Solution.

FINAL EXAM-MATH 1300
FALL TERM, 2006

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Name(Print LEGIBLY)____

I.D. Number____

Instructions-This final examination consists of 10 multiple choice questions worth 3 points each. Your answers to the multiple choice questions must be clearly marked in the squares below. There are also 5 long answer questions worth a total of 70 points. For the long answer questions, you must show your work **on the exam itself** and clearly display your answers. **Do not unstaple these pages.**

NO CALCULATORS. NO NOTES OR BOOKS.

Multiple Choice Answers:

B	C	B	A
#1	#2	#3	#4
D	E	E	E
#5	#6	#7	#8
B	C		
#9	#10		

Question 1- Find the equation of the tangent line to the graph of $y = \frac{2}{x^2-2}$ when $x = 2$.

A) $y = -2x + 6$ $\boxed{\text{B) } y = -2x + 5}$ C) $y = -2x - 1$ D) $y = 2x - 1$ E) $y = 2x - 3$

Solution

slope: $\left(\frac{2}{x^2-2}\right)' = -2 \cdot \frac{2x}{(x^2-2)^2}$, $k = -2\,\frac{2\cdot 2}{(2^2-2)^2} = -\frac{8}{4} = -2$ $(= f'(2))$

So, $y = -2x + 6$

b = ? $f(2) = \frac{2}{2^2-2} = 1 \Rightarrow 1 = -2\cdot 2 + b,$ $b = 5$

$$\boxed{y = -2x + 5}$$

Question 2- Suppose $x^3 + 4xy^2 - 2 = 3y^4$ Find $\frac{dy}{dx}$ at the point $(1, 1)$.

A) $-\frac{1}{5}$ B) 1 $\boxed{\text{C) } \frac{7}{4}}$ D) $\frac{4}{9}$ E) -1

Solution

$$3x^2 + 4y^2 + 8xy\,y' = 12\cdot y^3 \cdot y'$$

$$(12\cdot y^3 - 8xy)\,y' = 3x^2 + 4y^2$$

$$(12 - 8)\,y' = 3 + 4 = 7$$

$$y' = \frac{7}{4}$$

Question 3- On what interval is the function $g(x) = -2x^3 + 15x^2 - 36x + 3$ increasing?

A) $(2, \infty)$ $\boxed{\text{B) } (2, 3)}$ C) $(-1, 1)$ D) $(-2, 2)$ E) $(-2, 4)$

Solution

$$g'(x) = -6x^2 + 30x - 36 = -6(x^2 - 5x + 6) = -6(x-3)(x-2)$$

x		2		3	
$g'(x)$	$-$	0	$+$		$-$
$g(x)$	$\searrow$		$\nearrow$		$\searrow$

Question 4- Which of the following statements is correct for the function $g(x) = xe^x$.

$\boxed{\text{A) } x = -1 \text{ is a local min.}}$ B) $x = -1$ is a local max. C) $x = -3$ is a local min.

D) $x = 1$ is a local min. E) $x = 1$ is a local max.

Solution

$$g'(x) = e^x + xe^x = (x+1)e^x.$$

$$g'(x) = 0 \Rightarrow x = -1$$

Note that $\operatorname{sgn}(g') = \operatorname{sgn}(x+1)$

x		-1	
$g'(x)$	$-$	0	$+$
$g(x)$	$\searrow$		$\nearrow$
		local min	

Question 5- Calculate:

$$\int_1^2 (x^3 - 3x^2 + 4x - 1)dx$$

A) $\frac{4}{5}$ B) $\frac{1}{2}$ C) $\frac{4}{3}$ $\boxed{\text{D) } \frac{7}{4}}$ E) $\frac{8}{3}$

Solution

$$I = \left[\frac{1}{4}x^4 - x^3 + 2x^2 - x\right]_1^2$$

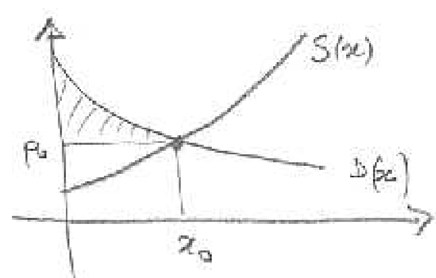
$$= \left(\frac{1}{4}\cdot 16 - \cancel{8} + \cancel{8} - 2\right) - \left(\frac{1}{4} - \cancel{1} + \cancel{2} - \cancel{1}\right)$$

$$= 4 - 2 - \frac{1}{4} = 2 - \frac{1}{4} = \frac{7}{4}$$

Question 6- Suppose that for a certain product, the demand function is given by $D(x) = (x-5)^2$ and the supply function is given by $S(x) = x^2 + x + 3$. Calculate the consumer surplus.

