Faculty of Mathematics

University of Waterloo

Math 137

Term Test 1 - Fall Term 2005

Time: 7:00 - 8:20 p.m. Date: October 3, 2005.

AIDS: 'PINK TIE' CALCULATORS ONLY

Family Name: ______________________ Initials: ______

I.D. Number: ____________ Signature: ______________________

Check the box next to your section:

- ☐ Section 01 C. B. Chua 11:30 a.m.
- ☐ Section 02 D. McKinnon 12:30 p.m.
- ☐ Section 03 A. Nayak 1:30 p.m.
- ☐ Section 04 X. Liu 2:30 p.m.
- ☐ Section 05 C. Small 1:30 p.m.
- ☐ Section 06 R. Khandekar 9:30 a.m.
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- ☐ Section 11 C. Struthers 10:30 a.m.
- ☐ Section 12 D. Wolczuk 8:30 a.m.
- ☐ Section 13 P. Balka 8:30 a.m.
- ☐ Section 14 A. Chau 2:30 p.m.

Your answers must be stated in a clear and logical form in order to receive full marks. Reference any theorems or rules used by their name, or the appropriate acronym.

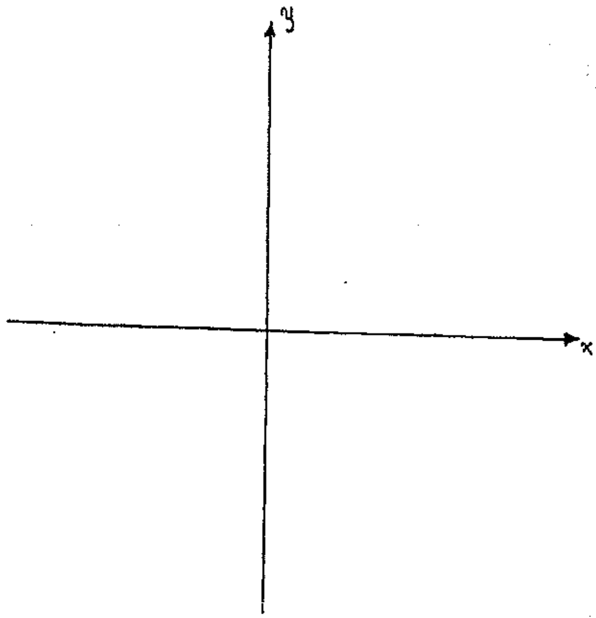
Note:

1. Complete the information section above, indicating your instructor's name by a checkmark in the appropriate box.
2. Place your initials and Id No. at the top right corner of each page.
3. Tear off the blank last page of the test to use for rough work.
4. Any 40 marks worth of questions constitutes a complete test. Excess marks will be treated as bonus.

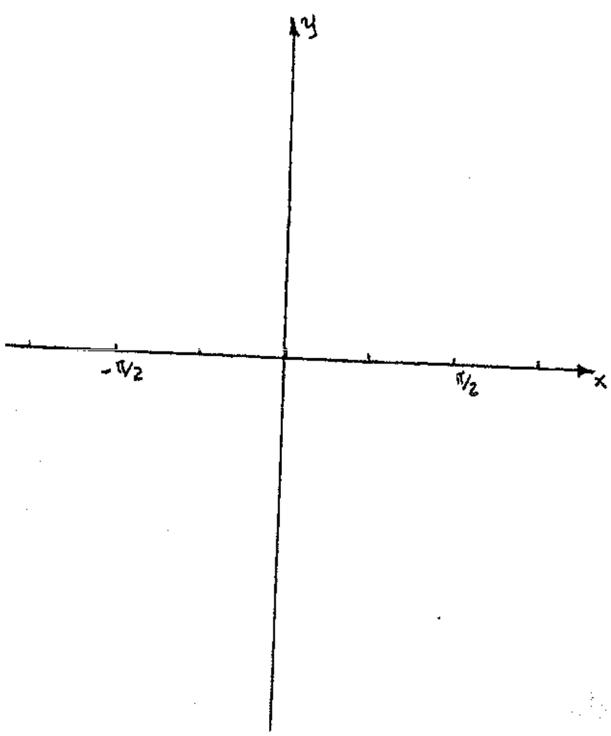
?	Mark	Out of
1		8
2		10
3		12
4		15
Total		40 + 5

