must be a positive integer.

zero vector: all entries are 0

properties of vectors pg 32

Linear Combination: b is a linear combination of $\bar{v}_1, \bar{v}_2, \cdots \bar{v}_p$ and $c_1, c_2, \cdots c_p$ (weights)

$$\bar{b} = c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots c_p\bar{v}_p$$

c is any real #

This can be determined by creating a matrix $\left[\, v_1 \quad v_2 \;\cdots\; v_p \;\vdots\; b \,\right]$ determine if weights exist.

If there is a solution to this vector, b is a combination of $V_1 \cdots$

Questions: Compute addition of vectors & scalar multiplications

↳ add corresponding entries, multiply c by all entries

Write a system of equations equal to a vector eq. (& visiversa)

↳ $x_1\begin{bmatrix}1\\2\end{bmatrix} + x_2\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$ => $\begin{matrix}x_1 + 3x_2 = 5\\2x_1 + 4x_2 = 6\end{matrix}$ => $\left[\begin{array}{cc|c}1 & 3 & 5\\2 & 4 & 6\end{array}\right]$ convert btw all 3

Determine if vector equation is a linear combination

↳ convert vector eq. or coeeficient matrix to augumented matrix

↳ solve, if it's consistant then yes, its a linear combo.

Determine if weights x_1 & x_2 exist such that vector eq. is true

$x_1\begin{bmatrix}1\\2\end{bmatrix} + x_2\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$ => put into matrix, consistant means yes.

Theorems

If A is an $m \times n$ matrix, these statements are all True or all false:

a) For each b in $\mathbb{R}^m$, $Ax = b$ has a solution
b) Each b in $\mathbb{R}^m$ is a linear combination of collumns of A
c) Collumns of A span $\mathbb{R}^m$
d) A has a pivot position in every row.

The invertible Matrix Theorem A is $n \times n$ (square)

a) A is invertible
b) A is row equivalent to I_n
c) Reduced EF of A has pivot in each row
d) Reduced EF of A has pivot in each collom
e) The equation $Ax = 0$ has only the trivial solution
f) Collumns of A form a linearly independant set (form a matrix w̄ vectors as collums, find if A^{-1} exists, if yes)
g) $Ax = b$ is consistant for every b in $\mathbb{R}_n$
h) The collumns of A span $\mathbb{R}_n$
i) A^T is an invertible matrix
j) There is an $n \times n$ matrix C such that $AC = CA = I$ $\quad C = A^{-1}$
k) The linear transformation $T(x) = Ax$ is one-to-one
L) The linear transformation $T(x) = Ax$ is onto

D

Homogeneous Systems: systems where every equation is equal to zero

This means there will always be a solution.

Form: $Ax = 0$

Trivial solution: $x = 0$, all variables are 0 **Nontrivial solution:** at least one x that $\neq 0$
that means has one free variable.

Questions: Determine if the system has a non trivial solution
↳ solve system and see if there is a free variable.

Span: The space the vectors cover  3D is a plane

It's the vectors $\bar{v}_1, \bar{v}_2, \cdots \bar{v}_p$ multiplied by any scalars $c_1, c_2 \cdots c_p$
↳ so span $\{v_1, v_2, \cdots v_p\} = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$

⇒ Span contains every scalar multiple of $\bar{v}$, it is the set of all scalar multiples.

⇒ The zero vector must be in the span

Questions: Is b in the span of $\bar{v}$?
↳ can the vector eq. $x_1 v_1, x_2 v_2 + \cdots x_p v_p = b$ be true?
↳ write $[v_1 \; v_2 \; \cdots \; v_p \,\vdots\, b]$ & see if it has a solution.

For what values of h is v_3 in span $\{v_1, v_2\}$
↳ asking if $[v_1 \; v_2 \,\vdots\, v_3]$ (where v_3 has an h) is consistent for an h value
↳ solve the matrix, solve h so its consistant

Determine if the system has a nontrivial solution
↳ solve to free variable (non trivial) or variable = 0 (trivial)

Write the solution to the system in parametric form:
↳ write $x = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$ where x_3 is free & $x_1 = 5x_3$, $x_2 = -2x_3$
↳ Let $x_3 = t$!! or lose marks

♪ Recall Homogeneous solutions => nontrivial & trivial solutions ♪

↳ The big question is if $c_1=0$, $c_2=0$, $c_p=0$ is the only solution...

Linearly Independent: A set of vectors in $\mathbb{R}^n$ is said to be L.I. if

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution. ($x_1=0$, $x_2=0$, ... $x_p=0$)

Linearly Dependent: A set of vectors in $\mathbb{R}^n$ is said to be L.D. if

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has a nontrivial solution (at least one of the weights $\neq 0$)

Linear Dependance Relation: If a set of vectors is L.D., there is a relationship btw the weights where the value of one or more of them determines the values of the others.

↳ to find this relationship, determine the free variable and write all others in relation to this variable. ♪ for each nonzero value of the free variable, the equation will have a non-trivial solution.

