Solutions - Final Exam - 2009

Question 1- What are the horizontal and vertical asymptotes of the function $y = \frac{2x^3-2}{x^3-x^2}$?

A) H.A. at $y = 0$, V.A. at $x = 1$
B) H.A. at $y = 0$, V.A. at $x = 0$
C) H.A. at $y = 2$, V.A. at $x = 1$
D) H.A. at $y = 2$, V.A. at $x = 0$
E) H.A. at $y = 2$, V.A at $x = 0$ and at $x = 1$.

$$y = \frac{2(x^3-1)}{x^3-x^2} = \frac{2(x-1)(x^2+x+1)}{x^2(x-1)} \qquad \lim_{x\to\infty} f(x) = 2$$

So there is an HA at $y=2$

and a VA at $x=0$ (not $x=1$)

Question 2- Which are the inflection points of $f(x) = x^4 - \frac{2}{3}x^3 + 1$?

A) $x = 0$ is the only inflection point.
B) $x = \frac{1}{3}$ is the only inflection point.
C) $x = \frac{1}{2}$ is the only inflection point.
D) $x = 0$ and $x = \frac{1}{2}$ are the inflection points.
E) $x = 0$ and $x = \frac{1}{3}$ are the inflection points.

$$f'(x) = 4x^3 - 2x^2$$

$$f''(x) = 12x^2 - 4x = 4(3x-1)x$$

There are possible IPs at $x = 0, 1/3$. They are both IPs

		0		1/3	
$f''(x)$	+		−		+

Question 3- Find the equation of the tangent line to the graph of $y = \frac{1}{2}e^{2x}$ when $x = \ln(2)$.

A) $y = 4x + 2 - 4\ln(2)$ B) $y = \frac{1}{2}x + \frac{1}{2}\ln(2)$ C) $y = 4x - 2$ D) $y = \frac{1}{2}x + \ln(2)$

E) $y = 4x - \frac{1}{2}\ln(2)$

$y' = e^{2x}$ $y'(\ln(2)) = e^{2\ln(2)} = 4$

$y(\ln(2)) = 2$. So $(\ln(2), 2)$ is on the line.

$y = 4x + b$

$2 = 4\ln(2) + b$

$b = 2 - 4\ln(2)$

Question 4- What are the critical points of the function $f(x) = \sqrt[3]{2x} - \frac{2}{3}x + 2$ on $[0,1]$?

A) f has no critical points B) $x = 0$ C) $x = 0, x = \frac{1}{2}$ D) $x = \frac{1}{2}$ E) $x = \frac{1}{4}, x = \frac{1}{2}$

$f'(x) = \frac{1}{3}(2x)^{-2/3}(2) - 2/3$

$= \frac{2}{3(2x)^{2/3}} - 2/3$

So $x = 0, \pm 1/2$ are CP's.

NB: 0 is a CP, since $f(x)$ is defined but $f'(x)$ is not.

Question 5- Calculate:

$$\int_1^e \ln(x)\mathrm{dx}$$

A) 0 **B)** 1 **C)** e **D)** $e-1$ **E)** $1-e$

$$x\ln(x) - x\Big|_1^e = [e\ln(e) - e] - [0 - 1] = 1$$

Question 6- Evaluate:

$$\int_1^\infty \frac{3}{x^2}\mathrm{dx}$$

A) 1 **B)** -1 **C)** 3 **D)** -3 **E)** divergent

$$\lim_{b\to\infty} \int_1^b \frac{3}{x^2}dx = \lim_{b\to\infty} \int_1^b 3x^{-2}dx = \lim_{b\to\infty} \frac{-3}{x}\Big|_1^b = \lim_{b\to\infty}\left[\frac{-3}{b} - (-3)\right]$$

$$= 3$$

Question 7- A company has determined that its marginal profit function is given as $P'(x) = 3\sqrt{x} - 3$. Currently the company is producing 4 units and is breaking even. How much profit will the company make when production is increased to 9 units?