A) 11 B) $\frac{4}{7}$ C) $\frac{9}{2}$ D) $\frac{7}{5}$ $\boxed{\text{E) } \frac{44}{3}}$

Solution



- $D(x) = S(x)$

$$(x-5)^2 = x^2 + x + 3$$

$$\cancel{x^2} - 10x + 25 = \cancel{x^2} + x + 3$$

$$11x = 22, \quad \boxed{x_0 = 2},$$

- $p_0 = D(x_0) = 3^2 = 9$

- $$C.S. = \int_0^2 (D(x) - 9)\,dx = \int_0^2 (x^2 - 10x + 16)\,dx$$

$$= \left[\frac{1}{3}x^3 - 5x^2 + 16x\right]_0^2 = \frac{8}{3} - 20 + 32 = \frac{8}{3} + 12$$

$$= \frac{44}{3}$$

Question 7- If $f(x)$ is a function such that $f'(x) = 4x^3 - 6x - 8$ and $f(2) = 2$, find $f(1)$.

A) 0 B) 1 C) 2 D) 3 $\boxed{\text{E) 4}}$

Solution

$$f(x) = \int (4x^3 - 6x - 8)\,dx = x^4 - 3x^2 - 8x + C$$

$$f(2) = 2 = 2^4 - 3 \cdot 2^2 - 8 \cdot 2 + C, \quad C = 2 - \cancel{16} + 12 + \cancel{16} = 14.$$

$$f(1) = \left(x^4 - 3x^2 - 8x + 14\right)_{\text{at } x=1} = 1 - 3 - 8 + 14 = 4.$$

Question 8-Calculate:

$$\int_1^\infty \frac{x\,dx}{x^2+1}$$

A) $\frac{3}{4}$ B) $\frac{1}{64}$ C) $\frac{1}{2}$ D) $\frac{5}{4}$ $\boxed{\text{E) The integral diverges.}}$

Solution

$$I = \lim_{b \to \infty} \int_1^b \frac{x\,dx}{x^2+1}$$

$$\left[\begin{array}{l} u = x^2+1 \\ du = 2x \end{array} \Rightarrow \int \frac{x\,dx}{x^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(x^2+1)\right] \Rightarrow$$

$$I = \lim_{b \to \infty} \left[\frac{1}{2}\ln(x^2+1)\right]_1^b = \lim_{b \to \infty} \frac{1}{2}\left(\ln(b^2+1) - \ln 2\right)$$

$= +\infty$, because

$$\lim_{b \to \infty} \ln(b^2+1) = \infty$$

Question 9- If $f(x,y) = x^2y - 3y^2x^3 - 4x$, calculate $f_y(2,1)$
A) -9 $\boxed{\text{B) 52}}$ C) -11 D) 76 E) 27

Solution

$f_y = x^2 + 6yx^3$

$f_y(2,1) = 2^2 + 6 \cdot 1 \cdot 2^3 = 4 + 48 = 52$

Question 10- If $f(x,y) = 2yx^2 - xy^3 - y^2 + 7$, calculate $f_{xy}(1,1)$.
A) 9 B) -2 $\boxed{\text{C) 1}}$ D) -13 E) -7

Solution

$f_x = 4yx - y^3$

$f_{xy} = 4x - 3y^2$

$f_{xy}(1,1) = 4 - 3 = 1$

Long Answer Question 1 (12 points)

Recall that radioactive substances decay exponentially, and that the *half-life* of a radioactive substance is the amount of time it takes for half of the substance to decay. Suppose a radioactive substance has a half-life of 5 years.

- If I begin with 7 grams of the substance, how much will I have after 3 years?
- How long does it take for 6 grams of the substance to decay to 1 gram?

Solution

$$P = P_0 \cdot b^t$$

$b = ?$ $\quad P(5) = \frac{P_0}{2} \Rightarrow \cancel{P_0} b^5 = \frac{\cancel{P_0}}{2}, \quad b^5 = \frac{1}{2}, \quad \boxed{b = \left(\frac{1}{2}\right)^{\frac{1}{5}}}$

- $P_0 = 7, \quad P(t) = 7 \cdot \left(\frac{1}{2}\right)^{\frac{t}{5}}$

  $P(3) = 7\left(\frac{1}{2}\right)^{\frac{3}{5}}$

- $P_0 = 6, \quad P(t) = 6\left(\frac{1}{2}\right)^{\frac{t}{5}}$; look for t s.th. $P(t) = 1$; so

$$6\left(\frac{1}{2}\right)^{\frac{t}{5}} = 1, \quad \left(\frac{1}{2}\right)^{\frac{t}{5}} = \frac{1}{6}, \quad \frac{t}{5} \ln\frac{1}{2} = \ln\frac{1}{6}$$

$$t = 5 \frac{\ln\frac{1}{6}}{\ln\frac{1}{2}} = 5 \frac{\ln 6}{\ln 2}$$

Question 2 (16 points)