Questions: Decide if the vectors are linearly dependent / have a nontrivial sol.

↳ write $Ax=0$ as $[A:0]$ and solve ... if there's a free variable, yes!

If v_3 is in span $\{v_1, v_2\}$ & has 'h' in it, when is $[v_1\ v_2\ v_3]$ L.D.

↳ set $[v_1\ v_2\ v_3 : 0]$ & solve so there's a free variable, solve 'h'

Given [ ], [ ], [ ], vectors, determine if L.D. or not

↳ just fill into $[v_1\ v_2\ v_3 : 0]$ & solve for nontrivial solution

Now if T([ ]), T([ ]), T([ ]) is L.D.

↳ write cT([ ]) = dT([ ]) + eT([ ]) from solution above to show there is a non trivial solution (definition works for transformations)

<u>Transformations</u>: Sometimes $Ax=b$ can arise in a way that is not exactly a linear combination of vectors.

↳ Think of matrix 'A' as 'acting' on vector 'x' by multiplication to produce Ax or, 'b'.

↳ IN SHORT: Multiplication by A transforms x into b

so, solving $Ax=b$ is like finding all the vectors x in $\mathbb{R}^n$ that are transformed into b in $\mathbb{R}^m$ under the 'action' of A.

$$\overrightarrow{[A]}\Big\downarrow \underset{\mathbb{R}^n}{[x]} = \underset{\mathbb{R}^m}{[b]}$$

A is a transformation (or function) that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

<u>Domain</u>: The set $\mathbb{R}^n$ is called the domain of T
<u>Codomain</u>: $\mathbb{R}^m$ is called the codomain of T

$$T: \underset{\text{(Domain)}}{\mathbb{R}^n} \to \underset{\text{(codomain)}}{\mathbb{R}^m}$$

<u>image</u>: the vector $T(x)$ in $\mathbb{R}^m$ is the image of x in $\mathbb{R}^n$
<u>Range</u>: The set of all images $T(x)$ is the Range of T

$T(x)$ is computed as Ax for each x in $\mathbb{R}^n$. A is $m \times n$ matrix

⇒ ♪ Domain T is $\mathbb{R}^n$ when A has n columns

♪ codomain of T is $\mathbb{R}^m$ when A has m rows.

♪ range T is set of all linear combinations of the columns of A

↳ because each image $T(x)$ is of form Ax

Questions: Find $T(u)$, the image of u under transformation T (given 'A', u)

↳ Note that $T(u) = Au$ so multiply A by u for answer

Find an x in $\mathbb{R}^2$ whose image under T is b (given 'A', b)

↳ create matrix $[A \mid b]$ and solve for set (x).

Is there more than one x for ↑

↳ If there is a free variable, yes, if not, no

Determine if c is in the range of the transformation T

↳ $[A \mid c]$ → if there is a solution, it is.

x-sheer or horizontal sheer $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + ay \\ y \end{bmatrix}$

↳ streatches graph horizontaly ← visual pg 86

y-sheer or vertical sheer $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ ax+y \end{bmatrix}$

↳ streutches graph vertically ♪ one vector moves, one doesn't ♪

Linear Transformations: Properties of matrix equation say:

$A(u+v) = Au + Av$ & $A(cu) = cAu$

↳ when it comes to Transformations, they are **linear** if:

$$T(u+v) = T(u) + T(v) \qquad T(cu) = cT(u)$$

Thus we say that linear transformations preserve the operations of vector addition and scalar multiplication

Also true:

$$T(0) = 0 \qquad T(cu + dv) = cT(u) + dT(v)$$

This can produce the generalized: $T(c_1v_1 + \cdots c_pv_p) = c_1T(v_1) + \cdots c_pT(v_p)$

Questions: Show that T is a linear transformation

↳ write $T(cu + dv) = \frac{\text{Actual}}{\text{Transformation}}(cu + dv)$ Also show $T(0) = 0$

& solve it out untill it shows $cT(u) + dT(v)$

D

Rotation Matrix rotates the vector about the origin by angle θ.
(counter clockwise)

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ given θ, the standard matrix of the transformation can be determined

Question: Given θ, write the standard matrix for the transformation & solve for the new vector.
↳ Fill θ into matrix, solve, write $[A][x]$ & multiply it out.

visuals of transformations are on pg 85 of the text book

onto A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each b in $\mathbb{R}^m$ is the image of at least one x in $\mathbb{R}^n$.
↳ If there is at least one solution to $T(x) = b$, T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ for each b in $\mathbb{R}^m$.
T is not onto when there is some b in $\mathbb{R}^m$ where $T(x) = b$ has no solution

one-to-one A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each b' in $\mathbb{R}^m$ is the image of at most one x in $\mathbb{R}^n$.
↳ If there is 1 unique solution or no solution for $T(x) = b$, T is one-to one
↳ T is not one-to-one if some b in $\mathbb{R}^m$ is the image of more than 1 vector in $\mathbb{R}^n$

onto: A matrix that is onto has a pivot in every row
one-to-one: A matrix that is one-to-one has a pivot in every collum

A matrix that is both onto and one-to-one must be square!

