Q.1

From Table: $\mathcal{L}\{\cos 3t\} = \dfrac{s}{s^2+(3)^2} = \dfrac{s}{s^2+9} = F(s)$

& $\mathcal{L}\{e^{2t}\cos 3t\} = F(s-2) = \dfrac{s-2}{(s-2)^2+9}$ Ans.(a).

Q.2 $\mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2+(2)^2} = \dfrac{s}{s^2+4} = F(s)$

From Table $\mathcal{L}\{t\cos 2t\} = (-1)\dfrac{d}{ds}(F(s)) = -\dfrac{1\cdot(s^2+4)-s(2s)}{(s^2+4)^2}$

$$= \frac{s^2-4}{(s^2+4)^2}$$ Ans. (d)

Q.3 $\dfrac{s+3}{s^2-6s+18} = \dfrac{s+3}{s^2-6s+9+9} = \dfrac{s+3}{(s-3)^2+9} = \dfrac{s-3+6}{(s-3)^2+9}$

$$= \frac{s-3}{(s-3)^2+9} + \frac{6}{(s-3)^2+9}$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{s^2-6s+18}\right\} = \mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{(s-3)^2+9}\right\} = e^{3t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + e^{3t}\mathcal{L}^{-1}\left\{\frac{6}{s^2+9}\right\}$$

$$= e^{3t}\cos 3t + 2e^{3t}\sin 3t$$ Ans. (b)

Q.4 $\dfrac{4s+7}{s^2+s-6} = \dfrac{4s+7}{(s-2)(s+3)} = \dfrac{A}{s-2} + \dfrac{B}{s+3} \Rightarrow A = \left.\dfrac{4s+7}{s+3}\right|_{s=2} = \dfrac{15}{5} = 3$

$B = \left.\dfrac{4s+7}{s-2}\right|_{s=-3} = \dfrac{-5}{-5} = 1$

$$\mathcal{L}^{-1}\Bigg\{\underbrace{\frac{4s+7}{s^2+s-6}}_{F(s)}\Bigg\} = \mathcal{L}^{-1}\left\{\frac{3}{s-2} + \frac{1}{s+3}\right\} = 3e^{2t} + e^{-3t} = f(t)$$

$$\mathcal{L}^{-1}\{e^{-2s}F(s)\} = u(t-2)\,f(t-2) = u(t-2)\left[3e^{2(t-2)} + e^{-3(t-2)}\right]$$ Ans. (d)

Q.5 $\mathcal{L}\{y''-2y''+5y\} = \mathcal{L}\{2\delta(t-2)\}$

$$s^2Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] + 5Y(s) = 2e^{-2s}$$

$$s^2Y(s) - s - 3 - 2sY(s) + 2 + 5Y(s) = 2e^{-2s}$$

$$(s^2-2s+5)Y(s) = 2e^{-2s} + s + 1 \Rightarrow Y(s) = \frac{2e^{-2s}+s+1}{s^2-2s+5}$$ Ans. (c)

Q.6 $Y(s) = \dfrac{2s+1}{s^2+s-2} = \dfrac{2s+1}{(s+2)(s-1)} = \dfrac{A}{s+2} + \dfrac{B}{s-1} \Rightarrow A = \left.\dfrac{2s+1}{s-1}\right|_{s=-2} = 1$

$B = \left.\dfrac{2s+1}{s+2}\right|_{s=1} = 1$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s+2} + \dfrac{1}{s-1}\right\}$

$y(t) = e^{-2t} + e^{t}$ Ans: (d).

Q.7 Aux. eqn: $4m(m-1) + 8m + 1 = 0$

$4m^2 + 4m + 1 = 0 = (2m+1)^2 \Rightarrow$ roots $m = -\frac{1}{2}$ with multiple 2

$y = c_1|x|^{-1/2} + c_2|x|^{-1/2}\ln|x|$. Ans. (b)

Q.8 Aux. eqn: $m(m-1) + 5m + 13 = 0$

$m^2 + 4m + 13 = 0 \Rightarrow$ roots $m = -2 \pm 3i$

$y = |x|^{-2}\left[c_1\cos(3\ln|x|) + c_2\sin(3\ln|x|)\right]$ Ans. (d)

Q.9 $y_1 = \sum_{n=0}^{\infty} a_n x^{n+1}$, $y_1' = \sum_{n=0}^{\infty} a_n(n+1)x^n$, $y_1'' = \sum_{n=0}^{\infty} n(n+1)a_n x^{n-1}$

$0 = x^2 y_1'' + (x^2 - x)y' + y = x^2 \sum_{n=0}^{\infty} n(n+1)a_n x^{n-1} + (x^2 - x)\sum_{n=0}^{\infty}(n+1)a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1}$