A) \$0 B) -\$12 C) \$27 D) \$23 E) -\$27

$P'(x) = 3x^{1/2} - 3$

$P(x) = 3\frac{x^{3/2}}{3/2} - 3x + C$

$= 2x^{3/2} - 3x + C$

$P(4) = 0$. So

$0 = 2(8) - 12 + C \Rightarrow C = -4$

$P(x) = 2x^{3/2} - 3x + C$

$P(9) = 2(\overset{27}{27}) - \cancel{\ } - 4 = 23$

Question 8- If the demand function is $D(x) = -x^2 + x + 10$ and the supply function is $S(x) = x + 1$, find the producer surplus.

A) 4 B) $\frac{11}{2}$ C) $\frac{9}{2}$ D) 3 E) $\frac{4}{3}$

$D(x) = S(x)$

$\Rightarrow -x^2 + x + 10 = x + 1$

$\Rightarrow x^2 - 9 = 0$

$\Rightarrow x = \pm 3$

So $(3,4)$ is the intersection

$PS = \int_0^3 [4 - (x+1)]\,dx$

$= 9/2$

Question 9- If $f(x,y) = \sqrt{xy^2}$, what is the domain of f?

A) $\{(x,y)|x \geq 0\}$ B) $\{(x,y)|x \geq 0, y \geq 0\}$ C) $\{(x,y)|x \geq 0, y \neq 0\}$

D) $\{(x,y)|x \neq 0, y > 0\}$ E) $\{(x,y)|x < 0, y \neq 0\}$

xy^2 must be greater than or equal to 0.
$y^2 \geq 0$ always. So $x \geq 0$.

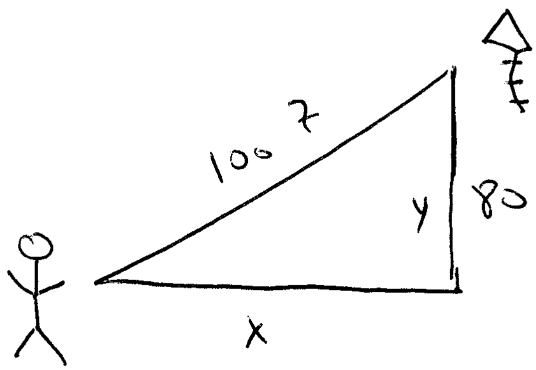
Question 10- If $f(x,y) = e^{x^2} + 2xy - 2y$, what are the critical points of $f(x,y)$?

A) $(0,0)$ B) $(1,-e)$ C) $(-1,1)$ D) $(0,e)$ E) $(1,1)$

$f_x = 2xe^{x^2} + 2y$
$f_y = 2x - 2 \Rightarrow x = 1$ is part of a CP.
If $x = 1$, then $2xe^{x^2} + 2y = 0 \Rightarrow$
$2e + 2y = 0 \Rightarrow$
$y = -e$

Long Answer Question 1 (14 points)

A girl is flying a kite. Initially the kite is directly above her at a height of 80 feet. The kite is moving horizontally away from her at a rate of 5 feet per second. At what rate is the string being let out when the distance from the girl to the kite is 100 feet? You may assume the string is on a straight line from the girl to the kite.



$x = 60$ (by Pythagoras)

$y = 80$

$z = 100$

$dx/dt = 5$

$dy/dt = 0$

$\frac{dz}{dt} = ?$

$x^2 + y^2 = z^2$

$\Rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$

$60(5) + 80(0) = 100 \frac{dz}{dt}$

$\Rightarrow \frac{dz}{dt} = 3$

Question 2 (14 points)

