UNIVERSITY OF WATERLOO

FINAL EXAMINATION

FALL TERM 2005

Student Name (Print Legibly) ______________________

Signature ______________________

Student ID Number ______________________

COURSE NUMBER	MATH 137
COURSE TITLE	Calculus 1 for Honours Mathematics
COURSE SECTION(s)	001 002 003 004 005 006 007 008 010 011 012 013 014
DATE OF EXAM	Monday, December 19, 2005
TIME PERIOD	9:00 a.m. - 11:30 a.m.
DURATION OF EXAM	2.5 hours
NUMBER OF EXAM PAGES	10

INSTRUCTORS (please indicate your section)

- ☐ 01 C. B. Chua (11:30 a.m.)
- ☐ 02 D. McKinnon (12:30 p.m.)
- ☐ 03 A. Nayak (1:30 p.m.)
- ☐ 04 X. Liu (2:30 p.m.)
- ☐ 05 C. Small (1:30 p.m.)
- ☐ 06 R. Khandekar (9:30 a.m.)
- ☐ 07 B. Marshman (10:30 a.m.)
- ☐ 08 J. Verstraete (11:30 a.m.)
- ☐ 10 D. Wolczuk (1:30 p.m.)
- ☐ 11 C. Struthers (10:30 a.m.)
- ☐ 12 D. Wolczuk (8:30 a.m.)
- ☐ 13 P. Balka (8:30 a.m.)
- ☐ 14 A. Chau (2:30 p.m.)

EXAM TYPE	Closed Book
ADDITIONAL MATERIALS ALLOWED	'Pink Tie' Calculators only

Notes:

1. Fill in your name, ID number and sign the paper.
2. Answer all questions in the space provided. Continue on the back of the preceding page if necessary. Show **ALL** your work.
3. Check that the examination has 10 pages.
4. **Your grade will be influenced by how clearly you express your ideas, and how well you organize your solutions.**

Marking Scheme:

Question	Mark	Out of
1		18
2		15
3		12
4		8
5		10
6		12
7		11
8		14
Total		100

[3] 1. a) Given $f(x) = \cos(x^2+1) + xe^{-x}$, find $f'(0)$.

[$f'(0) = 1$]

[4] b) Given $y = x^{\sin x}$, use logarithmic differentiation to find $\frac{dy}{dx}$.

$\frac{dy}{dx} = x^{\sin x} \left[\frac{\sin x}{x} + \cos x \ln x \right]$

[4] c) Find the equation of the tangent line at (2,0) to the curve defined implicitly by $y^2 + x^3 + \tan y = 8$

[$y = -12(x-2)$]

[7] d) Evaluate each limit, or show that it does not exist.

(i) $\lim\limits_{x \to 1} \frac{2x^2 + x - 3}{x - 1}$

[5]

(ii) $\lim\limits_{x \to 2} \frac{|x-2|}{x-2}$

[DNE]

(iii) $\lim\limits_{x \to 0} \quad x^2 \ln(x^2)$

[0]

2. a) Evaluate each integral or find the general antiderivative.

[2] (i) $\int (x+1)^{1/3}\,dx$

$[\ \frac{3}{4}(x+1)^{4/3}+C]$

[4] (ii) $\int_1^4 \frac{1}{x^2} e^{\frac{1}{x}}\,dx$

$[e - e^{1/4}]$

[3] (iii) $\int \frac{\cos x}{2+\sin x}\,dx$

$[\ln|2+\sin x|+C]$

[4] (iv) $\int_0^1 \frac{x}{1+x^4}\,dx$

$[\pi/8]$

[2] b) If $g(x) = \int_0^x \sqrt{1-t^2}\,dt$, use a suitable theorem to show that $g(x)$ is an increasing function on the interval $(-1,1)$.

[2] 3. a) Evaluate $\lim_{x \to +\infty} \frac{x^2}{e^x}$. Justify your answer.

[0]

[1] b) Determine whether the function $f(x) = x^2 e^{-x}$ is even, odd, or neither.

[neither]

[6] c) Find the intervals on which f is increasing, and on which f is decreasing, and the intervals on which $y = f(x)$ is concave up, and concave down.