$0 = \sum_{n=0}^{\infty} n(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty}(n+1)a_n x^{n+2} - \sum_{n=0}^{\infty}(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1}$

$= \sum_{n=0}^{\infty}\left[n(n+1) - (n+1) + 1\right]a_n x^{n+1} + \sum_{n=0}^{\infty}(n+1)a_n x^{n+2}$; let $k = n+1$, $k+1 = n+2$

$= \sum_{n=0}^{\infty} n^2 a_n x^{n+1} + \sum_{k=1}^{\infty} k\, a_{k-1} x^{k+1}$

$= \sum_{n=0} n^2 a_n x^{n+1} + \sum_{n=1} n\, a_{n-1} x^{n+1}$

You have to make all sums having the same power of x; i.e. x^{n+1} or x^{k+1}.

Q.9 (Cont'd)

$$0=\sum_{n=1}^{\infty}\left[n^2 a_n + n\, a_{n-1}\right]x^{n+1}$$

$\Rightarrow n^2 a_n + n\, a_{n-1} = 0$ for $n=1,2,\cdots$

$a_n = -\frac{1}{n}a_{n-1}$ for $n=1,2,\cdots$ Ans. (d)

or $a_{n+1} = \frac{-1}{n+1}a_n$ for $n=0,1,2,\cdots$

Q.10 $a_1 = \frac{2}{1^2}a_0$; $a_2 = \frac{2}{2^2}a_1 = \frac{2\cdot 2}{2^2\cdot 1^2}a_0$, $a_3 = \frac{2}{3^2}a_2$

$\cdots$ $a_n = \frac{2\cdot 2\cdots 2}{n^2\cdots 2^2\cdot 1^2}a_0 = \frac{2^n}{(n!)^2}a_0$ Ans: (a)

Q.11 $x=0$ is a regular singular point. Therefore the form of the solution is $y=\sum_{n=0}^{\infty} a_n x^{n+\lambda}$ where λ satisfies the indicial equation

$$\lambda(\lambda-1)+3\lambda+1=0=\lambda^2+2\lambda+1=(\lambda+1)^2$$

roots are $\lambda=-1$ with multiplicity 2.

$$y_1=\sum_{n=0}^{\infty} a_n x^{n-1} \quad \text{Ans:}$$

To find the indicial eq. for Q.11:

$xP(x) = x^2+3x$ (coefficient of y')

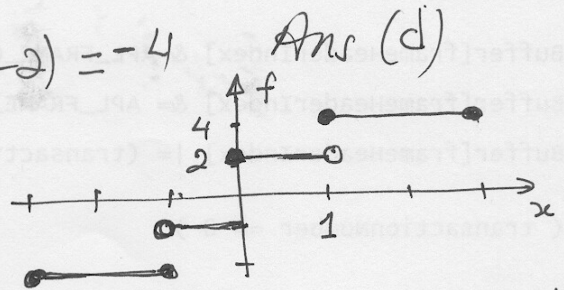
and $Q(x)=1$ (coefficient of y)

so $P(x) = x+3 = p_1 x + p_0 \Rightarrow p_0 = 3$

$Q(x) = 1 = q_0 \Rightarrow q_0 = 1$

Q.13 Period $= 6$; $59 = 6*9+3+2$

$$f(59) = f(5) = f(3+2) = f(-2) = -4$$ Ans (d)



Q.14 Fourier series with $L=1$.

$a_n = 0$ for $n = 0, 1, 2, \ldots$

$$b_n = \frac{2}{1}\int_0^1 x \sin\left(\frac{n\pi x}{1}\right) dx = 2\left[\frac{-1}{n\pi} x \cos(n\pi x) + \frac{1}{(n\pi)^2}\sin(n\pi x)\right]_0^1$$

$$b_n = 2\left\{\left[\frac{-1}{n\pi}\cos(n\pi) + 0\right] - \left[0 + 0\right]\right\} = \frac{-2}{n\pi}(-1)^n = \frac{2}{n\pi}(-1)^{n+1}$$

Used Table to evaluate the integral

Ans (c).

Q.15 Let $u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t}$

The boundary conditions $u(0,t) = u(1,t) = 0$ are already satisfied. One needs to make sure that u satisfies the initial cond.

$$u(x,0) = x \Rightarrow \underbrace{x = \sum_{n=1}^{\infty} b_n \sin(n\pi x)}_{\text{Fourier sine series for } f(x)=x \text{ on } (0,1)} \text{ for } 0 < x < 1$$

$$b_n = \frac{2}{1}\int_0^1 x \sin(n\pi x)\, dx = \frac{-2}{n\pi}(-1)^n \text{ for } n \geq 1$$ Ans (a)

for integral evaluation see Q.14.