Questions: Given a matrix, is it one-to-one or onto?
↳ look @ the pivots in rows & collums
↳ every row = onto every collum = one-to-one both = both

Theorem 10 ♪ T is completely determined by what it does to the collums of the $m \times n$ identity matrix I_n ♪

$$\Rightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{so} \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\therefore$ T is defined by what it does to e_1 and e_2

$\therefore$ $T(x) = Ax$ & $A = [\,T(e_1) \cdots T(e_n)\,]$

Proof: $x = [e_1 \cdots e_n] \Rightarrow x_1 = x_1 e_1 + \cdots x_n e_n$

using $T \Rightarrow$ $T(x) = T(x_1 e_1 + \cdots x_n e_n) = x_1 T(e_1) + \cdots + x_n T(e_n)$

$$\therefore \quad [\,T(e_1) \cdots T(e_n)\,] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Ax \qquad A \text{ is standard matrix for } T.$$

Questions: Given $T(e_1) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $T(e_2) = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$, write a formula for the image of x in $\mathbb{R}^2$

↳ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 e_1 + x_2 e_2$

$\therefore$ $T(x) = x_1 T(e_1) + x_2 T(e_2)$

↳ Fill in e_1 & e_2 under transformation & solve to $\begin{bmatrix} ax_1 + dx_2 \\ bx_1 + ex_2 \\ cx_1 + fx_2 \end{bmatrix}$ where A is coefficients $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$.

♪ given $e_1 = (a\ b\ c)$ $e_2 = (d\ e\ f)$ ... $A = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$

Given $T(e_1) = [\ ]$ & $T(e_2) = [\ ]$ & $T(e_3) = [\ ]$

Find the image of $[\ ]$ under T

↳ say $T([\ ]) = T(c\overset{e_1}{[\ ]} + d\overset{e_2}{[\ ]})$ where c & d make the $[\ ]$ vector & $\overset{e_1}{[\ ]}$ & $\overset{e_2}{[\ ]}$ are pieces of the identity matrix so you can assume $cT(\overset{e_1}{[\ ]}) + dT(\overset{e_2}{[\ ]}) = [\ ]$ and you know $T(e_1)$ & $T(e_2)$ so you can calculate.

Is T onto or one-to-one or both or none?

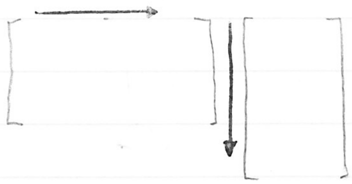
↳ row reduce $A = [e_1\ e_2\ e_3]$ to see if it has pivot in every row & or collum

<u>Notation</u>: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the (1, 1) entry is a, the (1, 2) entry is c etc... the main diagonal is (ad).

<u>Sum</u>: The matrices must be same size ($m \times n$) Add corresponding entries
↳ same goes for difference.

D

<u>Product</u>



$2 \times 3 \quad 3 \times 2$
give size of result

∫ $AB \neq BA$ always
but if $AB = AC$
$B \neq C$ always ∫

<u>Transpose</u>: The transpose is gained by making the collums of A the rows of A^T

ex/ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A \quad \begin{bmatrix} a & c \\ b & d \end{bmatrix} = A^T$

D

<u>Theorems</u>: $(A^T)^T = A$

$(A+B)^T = A^T + B^T$ These can be easily proven

$(rA)^T = rA^T$

$(AB)^T = B^T A^T$

<u>Powers of Matrices</u> $A^2 = A \cdot A$, $A^3 = A^2 \cdot A$ etc...

<u>Identity matrix</u> $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ etc for I_3 (REF basically)

<u>Inverse of a Matrix</u> A matrix A is **invertible** if there exists a C such that

$$AC = I \qquad I = CA$$

Proper notation says A's inverse is A^{-1} so: ∫ $A^{-1} = C$ ∫

$$A^{-1}A = I \qquad AA^{-1} = I$$

<u>2x2</u> The inverse of a 2x2 matrix can be calculated using the **determinant**

$\det A = ad - bc$ which $\neq 0$ & if it doesn't then

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

<u>Solving Systems w/ A^{-1}</u> if A is (an invertible) $m \times n$ matrix then $Ax = b$ can be solved to $x = A^{-1}b$ by multiplying both sides by A^{-1}.

♪ $(A^{-1})^{-1} = A$, $(AB)^{-1} = B^{-1}A^{-1}$, $(A^T)^{-1} = (A^{-1})^T$ ♪

<u>3x3</u> inverting these is more complicated.. must use I_3

$$\begin{array}{ccc} [A & \vdots & I] \\ \downarrow & & \downarrow \\ I & & A^{-1} \end{array}$$

if you reduce A to I then the same operations will reduce I to A^{-1}!