1. Use graphical operations to sketch a qualitative graph of each function. Explain clearly how you obtain your graph, and indicate asymptotes and intercepts, if they occur.

[4] a) $y = 2 - e^{-x}$



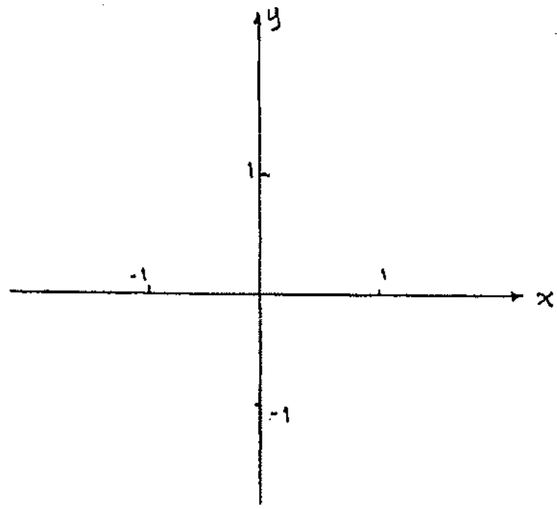
[4] b) $y = \frac{1}{2}\tan\left(x - \frac{\pi}{4}\right)$



[2] 2. a) (i) State the condition on a function f which guarantees that it has an inverse on an interval I of the real numbers.

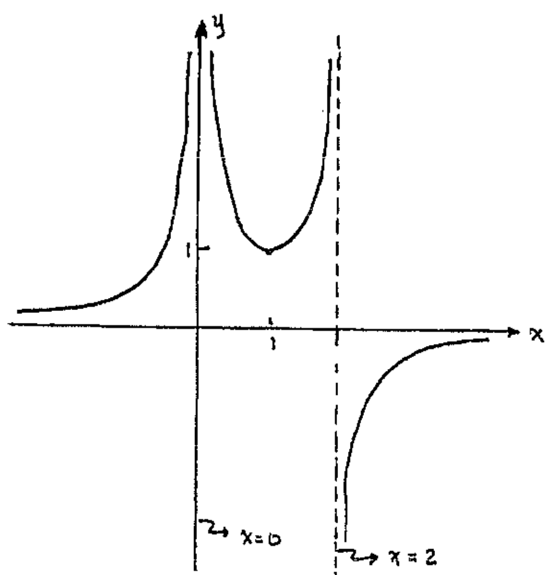
(ii) Explain how you know from its graph that the function $f(x) = 1 + x^3$ satisfies the condition in (i) for all $x \in \mathbb{R}$.

[5] b) If $f(x) = 1 + x^3$, find $f^{-1}(x)$ and sketch $y = f^{-1}(x)$ and $y = f(x)$ on the given axes.



