MATH 3705 Final Examination
April 2004

1. $\mathcal{L}\{e^{2t}\cos(3t)\} =$

   (a) $\dfrac{s-2}{(s-2)^2+9}$

   (b) $\dfrac{s}{(s-2)^2+9}$

   (c) $\dfrac{s-2}{s^2+9}$

   (d) $\dfrac{e^{-2s}}{s^2+9}$

   (e) None of the above.

2. $\mathcal{L}\{t\sin(2t)\} =$

   (a) $\dfrac{s}{(s^2+4)^2}$

   (b) $\dfrac{-s}{(s^2+4)^2}$

   (c) $\dfrac{2}{(s-1)^2+4}$

   (d) $\dfrac{4s}{(s^2+4)^2}$

   (e) None of the above.

3. $\mathcal{L}^{-1}\left\{\dfrac{3e^{-2s}}{s^2+s-2}\right\} =$

   (a) $u(t-2)\left[e^{t}-e^{-2t}\right]$

   (b) $u(t-2)\left[e^{t-2}-e^{-2t+4}\right]$

   (c) $e^{t-2}-e^{-2(t-2)}$

   (d) $u(t)e^{t}-u(t-2)e^{t-2}$

   (e) None of the above.

4. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2s+10}\right\} =$

(a) $e^{-t}\cos(3t)$

(b) $e^{-t}\sin(3t)$

(c) $e^{-t}\cos(3t) - e^{-t}\sin(3t)$

(d) $e^{-t}[\cos(3t) - \dfrac{1}{3}\sin(3t)]$

(e) None of the above.

5. The general solution of the differential equation $2x^2y'' - 5xy' + 3y = 0$, valid for $x \neq 0$, is given by

(a) $c_1|x|^3 + c_2|x|^{\frac{1}{2}}$

(b) $|x|^3[c_1 + c_2 \ln|x|]$

(c) $|x|^3\left[c_1\cos\left(\dfrac{1}{2}\ln|x|\right) + c_2\sin\left(\dfrac{1}{2}\ln|x|\right)\right]$

(d) $c_1|x|^{\frac{1}{2}} + c_2|x|^3\ln|x|$

(e) None of the above.

6. The general solution of the differential equation $x^2y'' + 2xy' + \frac{1}{4}y = 0$, valid for $x \neq 0$, is given by

(a) $|x|^{-1}\left[c_1\cos\left(\dfrac{\sqrt{3}}{2}\ln|x|\right) + c_2\sin\left(\dfrac{\sqrt{3}}{2}\ln|x|\right)\right]$

(b) $c_1|x|^{-1} + c_2|x|^{\frac{\sqrt{3}}{2}}$

(c) $c_1|x|^{-\frac{1}{2}} + c_2|x|^{-\frac{1}{2}}$

(d) $|x|^{-\frac{1}{2}}(c_1 + c_2\ln|x|)$

(e) None of the above.

7. The general solution of the differential equation $x^2y'' + xy' + (7x^2 - 4)y = 0$, valid for $x > 0$, is given by

(a) $c_1J_2(\sqrt{7}\,x) + c_2J_{-2}(\sqrt{7}\,x)$

(b) $c_1J_{\sqrt{7}}(2x) + c_2J_{-\sqrt{7}}(2x)$

(c) $c_1J_2(\sqrt{7}\,x) + c_2Y_2(\sqrt{7}\,x)$

(d) $c_1J_{\sqrt{7}}(2x) + c_2Y_{\sqrt{7}}(2x)$

(e) None of the above.

8. Let $f(x) = \left\{ \begin{array}{ll} 1, & 0 \le x < 1 \\ 2, & 1 \le x \le 2 \end{array} \right\}$ on $[0, 2]$. At $x = 79$, the Fourier sine series of f converges to

(a) 2

(b) $\frac{3}{2}$

(c) 0

(d) $-\frac{3}{2}$

(e) None of the above.

9. The differential equation $y'' - 2y' + xy + \lambda y = 0$, when placed in the Sturm-Liouville form $(py')' - qy + \lambda ry = 0$, has the weight function $r(x) =$

(a) $-xe^{-2x}$

(b) xe^{-2x}

(c) e^{-2x}

(d) x

(e) None of the above.