Questions: Be able to preform all these things.

<u>Elementary Matrix</u> is obtained by performing a single elementary row operation on an identity matrix.
↳ ex/ Adding 1 row to another, interchanging rows, multiply a row by a constant.
$E \cdot E^{-1} = I$ They are invertible!

<u>Properties of Invertible Matrices</u>
$(A^{-1})^{-1} = A$ $(AB)^{-1} = B^{-1}A^{-1}$ $(kA)^{-1} = \frac{1}{k}A^{-1}$ $(A^T)^{-1} = (A^{-1})^T$

Questions:

D

<u>Subspace</u> = a set of vectors in $\mathbb{R}^n$ (called H) which has these:

1) The **zero vector** is in H
2) for each u & v in H, $u+v$ is in H (subspace is closed under addition)
3) for each u in H, cu is in H (subspace is closed under scalar multiplication)

proove; $u = s_1v_1 + s_2v_2$ $\quad v = t_1v_1 + t_2v_2$ $\qquad u+v = (s_1+t_1)v_1 + (s_2+t_2)v_2$

if v_1 & v_2 are in $\mathbb{R}^n$, $H = \text{span}\{v_1, v_2\}$

A line through the origin is a subspace (if v_2 is a multiple of v_1)

$\mathbb{R}^n$ is a subspace of itself (satisfies conditions)

zero subspace = the set containing only the zero vector

span$\{v_1 \cdots v_p\}$ is the subspace **spanned** by $v_1 \cdots\cdots v_p$.

<u>Collumn Space</u> of a matrix A is the set of all linear combinations of the collums of A (called **Col A**)

↳ $A = [a_1 \cdots a_n]$, Col A is span$\{a_1 \cdots a_n\}$ so: **Col A** is a **subspace** of $\mathbb{R}^n$

↳ Col A is defined explicitly because vectors can be constructed by linear combination.

<u>Null space</u> of a matrix A is the set of all solutions to the equation $Ax=0$ (called **Nul A**)

↳ Solutions of $Ax=0$ belong to $\mathbb{R}^n$, so null space A is a subset of $\mathbb{R}^n$ (has subspace properties)

Proof => $A0=0$, $A(u+v) = Au+Av = 0+0=0$, if $Au=0$ & $Av=0$ (rule of null space)

=> Is v in Nul A? compute Av, does it equal 0?

Nul A must be checked for each vector so its define implicitly

↳ express answer in parametric vector form to give emplicit description

<u>Basis</u> a basis for a subspace H is a <u>linearly independant</u> set in H that spans H

↳ collumns of an invertible $n \times n$ matrix form a basis (linearly independant & span $\mathbb{R}^n$)

↳ set $\{e_1 \cdots e_n\}$ is standard basis for $\mathbb{R}^n$

writing $Ax=0$ in parametric form actually identifies a basis for Nul$A$

? write all frees in terms of pivots, these pivots now span Col$B$

Questions => Prove it's a subspace if $H = \text{span}\{e_1, e_2\}$ given e_1, e_2

↳ show $0 = \begin{bmatrix}0\\0\\0\end{bmatrix} = 0[e_1] + 0[e_2]$, show 0 is in span H

↳ show $u = a[e_1] + b[e_2]$, $v = c[e_1] + d[e_2]$ & show $u+v = (a+c)[e_1] + (b+d)[e_2]$

↳ show cu & cv equal $ca[e_1] + cb[e_2]$

$\Rightarrow$ Is b in the collumn space of A? given A and b

↳ see if $Ax=b$ is consistant (matrix & solve)

$\Rightarrow$ Is w in the subspace generated by v_1 & v_2, given v_1 v_2 & w

↳ Make $[v_1\ v_2 \,\vdots\, w]$, if consistant, yes, if inconsistant, no.

$\Rightarrow$ Is p in Col A? given v_1, v_2, v_3 p

↳ Make $[v_1, v_2\ v_3 \,\vdots\, p]$ if consistant, yes, if inconsistant, no.

? $\Rightarrow$ Is p in Nul A? given A & p

↳ solve $0 = p_1v_1 + p_2v_2 + p_2v_3$ where v are collumns of A, if $=0$ yes, if $\neq 0$, no.

? $\Rightarrow$ Is given sets a bases for $\mathbb{R}^2$ or $\mathbb{R}^3$?

↳ Make $A = [\text{sets}]$ & if it's invertible then yes. (memorize IMT)

$\Rightarrow$ Find a basis for Nul A, given A

write solution to $Ax=0$ in parametric vector form, the vectors are basis for Nul A

$\Rightarrow$ Find a basis for Col A, given A

reduce $\Rightarrow$ The pivot collumns form a basis for Col A

D

<u>Dimension of a subspace</u> dim H is the number of vectors in any basis for H

↳ or, if $\mathbb{R}^n$ has dimension n, every basis for $\mathbb{R}^n$ has n vectors.