Calculate the following two indefinite integrals:

- $\int x^4\sqrt{x^5+8}dx$

$$I = \frac{1}{5}\int \sqrt{u}\, du = \frac{1}{5}\cdot\frac{2}{3}u^{3/2} = \frac{2}{15}\left(x^5+8\right)^{\frac{3}{2}}$$

$$\begin{cases} u = x^5+8 \\ du = 5x^4 dx \end{cases}$$

- $\int x^3 \ln(x)dx$

$$I = \int x^3 \underline{\ln x}\, dx$$

$$\begin{cases} u = \ln x & du = \frac{1}{x}dx \\ du = x^3 dx, & u = \int x^3 dx = \frac{1}{4}x^4 \end{cases}$$

$$= \ln x \cdot \frac{1}{4}x^4 - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3\, dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\cdot\frac{1}{4}x^4 + C = \frac{1}{4}x^4\left(\ln x - \frac{1}{4}\right) + C.$$

Question 3 (10 points)

For the following function, find the *absolute* maximum and minimum on the interval $[0,3]$:

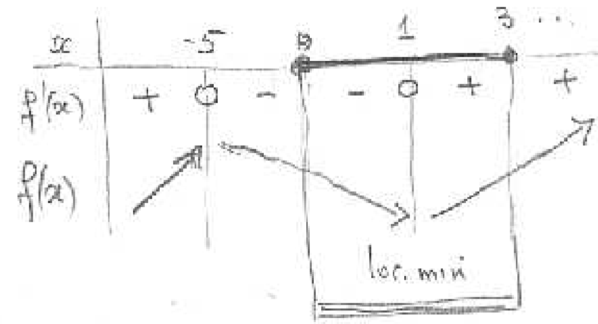
$$f(x) = \frac{1}{3}x^3 + 2x^2 - 5x - 1$$

Solution

- $f'(x) = x^2 + 4x - 5 = (x+5)(x-1)$

  C.P. $x = -5$, $x = 1$.

- Sign of $f'(x)$.



- In $[0, 3]$ we evaluate:

$0 \le x \le 3$

$\rightarrow f(1) = \frac{1}{3} + 2 - 5 + 1 = \frac{1}{3} - 2 = -\frac{5}{3}$

$\rightarrow f(0) = 1$

$\rightarrow f(3) = \frac{1}{3}3^3 + 2\cdot 3^2 - 5\cdot 3 + 1 = 9 + 18 - 15 + 1 = 13$

Comparing these values leads to:

- absolute max in $[0,3]$ is $f(3) = 13$
- absolute min in $[0,3]$ is $f(1) = -\frac{5}{3}$

Question 4 (16 points)

Consider the two functions:

$f(x) = 4 - x^2$ and $g(x) = x^2 - 4$

(a)(4 points) Find the intersection points of the graphs of the two functions.

(b)(6 points) On the next page, graph these functions, and shade the region bounded by $f(x), g(x), x = 0$ and $x = 4$.

(c)(6 points) Find the area of the shaded region.

Solution

(a) $f(x) = g(x), \quad 4 - x^2 = x^2 - 4, \quad x^2 = 4, \quad x = \pm 2$

(b) See the next page

because

$$\left|(4-x^2) - (x^2-4)\right| = \begin{cases} 8 - 2x^2, & |x| \le 2 \\ 2x^2 - 8, & |x| \ge 2 \end{cases}$$

(c) $A = \int_0^4 \left|(4 - x^2) - (x^2 - 4)\right| dx =$

$$= \int_0^2 \left((4-x^2) - (x^2-4)\right) dx + \int_2^4 \left((x^2-4) - (4-x^2)\right) dx$$

