

MATH3705 A — Test 1: Wed. 2:35pm–3:25pm, Jan 23

Name and Student Number:

Total points: 20. No partial marks for Questions 1-4.

Closed book! Non-programmer calculators are allowed!

[2] 1. $L\{t \sin 3t\} =$

(a) $\frac{3s}{(s^2+3)^2}$ (b) $\frac{3s}{s^2+9}$ (c) $\frac{2 \sin s}{(s^2+9)^2}$ (d) $-\frac{2e^s}{(s^2+3)^2}$ (e) $\frac{6s}{(s^2+9)^2}$

Solution: (e)

$$L\{\sin 3t\} = \frac{3}{s^2+9}, \Rightarrow L\{t \sin 3t\} = -\left(\frac{3}{s^2+9}\right)' = \frac{6s}{(s^2+9)^2}.$$

[2] 2. Find $L\{f(t)\}$, where $f(t) = 3t^2 - e^{-3t} \cos(4t)$.

(a) $\frac{12}{s^2} - \frac{(s+3)}{(s+3)^2+4}$ (b) $\frac{6}{s^3} - \frac{(s-3)}{(s-3)^2+16}$ (c) $\frac{12}{s^3} - \frac{(s-3)}{(s-3)^2+16}$
(d) $\frac{6}{s^3} - \frac{(s+3)}{(s+3)^2+16}$ (e) $\frac{3}{s^3} - \frac{(s+3)}{(s+3)^2+16}$

Solution: (d)

By linearity of LT and the First Shift Theorem, we have

$$F(s) = \frac{3(2!)}{s^3} - \frac{(s+3)}{(s+3)^2+16}.$$

[2] 3. Let $f(t)$ be 2-periodic for $t \geq 0$, and $f(t) = \begin{cases} 1, & 0 \leq t < 1; \\ 0, & 1 \leq t < 2. \end{cases}$ Find $L\{f(t)\}$.

(a) $\frac{-s(1-e^{-s})}{1-e^{-2s}}$ (b) $\frac{1-e^{-s}}{s(1-e^{-2s})}$ (c) $\frac{s(e^{-s}-e^{-2s})}{1-e^{-2s}}$ (d) $\frac{-e^{-s}+e^{-2s}}{s(1-e^{-2s})}$ (e) $\frac{e^{-s}-e^{-2s}}{s(1-e^{-2s})}$

Solution: (b)

$$\begin{aligned} L\{f(t)\} &= \frac{1}{1-e^{-2s}} \int_0^2 f(t)e^{-st} dt = \frac{1}{1-e^{-2s}} \int_0^1 e^{-st} dt \\ &= \frac{1}{1-e^{-2s}} \frac{e^{-st}}{-s} \Big|_{t=0}^1 = \frac{1-e^{-s}}{s(1-e^{-2s})}. \end{aligned}$$

[2] 4. Calculate $L\{4t\sqrt{t}\}$.

(a) $\frac{3\sqrt{\pi}}{s^{5/2}}$ (b) $\frac{3\pi}{s^{5/2}}$ (c) $\frac{3}{\sqrt{\pi}s^{5/2}}$ (d) $\frac{\sqrt{\pi}}{3s^{5/2}}$ (e) $\frac{\pi}{s^{5/2}}$

Solution: (a)

$$\begin{aligned}L\{4t\sqrt{t}\} &= L\{4t^{3/2}\} = \frac{4\Gamma(\frac{5}{2})}{s^{5/2}} \\ &= \frac{3\sqrt{\pi}}{s^{5/2}} = \frac{3\sqrt{\pi}}{s^{5/2}}.\end{aligned}$$

[7] 5. Using the Laplace transform solve the differential equation

$$f'' + f' - 6f = 70e^{4t}$$

with boundary conditions $f(0) = f'(0) = 0$.

Solution: Applying Laplace transform to the two sides, we get the equation

$$s^2F + sF - 6F = \frac{70}{s - 4}$$

or

$$F(s) = \frac{70}{(s - 4)(s + 3)(s - 2)}. \quad (2\text{points})$$

By partial fraction

$$\begin{aligned}\frac{70}{(s - 4)(s + 3)(s - 2)} &= \frac{A}{s - 4} + \frac{B}{s + 3} + \frac{C}{s - 2} \\ 70 &= A(s + 3)(s - 2) + B(s - 4)(s - 2) + C(s + 3)(s - 4).\end{aligned}$$

To find A, B, C , we use special s values: $s = 4$ gives $A = 5$, and $s = -3$ gives $B = 2$, and $s = 2$ gives $C = -7$. Putting all this together we have

$$F(s) = \frac{5}{s - 4} + \frac{2}{s + 3} + \frac{-7}{s - 2}, \quad (2\text{points})$$

Thus

$$f(t) = 5e^{4t} + 2e^{-3t} - 7e^{2t}. \quad (3\text{points})$$

[5] 6. Find $L^{-1} \left\{ \frac{s+4}{s^2+4s+5} \right\}$. Then use it to find $L^{-1} \left\{ \frac{(s+4)e^{-3s}}{s^2+4s+5} \right\}$.

Solution: Let $G(s) = \frac{s+4}{s^2+4s+5}$. We have

$$G(s) = \frac{s+4}{s^2+4s+5} = \frac{s+2}{(s+2)^2+1} + \frac{2}{(s+2)^2+1}. \quad (2points)$$

Thus

$$g(t) = e^{-2t} \cos(t) + 2e^{-2t} \sin(t). \quad (1point)$$

By the Second Shift Theorem,

$$\begin{aligned} L^{-1} \left\{ \frac{(s+4)e^{-3s}}{s^2+4s+5} \right\} &= u(t-3)g(t-3) \\ &= u(t-3) [e^{-2(t-3)} \cos(t-3) + 2e^{-2(t-3)} \sin(t-3)]. \end{aligned} \quad (2points)$$