<u>Dimension of Nul A</u> is the number of free variables in $Ax = 0$

♪ dimension of the zero subspace is 0 ♪

<u>Rank of A</u> rank A is the dimension of the collumn space of A

↳ The # of non-zero rows in echelon form of A.

↳ since pivot collumns of A form a basis for Col A, rank A = # of pivot collumns of A.

<u>The rank Theorem</u> if matrix A has n collumns,

$$n = \text{rank}\,A + \dim \text{Nul}\,A \quad \text{or} \quad \dim \text{Col}\,A + \dim \text{Nul}\,A = n$$

<u>The Basis Theorem</u> Any L.I. set of 'p' elements in H is a basis for H

Any set of 'p' elements of H that spans H is a basis for H.

<u>Coordinate Vector</u> if H is a subspace & $\mathcal{B} = \{v_1, v_2, \dots v_n\}$ is a basis for H

$[x]_\mathcal{B} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ is the coordinate vector of x relative to basis B. ♪ $x = c_1v_1 + c_2v_2 + \cdots c_nv_n$

Questions: Determine the dimension of subspace H of $\mathbb{R}^3$ spanned by v_1, v_2 & v_3 (given v_1 v_2 v_3)

↳ The basis is $[v_1\ v_2\ v_3]$ and then reduce it untill # of pivot collumns is known (dim H = pivot coll.)

Determine the rank of the matrix A, give A.

↳ row reduce ⇒ rank A = pivot collumns or # of non-zero rows.

Given A, verify rank theorem

↳ row reduce A ⇒ rank A = # pivot collumns

↳ w/ reduced A = 0, you'll have # of free varriables (non-pivot collumns) Nul A = those vectors

↳ dim Nul A is # of free variables so n = rank A + dim Nul A can be checked.

Given B = [1], [2] Find the coordinate vector of x = [ ] relative to basis B

↳ write $[x] = c_1[1] + c_2[2]$ so $[1\ 2\ \vdots\ x]$ gives c_1, c_2 $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ = coordinate vector.

Given B = [1], [2] and $[y]_\mathcal{B}$ = [#'s] Find y.

↳ write c_1 from y $[1] + c_2[2] = [y]$ & easily solved for.

<u>determinents</u> : 2×2 = $ad - bc \neq 0$

3×3 => $\sum_{j=1}^{n} (-1)^{1+j} a_{ij} \det(A_{ij})$

So... $\det A = (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12}) + \cdots + (-1)^{1+n} a_{1n} \det(A_{1n})$

where A_{11} = A w/out its 1^{st} row or collumn,
A_{12} = A w/out its 1^{st} row or 2^{nd} collumn,
A_{13} = A w/out its 1^{st} row or 3^{rd} collumn. etc...

<u>Another way</u> 3×3 =

multiply multiply

$$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} \begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

d e f a b c

$(a+b+c) - (d+e+f)$

<u>upper triangular</u> a square matrix w/ all entries below main diagonal as zero
<u>lower triangular</u> a square matrix w/ all entries above main diagonal as zero
<u>triangular</u> if a matrix is either upper or lower triangular

♪ If A is triangular, $\det A$ is the product of the entries on the main diagonal ♪

<u>Determinent Properties</u>

$\det(A) = \det(A^T)$
$\det(AB) = \det(A) \cdot \det(B)$
$\det(A^{-1}) = \frac{1}{\det A}$ (if A is invertible)
$\det(kA) = k^n \det A$

book says no. ? $\det A = \det B$ if B is obtained by interchanging 2 different rows of A.
$\det A = 0$ if A has 2 identical rows or collumns

♪ A is invertible <=> $\det A \neq 0$
♪ $Ax = 0$ has non-trivial solutions <=> $\det A = 0$

Questions: be able to find determinents & work w/ properties

Cramer's Rule $\quad x_i = \dfrac{\det A_i(b)}{\det A} \quad i = 1, 2, \ldots n$

where $A_i(b)$ is the matrix A w/ b in the place of collumn i
↳ repeat untill all collumns are used $[x_1\ x_2\ x_3$ etc...$]$

ex/ $A = \begin{bmatrix} a & c \\ d & e \end{bmatrix}$ $b = \begin{bmatrix} f \\ g \end{bmatrix}$ $\boxed{Ax = b} \Rightarrow A_1 b = \begin{bmatrix} f & c \\ g & e \end{bmatrix}$ $A_2 b = \begin{bmatrix} a & f \\ d & g \end{bmatrix}$

$x_1 = \dfrac{\det A_1(b)}{\det A}$ $\quad x_2 = \dfrac{\det A_2(b)}{\det A}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

A^{-1} using Cofactors a cofactor of an entry is given by: $C_{ij} = (-1)^{i+j} \det(A_{ij})$
so if we take the cofactor of each entry we make a cofactor matrix (remember $\det(A_{ij})$ if found by finding the determinent of A w/ the i^{th} row & j^{th} collumn crossed out) &:
Taking the transpose of the cofactor matrix you get the adjoint matrix

$A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$ $\det A \neq 0$ & $\operatorname{adj}(A) = [\text{cofactor matrix}]^T$

Finding Area & Volume: given A is a 2×2 matrix, Area $= |\det A|$ gives the area of the parallelogram. (collumns of A are the vectors)
: given A is a 3×3 matrix, Volume $= |\det A|$ gives the volume of the parallelepiped. (collumns of A are the vectors)

With Transformations: If 'S' is a parallelogram in $\mathbb{R}^2$ Then the area of $T(S)$ is:
Area of $T(S) = |\det A|$ (area of S) where 'S' is the original matrix and A is the transformation $T(x) = Ax$ applied.