Calculate the following two indefinite integrals:

$$\int x^2 e^{3x^3}\,dx$$

$$u = 3x^3$$
$$du = 9x^2\,dx$$

$$= \frac{1}{9}\int e^{3x^3}\, 9x^2\,dx$$

$$= \frac{1}{9}\int e^u\,du$$

$$= \frac{1}{9} e^u = \frac{1}{9} e^{3x^3} + C$$

$$\int x^7 \ln(3x)\,dx$$

$$u = \ln(3x) \qquad v = \frac{x^8}{8}$$
$$du = \frac{3}{3x}dx = \frac{1}{x}dx \qquad dv = x^7\,dx$$

$$= \frac{x^8}{8}\ln(3x) - \int \frac{x^8}{8}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^8}{8} - \frac{1}{8}\int x^7\,dx = \frac{x^8}{8} - \frac{1}{64}x^8 + C$$

Question 3 (14 points)

A manufacturer can produce T-shirts at a cost of 2 dollars each. They have been selling the shirts at 5 dollars each. At this price they sell 4,000 shirts per month. The manufacturer is planning on raising the price and estimates that for each 1 dollar increase in price, 400 fewer shirts will be sold.

(a) **(3 points)** Find the demand function.

(b) **(3 points)** Find the profit function.

(c) **(5 points)** How many shirts should be produced in order to maximize profit?

(d) **(3 points)** Explain why your answer in (c) is an absolute maximum.

p	x
5	4,000
6	3,600

$$m = \frac{\Delta p}{\Delta x} = \frac{-1}{400}$$

a) $y = mx + b = -\frac{1}{400}x + b \Rightarrow 5 = \frac{-1}{400}(4{,}000) + b$

$\Rightarrow b = 15$ So $D(x) = \frac{-1}{400}x + 15$

b) $P(x) = R(x) - C(x) = \left[-\frac{1}{400}x^2 + 15x\right] - 2x$

$= -\frac{1}{400}x^2 + 13x$

c) $P'(x) = \frac{-1}{200}x + 13 \Rightarrow x = 2600$ is a CP.

It is an absolute max, since $P(x)$ is a C.D. parabola

Question 4 (16 points)

Consider the two functions:
$f(x) = x^2 - 2x$ and $g(x) = x$

(a) **(4 points)** Find the intersection points of the graphs of the two functions.

(b) **(6 points)** On the next page, graph these functions, and shade the region bounded by $f(x), g(x), x = 0$ and $x = 4$.

(c) **(6 points)** Find the area of the shaded region.