Q.12 The eq. can be written as

$$y'' + \frac{1}{x}y' + \left(\underset{\lambda}{3} - \frac{\overset{\nu^2}{4}}{x^2}\right)y = 0$$ so $\lambda = 3$ & $\nu^2 = 4$

So $y = c_1 J_2(\sqrt{3}x) + c_2 Y_2(\sqrt{3}x)$ Ans (c)

Q.16 let $u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{3}\right)\left[a_n \cos\left(\frac{2n\pi t}{3}\right) + b_n \sin\left(\frac{2n\pi t}{3}\right)\right]$

$u(x,t)$ satisfies both boundary conditions. It needs to satisfy the initial conditions

IV#1

$$u(x,0) = 0 \Rightarrow 0 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{3}\right)\left[a_n \cdot 1 + b_n \cdot 0\right]$$

$$0 = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{3}\right) \Rightarrow a_n = 0 \text{ for } n = 1, 2, \ldots$$

is the only sol. to make both sides equal

So the sol. has the form (after replacing a_n by 0)

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right) \sin\left(\frac{2n\pi t}{3}\right)$$

I.V. #2

$$u_t(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right) \cdot \frac{2n\pi}{3} \cos\left(\frac{2n\pi t}{3}\right)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{2n\pi}{3} b_n \sin\left(\frac{n\pi x}{3}\right) \cos\left(\frac{2n\pi t}{3}\right)$$

$$u_t(x,0) = \underbrace{2\sin(\pi x) - 3\sin(2\pi x)}_{\text{Side \#1}} = \underbrace{\sum_{n=1}^{\infty} \frac{2n\pi}{3} b_n \sin\left(\frac{n\pi x}{3}\right) \cdot 1}_{\text{Side \#2}}$$

Expanding the sum in side #1 and then comparing the sides 1 & 2 we observe

$n=3$: $\quad 2\sin(\pi x) = \frac{2 \cdot 3 \cdot \pi}{3} b_3 \sin(\pi x) \Rightarrow b_3 = 1/\pi$

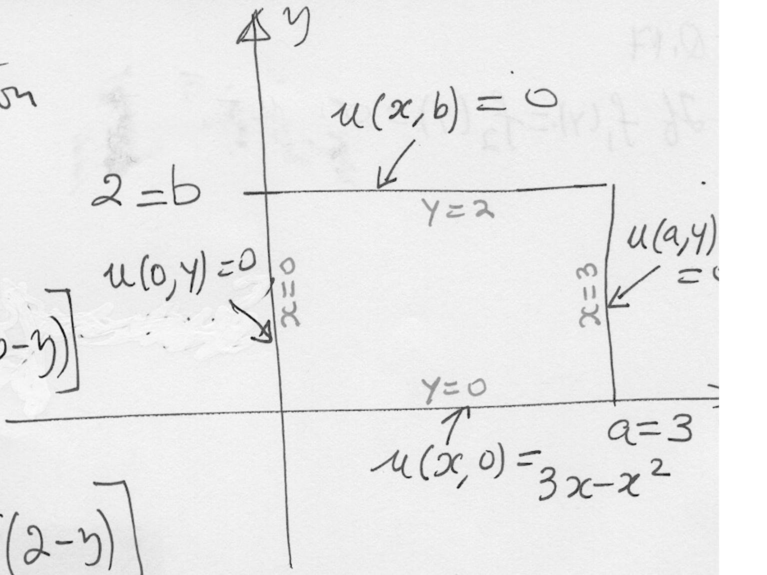
$n=6$: $\quad -3\sin(2\pi x) = \frac{2 \cdot 6 \cdot \pi}{3} b_6 \sin(2\pi x) \Rightarrow b_6 = \frac{-3}{4\pi}$

Otherwise $b_n = 0$, for $n \geq 1$

Ans (e)

Q.17

Base on the Boundary Condition then the form of the sol. is



$$u(x,y)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\cdot \sinh\left[\frac{n\pi}{a}(b-y)\right]$$

or

$$u(x,y)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{3}x\right)\cdot \sinh\left[\frac{n\pi}{3}(2-y)\right]$$

Ans. (b)

Other possibilites:

If $u(0,y)=u(a,y)=u(x,0)=0$ & $u(x,b)=f(x)$ then the sol. has the form $$u(x,y)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\cdot \sinh\left[\frac{n\pi}{a}y\right]$$