If 'S' is a parallelpiped in $\mathbb{R}^3$, The volume of $T(S)$ is:
Volume of $T(S) = |\det A|$ (area of S) where 'S' is the original matrix and A is the transformation $T(x) = Ax$ applied.

Question relating to Cramer's, coefactors Area/volume ū transf.

D

Determinents and Row Operations

- you can row reduce a matrix before computing the derivative as long as the row reductions are legal.
  - The operations affect the determinent:
    - regular row operations => no effect ex/ $R_1' = R_1 - 2R_2$
    - Multiply a row by a scalar => Multiply determinent by inverse ex/ $R_1 \cdot a \Rightarrow \frac{1}{a} \cdot \det$
    - swap 2 rows => Multiply determinent by -1 ex/ $R_1 \leftrightarrow R_2 \Rightarrow -\det$

D

ex/ $\det \begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} \underset{R_1' = \frac{1}{2}R_1}{=} 2\det \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \underset{R_2' = R_2 - 3R_1}{=} 2\det \begin{vmatrix} 1 & 2 \\ 0 & -5 \end{vmatrix}$

$= 2(1 \cdot -5) = -10$

Eigenvector if A is $n \times n$ matrix, a non-zero vector x such that $Ax = \lambda x$ for some scalar λ. 'x' is an eigenvector of A

Eigenvalue λ is the eigenvalue of A & is coorsponding to eigen vector x.

Results λ is eigen value of $A \iff Ax = \lambda x$ for some vector x

& $\iff Ax - \lambda x = 0 \iff (A - \lambda I)x = 0 \iff \det(A - \lambda I) = 0$

characteristic equation of A $\det(A - \lambda I) = 0$ when solved for λ will give the **eigenvalues** of A.

♪ $\det(A - \lambda I) = 0$ will give a polynomial of λ with degree n. ♪
characteristic polynomial

~~eigen~~

eigenspace the set of all solutions 'x' of $(A - \lambda I)x = 0$ coorsponding to each λ value. This is how we find the **eigenvectors**

♪ $\mathrm{Null}(A - \lambda I)$ = eigenspace of A.

♪ if 0 is an eigenvalue of A, A is not invertible! ♪

- The set of eigenvectors $\{v_1, v_2 \ldots v_n\}$ coorsponding to distinct eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_n$ form a **linearly independant** set.

<u>Diagonalization</u> $\overset{n \times n}{A}$ is diagonalizable in 'A' has n linearly independant eigenvectors

$A = PDP^{-1}$ $P = [\text{eigenvectors of } A]$ $D = [\diagdown]$ eigenvalues of A

↳ to show A is diagonalizable, show $AP = PD$

∫ nxn matrix A is diagonalizable <=> it has n linearly independant eigenvectors
" " a basis consisting of eigenvectors of A

An nxn matrix w n distinct eigenvalues is diagonalizable.

D

Complex #'s is a # of the form $a + ib$ where $i = \sqrt{-1}$, $i^2 = -1$

↳ 'a' is the real part $Re(z) = a$

↳ 'b' is called the imaginary part $Im(z) = b$

Conjugate $\bar{z}$ $z = a + ib$ $\bar{z} = a - ib$

$z + \bar{z} = 2Re(z) \cdots 2a$ $\bar{\bar{z}} = z$

$z - \bar{z} = 2i\,Im(z) \cdots 2ib$ $z \cdot \bar{z} = a^2 + b^2$

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

$\overline{z_1 \div z_2} = \bar{z}_1 \div \bar{z}_2$

Addition / Subtraction just add & subtract like terms

Multiplication Multiply out like a, ib are variables $(a + ib)(a + ib) \Rightarrow (x+y)(x+y)$

Devision Multiply & devide by conjugate of denom. expand & simplify.

Absolute Value $|z| = \sqrt{a^2 + b^2}$ $|z| = \sqrt{z \cdot \bar{z}}$ & $|zw| = |z||w|$

Polar form $a = |z|\cos\theta$ $b = |z|\sin\theta$ find the angle $\theta = \cos^{-1}(a/|z|)$

↳ $a + ib = |z|(\cos\theta + i\sin\theta)$ $\theta = \sin^{-1}(b/|z|)$

De Moiver's Theorem

$$z^n = |z|^n(\cos(n\theta) + i\sin(n\theta))$$

D

Complex Eigenvalues → when solving $\det(A-\lambda I)=0$ ... what happens if λ is a complex #?