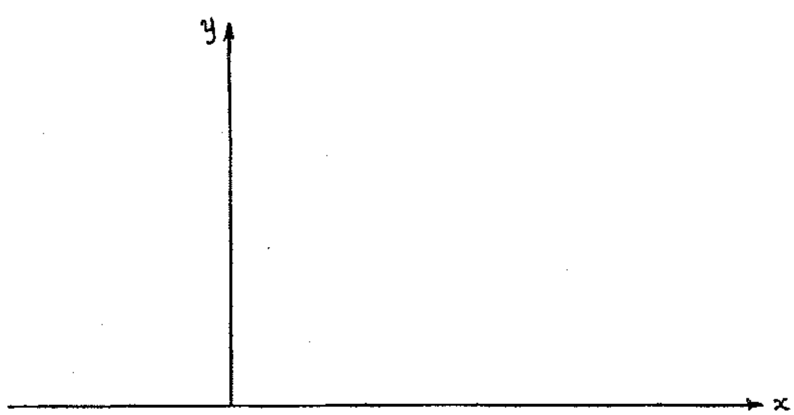
incr $(0, 2)$

decr $(-\infty, 0) \cup (2, \infty)$

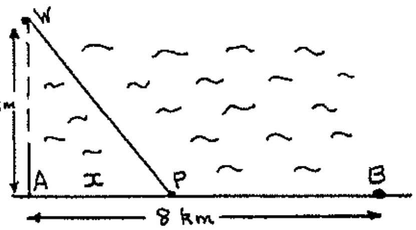
concave up $(-\infty, 2-\sqrt{2}) \cup (2+\sqrt{2}, \infty)$

" down $(2-\sqrt{2}, 2+\sqrt{2})$

[3] d) Sketch a graph of $y = f(x)$, indicating any intercepts, extremes, and asymptotes.



4.



An offshore oil well is located at W, 6 km from the nearest point A on a straight shoreline. Oil is to be piped from W to a refinery at B, 8 km from A, via an underwater pipeline from W to P, and then to B by an overland pipe along the shoreline.

[2] (a) If the cost of laying pipe is \$M per km underwater, and \$0.5 M overland (where M is a constant), show that the total cost $C(x)$ in dollars is

$$C(x) = M\left(\sqrt{x^2+36}+4-\frac{x}{2}\right), \quad \text{where} \quad x = AP.$$

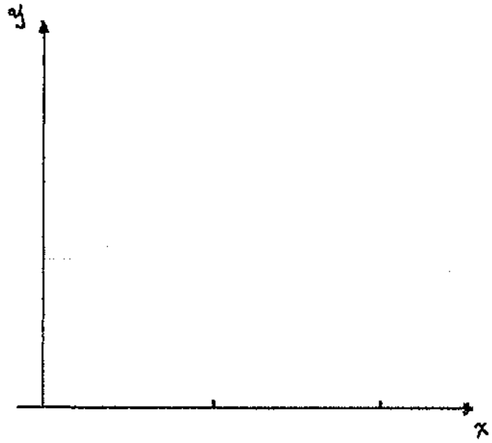
Specify the interval of values of x relevant to this situation.

$[0 \leq x \leq 8\,\text{km}]$

[6] (b) Determine the location of P which minimizes the total cost $C(x)$. Justify your method.

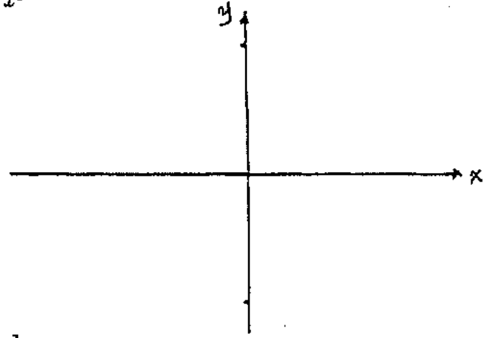
$[x = 2\sqrt{3}\ \text{km}]$

[10] 5. Find the area A bounded by the curves $y = e^x$, $y = \sin(\pi x)$, $x = 0$ and $x = 2$. Your answer must include an appropriately labelled sketch of the region, and a brief explanation of how the definite integral for A is derived, starting from a regular partition of $[0, 2]$ and suitably chosen approximating rectangles.