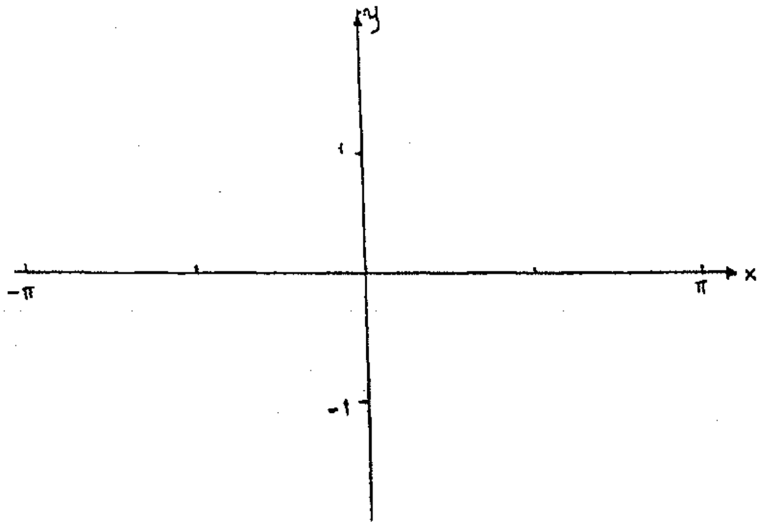
$[\,f^{-1}(x) = (x-1)^{1/3}\,]$

[3] c) The graph below is the graph of the reciprocal of a function of the form $ax^3 + bx^2$. Find a and b.



$\left[y = \dfrac{1}{x^2(2-x)}; \right.$
$\left. a = -1,\ b = 2 \right]$

[7] 3. a) (i) Sketch the graphs of $y = |\sin 2x|$ and the graph of $y = \ln x + 1$ on the given axes.



(ii) Use your graph in (i) to find a value of a such that the interval $[0, a]$ contains the single real x satisfying $|\sin 2x| = \ln x + 1$.

[0, 1]

[5] b) The radioactive isotope Carbon-14, which is present in all living things, is often used to detect the age of archeological artifacts, e.g., charcoal from an ancient campfire. Assuming the wood was burned at $t = 0$, the amount $C(t)$ of Carbon-14 undergoes exponential decay

$$C(t) = C_0 e^{-kt},$$

where C_0 is the amount present at $t = 0$, (i.e., in the living tree) and $k > 0$ is determined by the half-life T_h.

(i) Explain the meaning of half-life T_h and show that $k = \dfrac{\ln 2}{T_h}$.

(ii) If a charcoal sample has 10 % of the amount of Carbon-14 in living wood, and $T_h = 5730$ years for Carbon-14, determine the age of the sample.

[$t \approx 19{,}035$ yr]

4. Label each statement as TRUE or FALSE in the blank provided. (Use the space between questions to justify for your answer. Guesses will not be graded.)

[2] a) ______ If $f(x) = \sqrt{x-1}$ and $g(x) = x^2$, then the domain of $(g \circ f)(x)$ is $\mathbb{R}$.

[False]

[3] b) ______ The function $f(x) = \dfrac{1}{e^x + 1}$ has range $(0, 1)$.

[True]

[2] c) ______ The inequality $|2x - 3| < 7$ has solution $-1 < x < 5$.

[False]

[3] d) ______ The equation $(e^x)^2 = 2 + e^x$ has two real solutions.

[False]

[2] e) ______ The expression $\ln(e^{2x} \cdot e^{\pi/4}) - \log_2(4^x)$ has value $\frac{\pi}{4}$ for any $x > 0$.

[True]

[3] f) ______ If f is an even function then $f(|\, x\,|) = f(x)$ for any x in the domain of f.

[True]

Faculty of Mathematics

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Math 137

Term Test 2 - Fall Term 2005

Time: 7:00 - 8:30 p.m. Date: October 24, 2005.

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Family Name: ____________________ Initials: ______

I.D. Number: ______________ Signature: ____________________

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- ☐ Section 13 P. Balka 8:30 a.m.
- ☐ Section 14 A. Chau 2:30 p.m.

Your answers must be stated in a clear and logical form in order to receive full marks. Reference any theorems or rules used by their name, or the appropriate acronym.