λ = complex eigenvalue } if $\det(A-\lambda I)=0$ gives λ
x = complex eigenvector } and λ satisfies $Ax=\lambda x$

↳ ∫ $x = Re(x) - i\,Im(x)$ then $\bar{x} = Re(x) + i\,Im(x)$ conjugates = conjuagets.

↳∫ if A has eigenvalue λ (complex) w/ eigenvector x (complex) then:

$\bar{\lambda}$ is another eigenvalue & $\bar{x}$ is it's eigenvector (in A).

Questions: Given A w/ knowledge of complex eigenvalues

↳ solve $\det(A-\lambda I)=0$ for complex eigenvalues

↳ solve $(A-\lambda_n I)x=0$ w/ all λ's for all vectors 'x'

⇒ To diagonalize D= P= $AP=PD$

Inner Product (dot product) ⇒ $U \cdot V = U^T V$ or multiply coorsponding elements and take sum.

properties

⊙ $U\cdot V = V \cdot U$ ⊙ $(U+V)\cdot W = U\cdot W + V\cdot W$ ⊙ $(CU)V = C(U\cdot V) = U(CV)$ ⊙ $U\cdot U \geq 0$ [only =0 if U=0]

Length (norm) $\|v\| = \sqrt{v\cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2}$ and thus $\|v\|^2 = v\cdot v$

↳ unit vector

A vector with length 1

normalizing v (creating a unit vector from 'v')

creating a vector 'u' by multiplying V by $\frac{1}{\|v\|}$ ⇒ u is in the same direction as 'v'

-u is another unit vector

Distance the distance btw u & v is: $\text{dist}(u,v) = \|u-v\|$

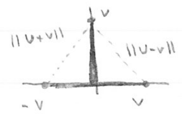
Questions: Compute Inner product, norm & distance of given vectors.

↳ just plug in and solve... same w/ finding unit vectors.

D

D

Orthogonal Vectors How to determine if 2 lines are perpendicular?



The lines u & v are perpendicular (orthogonal) if the squares of the dotted lines are equivalent...

$$[\text{dist}(u,-v)]^2 = \|u+v\|^2$$
$$= (u+v)\cdot(u+v)$$
$$= u\cdot(u+v) + v\cdot(u+v)$$
$$= u\cdot u + u\cdot v + v\cdot u + v\cdot v$$
$$= \|u\|^2 + \|v\|^2 + 2u\cdot v$$

→ similar calculations show that
$$[\text{dist}(u,v)]^2 = \|u\|^2 + \|v\|^2 - 2u\cdot v$$

♪ The only way the 2 distances equal is if $u\cdot v = 0$

so: 2 vectors are orthogonal ($u \perp v$) if $u\cdot v = 0$ or if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

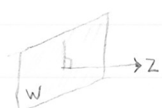
♪ The <u>zero vector</u> is orthogonal to all others because $0^T v = 0$ ♪

Orthogonal Complements W = a subspace in $\mathbb{R}^n$ z = a vector

=> if z is orthogonal (perpendicular) to every vector in W, z is orthogonal to W.

z is the <u>orthogonal compliment</u> of W. (called $W^\perp$)

♪ Think of a plane W and a line z such that z is perpendicular to W.



z is always perp. to W so z is orthogonal compliment.

Questions: Determine if 2 vectors are orthogonal

↳ does $u\cdot v = 0$? if yes, yes!

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The set is also linearly independant

Orthogonal Set let $S = [u_1, u_2\ u_3]$ where u_i are vectors
↳ 'S' is an orthogonal set if every vector combination is 'S' is orthogonal

$u_i \cdot u_j = 0 \quad i \neq j.$ so $u_1 \cdot u_2 = 0 \quad u_1 \cdot u_3 = 0 \quad u_2 \cdot u_3 = 0$

Orthonormal set when each vector in an orthogonal set has length 1. (unit vectors)

Orthogonal Basis A basis consisting of (orthogonal set) orthogonal vectors.

Subspace spanned by orthogonal set has orthogonal basis. (linear independance)

orthonormal basis A basis consisting of orthonormal vectors.

∫ A matrix has orthonormal collumns (is an orthonormal set) <=> $U^T U = I$

Properties or orthonormal matrices

- $\|Ux\| = \|x\|$
- $(Ux) \cdot (Uy) = x \cdot y$
- $(Ux) \cdot (Uy) = 0 \iff x \cdot y = 0$

orthogonal matrix => square, invertible matrix such that $U^T U = I$
↳ collumns of this matrix are orthonormal (not just orthogonal!)

∫ $c_j = \dfrac{x \cdot u_j}{u_j \cdot u_j}$ if $\{u_1, u_2, \dots u_n\}$ is an orthogonal basis, the 'c' gives
$x = c_1 u_1 + c_2 u_2 + \cdots c_n u_n$ constants for linear combo

Questions: are given vectors an orthonormal set?
↳ check if $u \cdot v = 0$ for every possible pair, if yes, yes!