If $u(a,y)=u(x,0)=u(x,b)=0$ & $u(0,y)=f(y)$ then the sol. will have the form $$u(x,y)=\sum_{n=1}^{\infty} A_n \sinh\left[\frac{n\pi}{b}(a-x)\right]\cdot \sin\left(\frac{n\pi}{b}y\right)$$

If $u(0,y)=u(x,0)=u(x,b)=0$ & $u(a,y)=f(y)$ then we will have $$u(x,y)=\sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{b}x\right)\cdot \sin\left(\frac{n\pi}{b}y\right)$$

Q.18 $u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}\left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$

B.C.: $u(3,\theta) = 3 + 2\sin(2\theta) - \cos(3\theta)$

$\Rightarrow 3 + 2\sin(2\theta) - \cos(3\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} 3^{-n}\left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$

$$\underbrace{3 + 2\sin(2\theta) - \cos(3\theta)}_{\text{side \#1}} = \underbrace{\frac{a_0}{2} + 3^{-1}\left[a_1 \cos\theta + b_1 \sin\theta\right] + 3^{-2}\left[a_2 \cos(2\theta) + b_2 \sin(2\theta)\right] + 3^{-3}\left[a_3 \cos(3\theta) + b_3 \sin(3\theta)\right] + \dots}_{\text{side \#2}}$$

Comparing side #1 & 2
we observe that

$$\begin{cases} \frac{a_0}{2} = 3 \\ 2\sin(2\theta) = 3^{-2} b_2 \sin(2\theta) \\ -\cos(3\theta) = 3^{-3} a_3 \cos(3\theta) \\ \text{otherwise } a_n = b_n = 0 \text{ for } n \geq 1 \end{cases} \Rightarrow \begin{cases} a_0 = 6 \\ b_2 = 18 \\ a_3 = -27 \\ \text{otherwise } a_n = b_n = 0 \text{ for } n \geq 1 \end{cases}$$

Ans: (a)

Q.19 Write the eq. in the form $xy'' + 2y' + xy = -\lambda x y$

Divide by x (coeff. of y''): $y'' + \frac{2}{x}y' + y = -\lambda y$

$p(x) = e^{\int \frac{2}{x} dx} = e^{2\ln|x|} = e^{\ln|x|^2} = x^2$ (is the integra[ting] factor)

Multiply both ~~side~~ side of the eqns by x^2

$x^2 y'' + 2x y' + x^2 y = -\lambda \underbrace{x^2}_{} y$. Ans. (c)

$x^2 y'' + 2x y' + x^2 y + \lambda \overset{r(x)}{x^2} y = 0$

Q.20 Using integration by parts $\int u\,dv = uv - \int v\,du$

$$I=\int_0^2 x^4 J_1(3x)\,dx = \int_0^2 \underbrace{x^2}_{u}\cdot \underbrace{x^2 J_1(3x)\,dx}_{dv}$$

for $\nu=2$; Bessel identity becomes: $\frac{1}{\alpha}\frac{d}{dx}\left[x^2\cdot J_2(\alpha x)\right] = x^2 J_1(\alpha x)$

or if $\alpha=3$: $\frac{1}{3}\frac{d}{dx}\left[x^2\cdot J_2(3x)\right] = \underbrace{x^2 J_1(3x)}_{dv/dx}$

$\Downarrow$

$v=\frac{1}{3}x^2 J_2(3x)$

$$I = \left[x^2\cdot\frac{1}{3}x^2 J_2(3x)\right]_0^2 - \int_0^2 \frac{1}{3}x^2 J_2(3x)\cdot 2x\,dx$$

$$I = \left[\frac{16}{3}J_3(3x) - 0\right] - \frac{2}{3}\int_0^2 x^3 J_2(3x)\,dx$$

for $\nu=3$; Bessel identity becomes: $\frac{1}{\alpha}\frac{d}{dx}\left[x^3 J_3(\alpha x)\right] = x^3 J_2(\alpha x)$

$$I = \frac{16}{3}J_3(6) - \left[\frac{2}{3}\cdot\frac{1}{3}x^3 J_3(3x)\right]_{x=0}^{2}$$

Ans. (a)

$$I = \frac{16}{3}J_3(6) - \frac{2}{9}\,8\,J_3(6) = \frac{16}{3}J_3(6) - \frac{16}{9}J_3(6)$$

Note: If $dv = x^{\nu} J_{\nu-1}(\alpha x)$ then $v = \frac{1}{\alpha}x^{\nu} J_{\nu}(\alpha x)$

i.e. $\int x^{\nu} J_{\nu-1}(\alpha x)\,dx = \frac{1}{\alpha}x^{\nu} J_{\nu}(\alpha x)$