Note:

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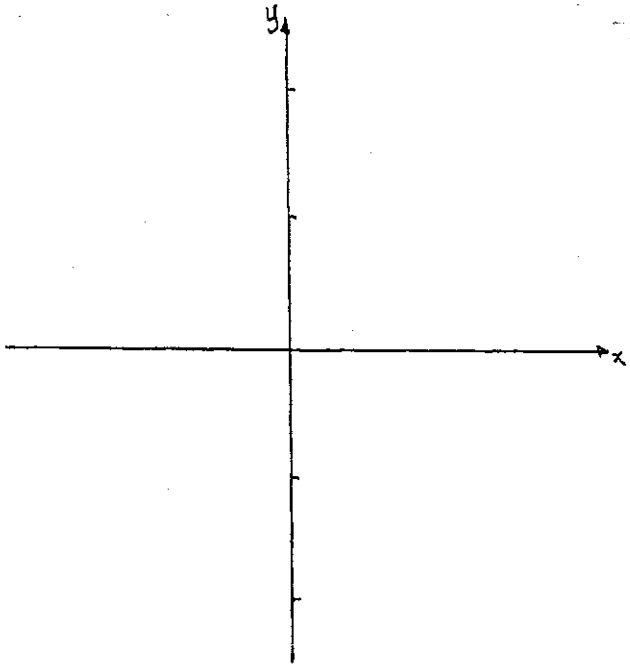
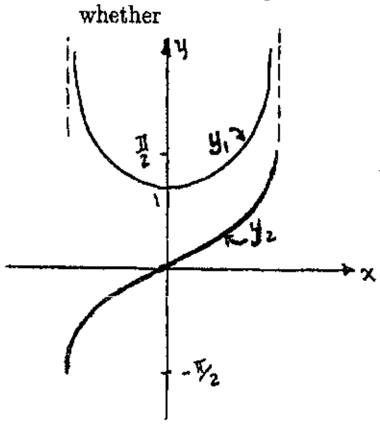
?	Mark	Out of
1		8
2		12
3		12
4		10
Total		40 + 2

[5] 1. a) Use graphical operations to sketch a graph of $y = |2\arctan(x-1)|$. Explain clearly how you obtain your graph, and indicate asymptotes and intercepts if they occur.



[3] b) Based on the graph below and known properties of the functions sec and arcsin, decide whether



(i) $y_1 = \sec x$ and $y_2 = \arcsin\left(\frac{\pi}{2}x\right)$,

OR

(ii) $y_2 = \arcsin x$ and $y_1 = \sec\left(\frac{\pi}{2}x\right)$.

Justify your answer.

[11]

[6] 2. a) Evaluate each limit, or show that it does not exist. Indicate limit rules used (e.g., Limit Sum Rule (LSR), Limit Product Rule (LPR), etc.)

(i) $\lim_{x\to 4} \frac{x^2-3x-4}{x-4}$

[5]

(ii) $\lim_{x\to\infty} \frac{1-x^2}{x+3}$

[$-\infty$]

(iii) $\lim_{x\to 0} \frac{\tan x}{x}$

[6] b) Find the domain $D(f)$ for each of the following functions, and then determine whether f is continuous at each point of $D(f)$. Indicate any theorems or rules used (e.g. Squeeze Theorem, Continuity Sum Rule (CSR), Continuity Composite Rule (CCR), etc.)

(i) $f(x) = \ln(4-x^2)$

[$(-2, 2)$]

(ii) $f(x) = \begin{cases} x^2\sin(\frac{1}{x}) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

[$\mathbb{R}$]

[4] 3. a) (i) Find $f'(1)$ for the function $f(x) = x^{4/3}e^x - e\sqrt{x}$ at $x = 1$.

$\left[\frac{11}{6}e\right]$

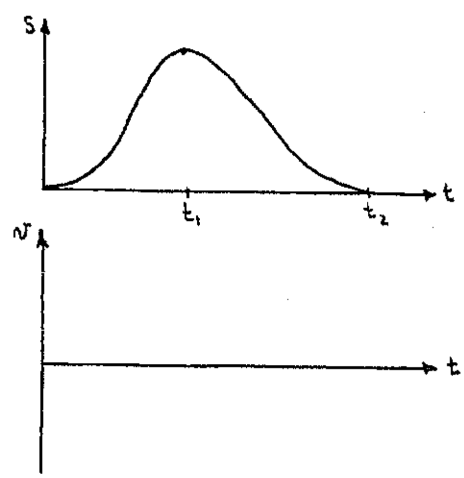
(ii) Find the equation of the tangent line to $y = f(x)$ at $x = 1$ for f as in (i).