a) $x^2 - 2x = x \Rightarrow x^2 - 3x = 0 \Rightarrow x = 0, 3$

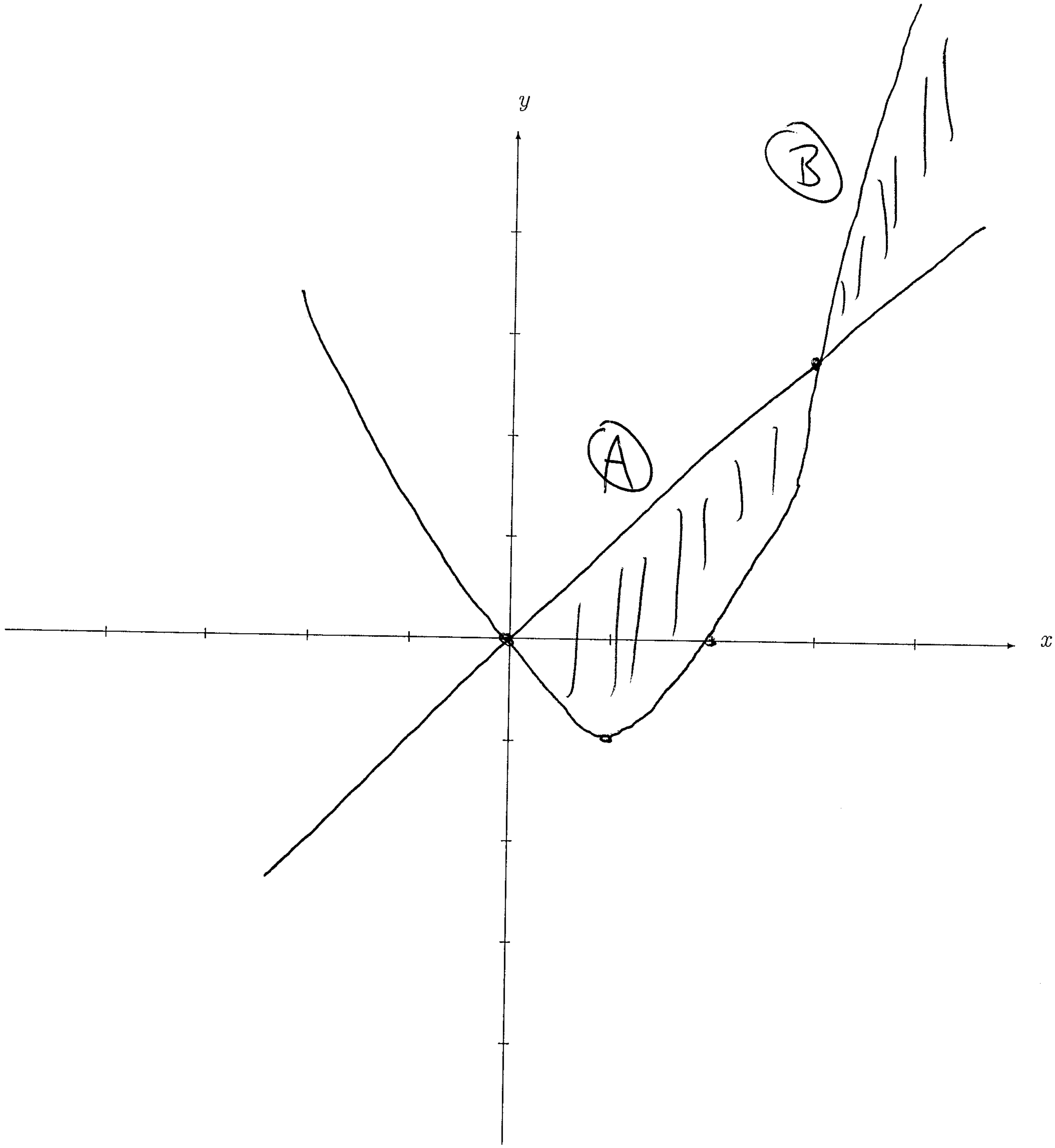
c) Area of A =

$$\int_0^3 x - (x^2 - 2x)\, dx = \int_0^3 (3x - x^2)\, dx = 9/2$$

Area of B =

$$\int_3^4 [(x^2 - 2x) - x]\, dx = \int_3^4 (x^2 - 3x)\, dx = 11/6$$

Total Area $= 9/2 + \frac{11}{6} = \frac{38}{6} = \frac{19}{3}$


y
x
A
B

Question 5 (12 points)
Consider the function of two variables $f(x,y) = 2x^3 + y^2 + 3x^2 - 8y - 12x + 4$.

(a) **(3 points)** Calculate the first-order partial derivatives.

(b) **(3 points)** Find all critical points.

(c) **(6 points)** Identify what type of critical points they are (local max, local min or saddle point).

$f_x = 6x^2 + 6x - 12 = 0 \Rightarrow 6(x^2 + x - 2) = 0$

$\Rightarrow 6(x+2)(x-1) = 0$

$f_y = 2y - 8 = 0$

$\Rightarrow y = 4$

There are 2 CPs at $(-2,4)$ and $(1,4)$

$$\left.\begin{aligned} f_{xx} &= 12x+6 \\ f_{yy} &= 2 \\ f_{xy} &= 0 \end{aligned}\right\} \Rightarrow D = 24x+12$$

For $(-2,4)$, $D = -36 < 0$. So saddle point

For $(1,4)$ $D = 36$ and $f_{xx} > 0$. So local min