Show that $\{u, v\}$ is an orthogonal basis
↳ check all pairs for $u \cdot v = 0$

Express 'x' as a linear combination of $\{u, v\}$
↳ compute $c_j = \dfrac{x_j \cdot u_j}{u_j \cdot u_j}$ for each constant. $x = c_1 u_1 \dots$

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Orthogonal Projection decomposing one vector (y) into:

An orthogonal projection of 'y' onto 'u' => $\boxed{\hat{y} = \frac{y \cdot u}{u \cdot u} u}$ (called $\text{proj}_u\, y$) &

A component of 'y' orthogonal to 'u' => $\boxed{z = y - \hat{y}}$ such that

$$y = \hat{y} + z$$

& $\hat{y} = \alpha u$ for any scalar α so the projection is onto the subspace spanned by 'u' (the line or plane through 0) => vector 'u', subspace W spanned by u => $\hat{y} = \text{proj}_W\, y$

=> All calculations are the same

~~& $\hat{y}$ is in W & z is in $W^\perp$ (as $z \perp u$)~~

orthogonal decomposition write the expression $y = \hat{y} + z$ w/ calculated vectors.

Questions: Calculate the orthogonal projection of x onto u (give x, u)

↳ $\hat{x} = \frac{x \cdot u}{u \cdot u} u$ ... $\hat{x}$ is $\text{proj}_u x$

Decompose vector y into sum of vector in span{u} & vector ⊥ to u

↳ $y = \hat{y} + z$, $\hat{y}$ above ... $z = y - \hat{y}$ ... write as sum.

Projection of a vector onto a subspace

W is a subspace w/ orthogonal basis $\{u_1, u_2 \cdots u_n\}$, x is a vector

Orthogonal projection of x onto W is

$$\text{proj}_W\, x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots \frac{x \cdot u_n}{u_n \cdot u_n} u_n$$

& $\hat{x}$ is in W

z is in $W^\perp$ ($z \perp$ to u)

Angle between 2 vectors $\cos\theta = \frac{x \cdot y}{\|x\|\|y\|}$ Solve for θ

$u \cdot v$ (inner)

$\|u\| = \sqrt{v \cdot v}$ (length)

$\frac{1}{\|v\|} v$ (normalize)

$\|u - v\|$ (distance)

$\cos\theta = \frac{x \cdot y}{\|x\|\|y\|}$ (angle)

$x_i = \frac{\det A_i(b)}{\det A}$ (cramer)

$|\det A|$ (area/volume)

$[x]_B \begin{bmatrix} c_1 \\ c_n \end{bmatrix}$ (coordinate vector)

$\det A = ad - bc$ (2x2)

$\sum_{j=1}^{n} (-1)^{1+j} a_{ij} \, de(A_{ij})$ (square nxn)

$C_{ij} = (-1)^{i+j} \det(A_{ij})$ (coefactors) $\Rightarrow A^{-1} = \frac{adj(A)}{\det A}$ (inverse) $\Rightarrow C^T$

$r(\cos\theta + i\sin\theta)$ (polar form) $\Rightarrow r^n(\cos(n\theta) + i\sin(n\theta))$ (Demoivers theorem)

$|z| = \sqrt{a^2 + b}$

$i^2 = -1$

$\det|(A - \lambda I)| = 0$ (eigenvalues)

$(A - \lambda_n I)x = 0$ (eigenvectors)

$A = PDP^{-1}$

$A^n = P\begin{bmatrix} \lambda^n & 0 \\ 0 & \lambda^n \end{bmatrix}P^{-1}$

$AP = PD$ (diagonalizable)

$u \cdot v = 0$ (orthogonal)

$z_1 \cdot v_1 = 0$

$z \cdot v_2 = 0$ (orthogonal compliment)

$U^T U = I$ (orthogonal matrix)

$c_j = \frac{x \cdot u_j}{u_j \cdot u_j}$ (gives linear constants)

$\hat{y} = \frac{y \cdot u}{u \cdot u} u$ (orthogonal proj. y onto u)

$z = y - \hat{y}$

$y = \hat{y} + z$ (decomposition)

$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 \dots$ (projection of $\hat{y}$ onto subspace $W = (u_1, u_2)$)

Basis Col A = vectors of pivots (in reduced)

Basis Null A = solution to $Ax = 0$ in parametric form

dim Col A = # of pivots (in reduced)

dim Null A = # free variables (reduce)

Subspace = can it contain 0 vector?

Basis = linearly independant

dim Subspace = # vectors in basis

$\mathbb{R}^n$ $(n-1)$ (1 less than Dimension)

Rank A = # pivots (in reduced)

n = rank A + dim Null A

$|\det A|$ - Area S (Area/Volume in Transformation)

one-to-one = every collumn

onto = every row

$T(x) = Ax$

Formula's *party*