$\left[y = \frac{11}{6}e\,(x-1)\right]$

[4] b) Use the definition of $f'(a)$ to show whether $f'(0)$ exists or not for the function $f(x) = x|x|$.

[exists]

[4] c) At right is a graph of the altitude $s(t)$ of a hot air balloon during a flight. Sketch a graph of its vertical velocity $v = \frac{ds}{dt}$ on the given axes. Describe what is happening physically to the balloon's altitude and velocity at the times t_1 and t_2.



[10] 4. In each of the following, circle the correct answer. (No justification is required.)

a) The functions $f(x) = \dfrac{x^2-9}{x-3}$ and $g(x) = x+3$ are the same functions.

(i) True (ii) False

[(ii)]

b) $\sec(\arcsin x)$ is equal to

(i) $\dfrac{1}{x}$ (ii) $\dfrac{1}{\sqrt{1-x^2}}$ (iii) $\sqrt{1-x^2}$.

[(ii)]

c) If $f(x) = x^3 + x - 2$, then $f^{-1}(0)$ is equal to

(i) $-\dfrac{1}{2}$ (ii) -1 (iii) 1 .

[(iii)]

d) $\lim\limits_{x\to-\infty} e^{\arctan(x^2-x)}$ is equal to

(i) 1 (ii) $e^{\frac{\pi}{2}}$ (iii) $e^{-\frac{\pi}{2}}$.

[(ii)]

e) $(\sin x)\,(\sin 2x) + (\cos x)\,(\cos 2x) = \cos x$ for all x.

(i) True (ii) False

[(i)]

f) The function $f(x) = \arcsin(x^3-1)$ is continuous at $x=-1$.

(i) True (ii) False

[(ii)]

g) The function $f(x) = \begin{cases} x+1 & \text{for} \quad x > 1 \\ 3-x^2 & \text{for} \quad x \le 1 \end{cases}$

(i) has a jump discontinuity at $x=1$. (ii) is continuous for all x.

[(ii)]

h) If $\lim\limits_{x\to 0} f(x) = 0$, and $g(x)$ is defined for all x near 0 (but not necessarily at $x=0$), then $\lim\limits_{x\to 0} f(x)g(x) = 0$.

(i) Always. (ii) Sometimes.

[(ii)]

i) If $f(x) = x^3 + \arctan x$ for $0 < x < 1$, then $x = 3$ is in the domain of f^{-1}.

(i) True (ii) False

[(ii)]

j) For $f(x) = x^{3/4}$, the domain of $f'(x)$ is $\{x|x \ge 0\}$.

(i) True (ii) False

[(ii)]

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Math 137

Term Test 3 - Fall Term 2005

Time: 7:00 - 8:30 p.m. [illegible] pts for Fall 2006 Date: October 24, 2005.

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Family Name: ______________________ Initials: ______

I.D. Number: ______________ Signature: ______________________

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?	Mark	Out of
1		14
2		9
3		10
4		7
Total		40

[14] 1. a) For $f(x) = xe^{(x^2)} + \ln(\cos x)$, find $f'(0)$.

[1]

b) Find the equation of the tangent line at the point $(0, -1)$ to the curve defined implicitly by $xy + y^3 = \arctan x - 1$.

[$y = -1 + \frac{2}{3}x$]

c) Evaluate each limit. Justify your method.