10. Given the Bessel identity $\frac{1}{\alpha}\frac{d}{dx}\left[x^{\nu}J_{\nu}(\alpha x)\right] = x^{\nu}J_{\nu-1}(\alpha x),\ \nu > 0,\ \alpha \neq 0,\ \int_0^3 x^4 J_1(2x)\,dx =$

(a) $\frac{1}{2}[81J_2(6) - 27J_3(6)]$

(b) $81J_2(6) - 27J_3(6)$

(c) $27J_3(6) - 81J_2(6)$

(d) $\frac{243}{5}J_2(6)$

(e) None of the above.

11. $\mathcal{F}\{e^{-2ix-|x+3|}\} =$

(a) $\dfrac{2e^{-3i\lambda}}{1+(\lambda-2)^2}$

(b) $\dfrac{2e^{-3i(\lambda-2)}}{1+\lambda^2}$

(c) $\dfrac{2e^{-3i(\lambda-2)}}{1+(\lambda-2)^2}$

(d) $\dfrac{2e^{-3i\lambda}}{1+(\lambda+2)^2}$

(e) None of the above.

12. $\mathcal{F}\{xe^{-3x^2}\} =$

(a) $\dfrac{\sqrt{\pi}}{3}\lambda e^{-\frac{\lambda^2}{12}}$

(b) $\dfrac{i\sqrt{\pi}}{6\sqrt{3}}\lambda e^{-\frac{\lambda^2}{12}}$

(c) $\dfrac{\sqrt{\pi}}{3}e^{-3\lambda^2}$

(d) $\dfrac{1}{2\pi}\lambda e^{-3\lambda^2}$

(e) None of the above.

13. $\mathcal{F}^{-1}\left\{\dfrac{e^{2i\lambda}}{1+(\lambda+3)^2}\right\} =$

(a) $\frac{1}{2}e^{3ix-|x-2|}$

(b) $\frac{1}{2}e^{3i(x-2)-|x|}$

(c) $\frac{1}{2}e^{3i(x+2)-|x+2|}$

(d) $\frac{1}{2}e^{3i(x-2)-|x-2|}$

(e) None of the above.

14. $\mathcal{F}^{-1}\{\lambda e^{-|\lambda|}\} =$

(a) $\dfrac{2ix}{\pi(1+x^2)^2}$

(b) $\dfrac{-2ix}{\pi(1+x^2)^2}$

(c) $\dfrac{x}{\pi(1+x^2)}$

(d) $\dfrac{-x}{\pi(1+x^2)}$

(e) None of the above.

15. Employ the Laplace transform to solve the initial-value problem
$y'' - 4y' + 13y = \delta(t-3),\ y(0) = 1,\ y'(0) = 5.$

16. Find one (non-zero) series solution y_1 of the differential equation $xy'' + 2xy' + 2y = 0$, valid for $x > 0$ near $x_0 = 0$. Express the solution as an elementary function.

17. Find the Fourier cosine series of $f(x) = \left\{ \begin{array}{ll} x, & 0 \le x < 1 \\ 0, & 1 \le x \le 2 \end{array} \right\}$ on $[0, 2]$.

18. The solution of the heat equation $u_{xx} = \frac{1}{\alpha^2} u_t,\ 0 < x < L$, which satisfies the boundary conditions $u_x(0,t) = u_x(L,t) = 0$, has the form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}.$$

Find the solution of $u_{xx} = \frac{1}{9} u_t,\ 0 < x < \pi$, which satisfies the boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$ and the initial condition $u(x,0) = 3\cos(2x) - 2\cos(5x)$. Write down the complete solution $u(x,t)$.

19. The solution of Laplace's equation $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ inside the circle $r = a$ has the form

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

Find the solution of Laplace's equation inside the circle $r = 2$, which satisfies the boundary condition $u(2,\theta) = 2 + \sin(2\theta) - \cos(\theta)$. Write down the complete solution $u(r,\theta)$.