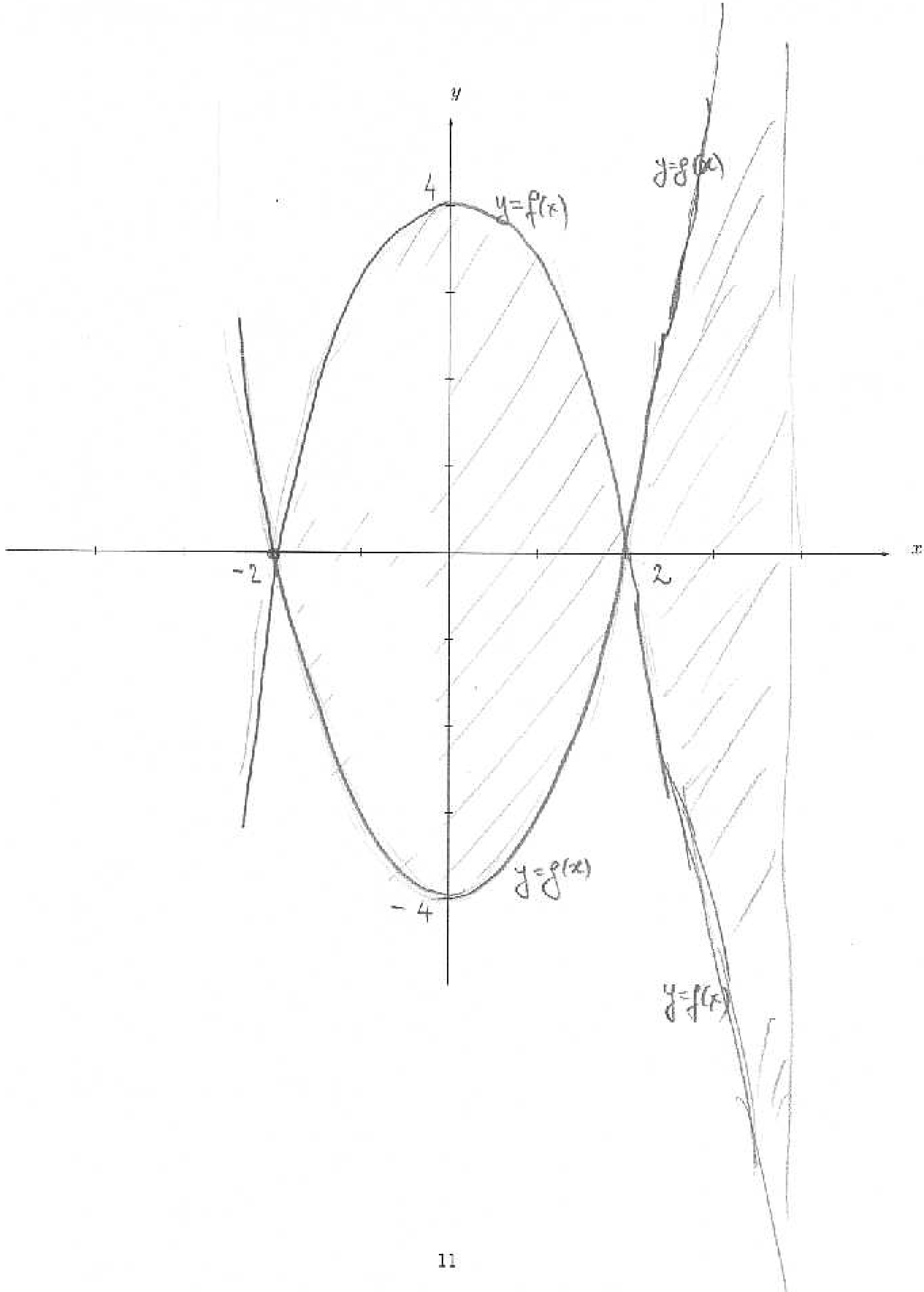
$$= \int_0^2 (8 - 2x^2)\, dx + \int_2^4 (2x^2 - 8)\, dx$$

$$= \left[8x - \frac{2}{3}x^3\right]_0^2 + \left[\frac{2}{3}x^3 - 8x\right]_2^4 =$$

$$= \left(\cancel{16} - \frac{16}{3}\right) + \left(\frac{128}{3} - \cancel{32} - \frac{16}{3} + \cancel{16}\right)$$

$$= \frac{128 - 32}{3} = \frac{96}{3}$$

$$= 32$$


y
x
4
-2
2
-4
y=f(x)
y=g(x)
y=g(x)
y=f(x)

Question 5 (16 points)

Consider the function of two variables $f(x,y) = \frac{x^3}{3} + y^2 - 3x - 2xy$.

- Calculate the first-order partial derivatives.
- Find all critical points.
- Identify what type of critical points they are (local max, local min or saddle point).

Solution

- $f_x = x^2 - 3 - 2y$, $f_y = 2y - 2x$.

- $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$, $\begin{cases} x^2 - 3 - 2y = 0 \\ 2(y-x) = 0 \end{cases}$, $\Rightarrow \begin{cases} x^2 - 2x - 3 = 0 \\ y = x \end{cases}$

$\begin{cases} x = -1 \\ y = -1 \end{cases}$, $\begin{cases} x = 3 \\ y = 3 \end{cases}$

- $f_{xx} = 2x$, $f_{xy} = -2$, $f_{yy} = 2$

$d = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 4x - 4 = 4(x-1)$.

$\rightarrow (-1,-1)$: $d = -8$; so $(-1,-1)$ is saddle point.

$\rightarrow (3,3)$: $d = 8$, $f_{xx} = 6$; so $f(3,3)$ is local min