(i) $\lim_{x\to 0} \dfrac{\tan x + x^2 - x}{\sin^2 x}$

[1]

(ii) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{2x}$

[e^2]

d) If $h(x) = \arcsin x + \sqrt{x^2 - 1}$, for what values of x does $h'(x)$ exist?

[none]

[9] 2. Consider the function $f(x) = \frac{1}{3}x^3 + x - 1$.

a) Using a suitable theorem, show that $f(c) = 0$ for some c in $[0, 1]$.

b) Use the method of bisection to estimate c with an error of at most 0.25.

$[c \sim \frac{3}{4}]$

c) Find the linear (tangent line) approximation $\mathcal{L}_0(x)$ for $f(x)$ at $x = 0$ and hence estimate $f(0.11)$.

$[\sim -0.89]$

d) Explain how you know that f has an inverse f^{-1} on $\mathbb{R}$. Using the Inverse Function Derivative Theorem (INVDR) or any other method, find the slope of the tangent line to $y = f^{-1}(x)$ at $x = -1$.

$[1]$

[10] 3. Consider the function $f(x) = x\sqrt{2-x^2}$.

a) State the domain D of f.

[$-\sqrt{2} \leq x \leq \sqrt{2}$]

b) Find all critical numbers (points) of f in D.

[$x = \pm 1$,
$x = \pm\sqrt{2}$]

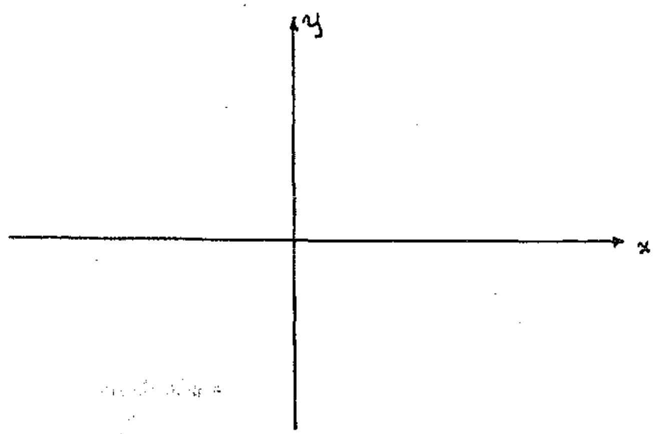
c) Find the intervals on which f is increasing, and on which f is decreasing.

[inc. $-1 < x < 1$,
decr $1 < |x| < \sqrt{2}$]

d) Find the intervals on which the graph of $y = f(x)$ is concave up and concave down.
$\left[\text{NOTE: } f''(x) = \dfrac{2x(x^2-3)}{(2-x^2)^{3/2}} \right]$

[down $(0, \sqrt{2})$,
up $(-\sqrt{2}, 0)$]

e) Use the results of a)-d) to sketch the graph of $y = f(x)$ on the given axes, indicating any local extremes.



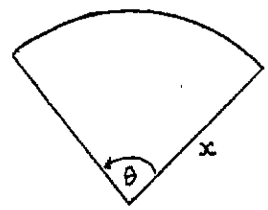
f) Describe the shape of the resulting figure if you add to your sketch the graph of $y = -f(x)$. Write the equation of the complete curve in the form $y^2 = h(x)$ for suitable h.

[$y^2 = x^2(2-x^2)$]

[7] 4. You wish to design a garden in the shape of a circular sector having radius x and central angle θ (in radius), as shown. The plantings you have in mind require an area of $A = 64$ m^2. Space restrictions and design considerations require the radius x to be no more than 10 m and no less than 2m.

a) Show that, given $A = 64\text{m}^2$, the total perimeter of the garden is

$$P = 2x + \frac{128}{x} \text{ m, where } 2 \le x \le 10.$$



NOTE: For the segment shown, the area is $\frac{1}{2}x^2\theta$ and the arc length is $x\theta$.

b) Find the minimum length of fencing required to go around the perimeter of the garden. Justify your method.

[32 m]