Q.21 Gen. Sol. is $y = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$y(0)=0 \Rightarrow 0 = c_1 \cdot 1 + c_2 \cdot 0 \Rightarrow c_1 = 0$ So $y = c_2 \sin(\sqrt{\lambda}\, x)$

$y' = \sqrt{\lambda}\, c_2 \cos(\sqrt{\lambda}\, x)$

$y'(2) = 0 \Rightarrow 0 = \sqrt{\lambda}\, c_2 \cos(\sqrt{\lambda} \cdot 2) \Rightarrow 2\sqrt{\lambda} = (2n+1)\frac{\pi}{2}$ for $n \geq 1$

$\sqrt{\lambda} = (2n+1)\frac{\pi}{4} \Rightarrow \lambda = (2n+1)^2 \frac{\pi^2}{16}$ for $n \geq 0$

eigenvalue: $\lambda = \frac{(2n+1)^2}{16}\pi^2$ Ans: (d)

Corresponding eigenfunctions: $y = K \cos\left(\frac{2n+1}{4}\pi x\right)$

Q.22 $\mathcal{F}\left\{e^{-2ix - |x-3|}\right\} = \tilde{\mathcal{F}}\left\{e^{-2ix} \cdot \underbrace{e^{-|x-3|}}_{f(x)}\right\} = \hat{f}(\lambda - 2)$, (From Table) line #6

$\hat{f}(\lambda) = \tilde{\mathcal{F}}\left\{e^{-|x-3|}\right\} = e^{i3\lambda} \frac{2}{1+\lambda^2}$ (from Table line # 5 & 7)

$\mathcal{F}\left\{e^{-2ix - |x-3|}\right\} = e^{i3(\lambda-2)} \cdot \frac{2}{1+(\lambda-2)^2}$ Ans (b)

Q.23. $\mathcal{F}\left\{2xe^{-x^2}\right\} = 2\tilde{\mathcal{F}}\left\{x\underbrace{e^{-x^2}}_{f(x)}\right\} = 2 \cdot \left[-i\frac{d\hat{f}}{d\lambda}\right]$ (line#9)

$\hat{f}(\lambda) = \tilde{\mathcal{F}}\left\{e^{-x^2}\right\} = \sqrt{\frac{\pi}{1}}\, e^{-\lambda^2/4}$ (line# 11)

$\mathcal{F}\left\{2xe^{-x^2}\right\} = -2i \cdot \sqrt{\pi} \cdot \left(-\frac{2\lambda}{4}\right) e^{-\lambda^2/4} = i\sqrt{\pi}\,\lambda\, e^{-\lambda^2/4}$

Ans. (a)

Q.24

$$\mathcal{F}^{-1}\left\{\frac{e^{-3i\lambda}}{1+(\lambda+2)^2}\right\} = \mathcal{F}^{-1}\left\{e^{i\lambda(-3)} \cdot \underbrace{\frac{1}{1+(\lambda+2)^2}}_{\hat{f}(\lambda)}\right\}$$

$$= f(x+3) \quad (\text{line \# 7 with } a=-3)$$

We need to fin f.

$$f(x) = \mathcal{F}^{-1}\left\{\frac{1}{1+(\lambda+2)^2}\right\} = e^{i(+2)x}\,\mathcal{F}^{-1}\left\{\frac{1}{1+\lambda^2}\right\} \quad (\text{line \#, } a=2$$

$$= e^{2ix} \cdot \frac{1}{2} e^{-|x|} \quad (\text{line \# 5})$$

$$\text{So } \mathcal{F}^{-1}\left\{\frac{e^{-3i\lambda}}{1+(\lambda+2)^2}\right\} = e^{2i(x+3)} \cdot \frac{1}{2} e^{-|x+3|}$$

$$= \frac{1}{2} e^{-|x+3| + 2i(x+3)}$$

Ans (a)

Q.25

$$\mathcal{F}^{-1}\left\{\lambda e^{-|\lambda|}\right\} = \frac{-1}{i}\mathcal{F}^{-1}\left\{-i\lambda \underbrace{e^{-|\lambda|}}_{\hat{f}(\lambda)}\right\} = i f'(x) \quad \text{line \# 8}$$

$$f(x) = \mathcal{F}^{-1}\left\{e^{-|\lambda|}\right\} = \frac{1}{\pi(1+x^2)} \quad (\text{This formula is not available in the Table})$$

$$\mathcal{F}^{-1}\left\{\lambda e^{-|\lambda|}\right\} = i \cdot \frac{1}{\pi} \frac{-2x}{(1+x^2)^2} = \frac{-2ix}{\pi(1+x^2)^2}$$

Ans (c)