[$A = e^2 - 1$]

[3] 6. a) Sketch the graphs of $y = \arctan x$ and $y = \dfrac{1}{x^2}$ on the given axes, showing any asymptotes.



[1] b) Consider the function $f(x) = \arctan x - \dfrac{1}{x^2}$. Explain how you know that there is only one point c in the domain of f such that $f(c) = 0$. [HINT: Look at your sketch.]

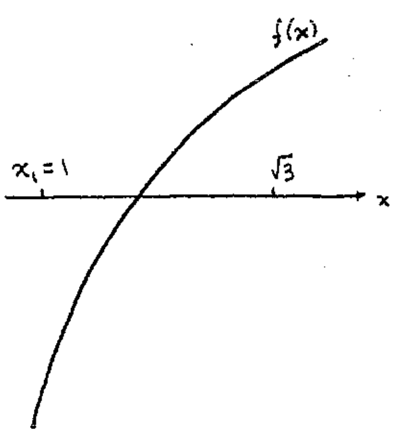
[2] c) Use a suitable theorem to show that $1 < c < \sqrt{3}$.

[IVT]

[3] d) Find the linear (tangent line) approximation $\mathcal{L}(x)$ to $f(x)$ at 1 (i.e., $\mathcal{L}_1(x)$).

$\left[y = \mathcal{L}_1(x) = \frac{\pi}{4} - 1 + \frac{5}{2}(x-1)\right]$

[3] e)



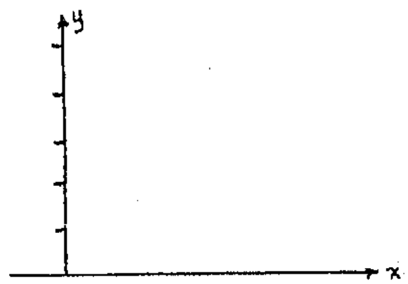
On the graph, show how Newton's method would use $x_1 = 1$ and $\mathcal{L}_1(x)$ to compute the next iterate x_2. Then use the formula for Newton's method to calculate x_2.

$\left[x_2 = \dfrac{7 - \pi/2}{5} \simeq 1.09\right]$

[3] 7. a) (i) State the Mean Value Theorem. Be sure to include all the hypotheses.

[4] (ii) Use the Mean Value Theorem to prove that if f is differentiable on an interval I and $f'(x) > 0$ on I, then f is increasing on I.

[4] b) (i) Draw a sketch on the given axes which illustrates the Right Riemann Sum R_4 (i.e., 4 subintervals) for $f(x) = x + 1$ on the interval $[0, 4]$. Will R_4 give an underestimate or an overestimate of $\int_0^4 f(x)\,dx$?



[overestimate]

(ii) How does the Mid-point Rule M_4 compare to $\int_0^4 f(x)\,dx$?

$M_n = \int_0^4 f(x)dx$

8. Determine whether each of the following statements is true or false. Justify your choices by either citing a theorem, providing reasons why the statement is true, or providing a counter example that shows why it is false. Answers with no explanations will receive no marks.

[2] a) If f is continuous at $x = a$, then f is differentiable at $x = a$.

False

[2] b) If f is continuous on $[a, b]$ and f has an absolute maximum at some point c in (a, b), then $f'(c)$ must exist and equal zero.

False

[2] c) If $f''(x)$ exists and $y = f(x)$ is concave upwards on $\mathbb{R}$, then $y = e^{f(x)}$ is concave upwards on $\mathbb{R}$.

True

[2] d) There exist differentiable functions $f(x)$ and $g(x)$ satisfying $f(x) + g(x) = x$ for all $x \in \mathbb{R}$, and $f'(0) = 1 = g'(0)$.

False

[2] e) The function $f(x) = \begin{cases} \dfrac{\sin(x^2)}{x} & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$ has $f'(0) = 1$.

True

[2] f) The curve segments $y = e^x$ for $0 \le x \le 1$ and $y = \ln x$ for $1 \le x \le e$ have the same length.

True

[2] g) Suppose that $F'(x) \le G'(x)$ for all $x \in \mathbb{R}$. Then $F(x) \le G(x)$ for all $x \in \mathbb{R}$.

False