20. Find all eigenvalues and corresponding eigenfunctions of the Sturm-Liouville problem
$y'' + \lambda y = 0,\ y(0) = 0,\ y'(1) = 0.$

Table of Laplace Transforms

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt,\ s > 0$$

$$\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}},\ p > -1,\ \text{and } \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \text{ if } n \geq 0 \text{ is an integer}$$

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2 + a^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a),\ s > a$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s),\ s > a \geq 0$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0),\ n \geq 0$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s) \equiv (-1)^n \frac{d^n}{ds^n}F(s),\ n \geq 0$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(x)\,dx$$

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s}F(s)$$

$$\mathcal{L}\{f(t) * g(t)\} \equiv \mathcal{L}\left\{\int_0^t f(t-x)g(x)\,dx\right\} = F(s)G(s),\ \text{where } G(s) = \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as},\ a \geq 0$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-\omega s}}\int_0^\omega e^{-st}f(t)\,dt \text{ whenever } f \text{ is periodic with period } \omega$$

Summary of Fourier Series

1. The Fourier sine series of a function f defined on $[0, L]$ is given by

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1.$$

2. The Fourier cosine series of a function f defined on $[0, L]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 0.$$

3. The full Fourier series of a function f defined on $[-L, L]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right],$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 0, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1.$$

4. The full Fourier series of a function f defined on $[0, 2L]$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right],$$

$$a_n = \frac{1}{L}\int_{0}^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 0, \qquad b_n = \frac{1}{L}\int_{0}^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n \geq 1.$$

5. The Fourier series of an ω-periodic function f on $\mathbb{R} = (-\infty, \infty)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi x}{\omega}\right) + b_n \sin\left(\frac{2n\pi x}{\omega}\right)\right],$$

$$\begin{aligned}
a_n &= \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{\omega}\int_0^{\omega} f(x)\cos\left(\frac{2n\pi x}{\omega}\right)dx \\
&= \frac{2}{\omega}\int_{\alpha}^{\alpha+\omega} f(x)\cos\left(\frac{2n\pi x}{\omega}\right)dx, \quad n \geq 0, \\
b_n &= \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{\omega}\int_0^{\omega} f(x)\sin\left(\frac{2n\pi x}{\omega}\right)dx \\
&= \frac{2}{\omega}\int_{\alpha}^{\alpha+\omega} f(x)\sin\left(\frac{2n\pi x}{\omega}\right)dx, \quad n \geq 1,
\end{aligned}$$

where $L = \frac{\omega}{2}$ and α is any real number.

Table of Fourier Transforms

$$\mathcal{F}\{f(x)\} = \widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)\, e^{i\lambda x} dx$$

$$\mathcal{F}^{-1}\{F(\lambda)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{-i\lambda x}\, d\lambda$$

$$\mathcal{F}\left\{u(x-a) - u(x-b)\right\} = \frac{1}{i\lambda}\left[e^{i\lambda b} - e^{i\lambda a}\right],\ a < b$$

$$\mathcal{F}\{e^{-|x|}\} = \frac{2}{1+\lambda^2}$$

$$\mathcal{F}\{\delta(x-a)\} = e^{i\lambda a}$$

$$\mathcal{F}\{e^{iax} f(x)\} = \widehat{f}(\lambda + a)$$

$$\mathcal{F}\{f(x-a)\} = e^{i\lambda a}\widehat{f}(\lambda)$$

$$\mathcal{F}\{f'(x)\} = -i\lambda \widehat{f}(\lambda)$$

$$\mathcal{F}\{xf(x)\} = -i\frac{d\widehat{f}}{d\lambda}$$

$$\mathcal{F}\left\{f(\alpha x)\right\} = \frac{1}{|\alpha|}\widehat{f}\left(\frac{\lambda}{\alpha}\right),\ \alpha \neq 0$$

$$\mathcal{F}\{e^{-tx^2}\} = \sqrt{\frac{\pi}{t}}\, e^{-\frac{\lambda^2}{4t}},\ t > 0$$

$$\mathcal{F}\{(f*g)(x)\} \equiv \mathcal{F}\left\{\int_{-\infty}^{\infty} f(s)g(x-s)ds\right\} = \widehat{f}(\lambda)\widehat{g}(\lambda),\ \text{where } \widehat{g}(\lambda) = \mathcal{F}\{g(x)\}$$