4.2

Solutions # 8

#4. $f(x) = x\sqrt{x+6}$; $[-6,0]$

6pt.

$f(x)$ is continuous on $[-6,0]$ & diff. on $(-6,0)$ & $f(-6) = f(0) = 0$.

$$f'(x) = \sqrt{x+6} + \frac{x}{2\sqrt{x+6}} = \frac{2(x+6)+x}{2\sqrt{x+6}} = \frac{3x+12}{2\sqrt{x+6}}$$ 3

$\boxed{\therefore\ c = -4 \text{ satisfies conclusion of Theorem}}$ (RT) 3

#6 $f(0) = 1 = f(2)$. 1

6pt.

$$f'(x) = -2(x-1)^{-3} = \frac{-2}{(x-1)^3}$$ 2

$$\therefore \begin{cases} f' > 0 & \text{for } x < 1 \\ f' \text{ undefined} & \text{for } x = 1 \\ f' < 0 & \text{for } x > 1 \end{cases}$$ 1

$\therefore$ f' is never zero.

f not differentiable on $(0,2)$, $\therefore$ can't use Theorem (RT) 2

36. we assume f is differentiable here,
4pt.
So now suppose f has 2 fix points $x=a$, $x=b$
and suppose $a<b$. Then by Theorem (MVT)

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{b-a}{b-a} = 1 \quad 2$$

for some $c \in (a,b)$. But this contradicts 2
assumption. $\therefore$ cannot have 2 fix points
$\therefore$ at most 1 fixed point.

4.5

24. $y = f(x) = x\sqrt{2-x^2}$
8pt.

A; Domain $x^2 \leq 2$

$\therefore -\sqrt{2} \leq x \leq \sqrt{2}$ 1

B; intercepts $f(x) = 0$
$\Rightarrow x = 0, \pm\sqrt{2}$ $\Big\}$ x intercepts 1

$f(0) = 0$ $\}$ y intercepts

C; $f(x)$ is ODD

D; no horizontal asymptotes.
(vertical)

E; $f'(x) = \sqrt{2-x^2} + \frac{x\,(\cancel{2}x)}{\cancel{2}\sqrt{2-x^2}} = \frac{2-2x^2}{\sqrt{2-x^2}} = 0$

$\Rightarrow x = \pm 1$

	$f'(x)$	
$-\sqrt{2} \le x < -1$	$-$	decrease
$-1 < x < 1$	$+$	increase
$1 < x < \sqrt{2}$	$-$	decrease

2

F; ~~from E we have~~

- local min of $f(-1) = -1$ at $x = -1$
- local max of $f(1) = 1$ at $x = 1$

G; $f'' = \dfrac{\sqrt{2-x^2}\cdot 2(-2x) - \dfrac{\cancel{2}(1-x^2)(-2x)}{\cancel{2}\sqrt{2-x^2}}}{(2-x^2)}$

$= \dfrac{-(2-x^2)4x + 2x(1-x^2)}{(2-x^2)^{3/2}}$

$= \dfrac{2x(x^2-3)}{(2-x^2)^{3/2}} = 0 \Rightarrow x = 0, \cancel{\pm\sqrt{3}}$ not in domain

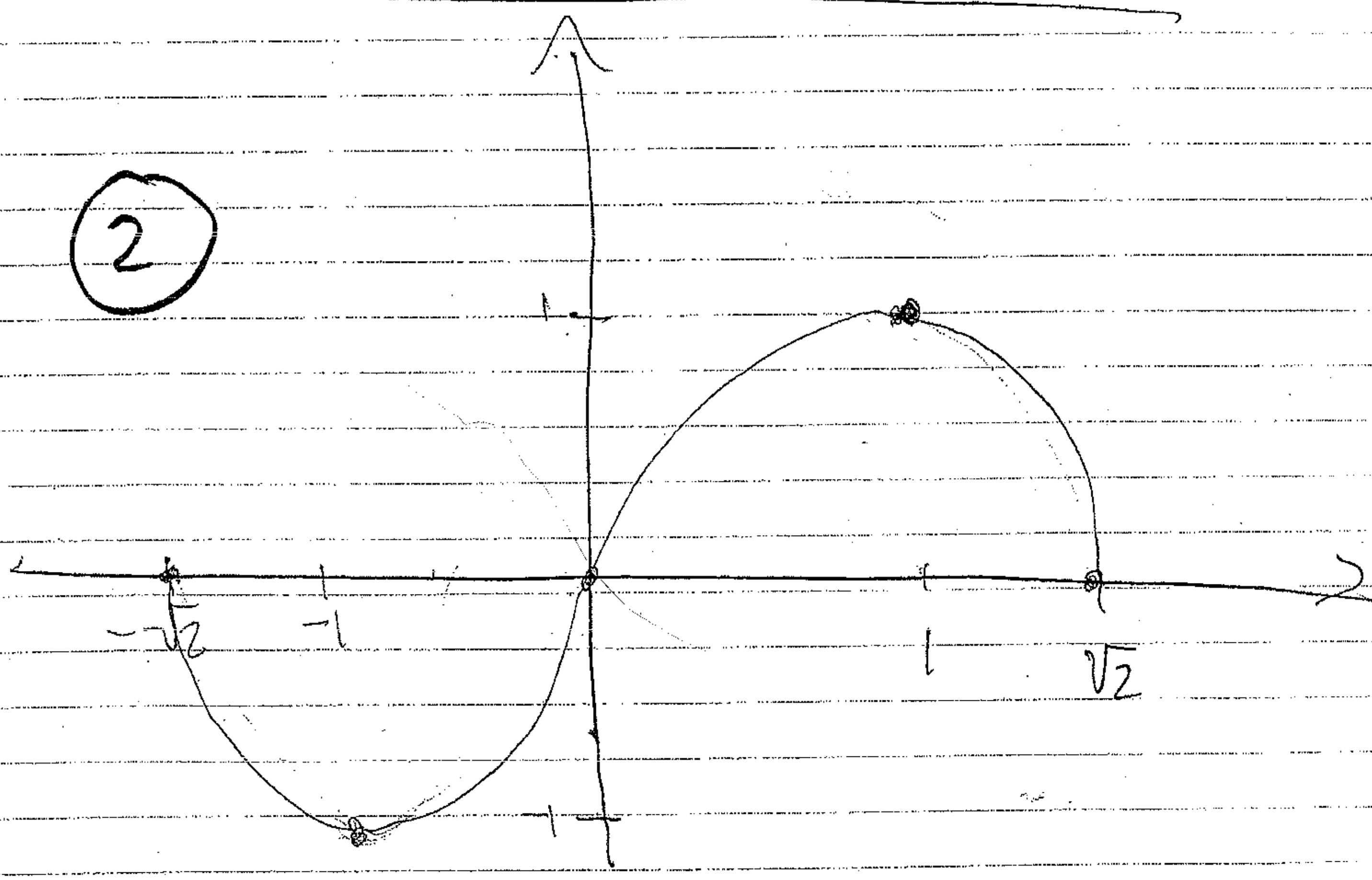
x	f''	
$-\sqrt{3}<x<0$	$+$	concave up
$\sqrt{3}>x>0$	$-$	concave down

2

~~wt~~

$\therefore$ inflection pt. at $x=0$

2



4.7

5pt. 2. ~~Q(x)~~ let numbers be $100+x$ and x.

Then minimize
$$Q(x) = (100+x)x \quad \text{for } x \in \mathbb{R}$$ 2

$$Q'(x) = 2x + 100 = 0$$

$$\Rightarrow x = -50$$

	$Q'(x)$
$x<-50$	$-$
$x>-50$	$+$

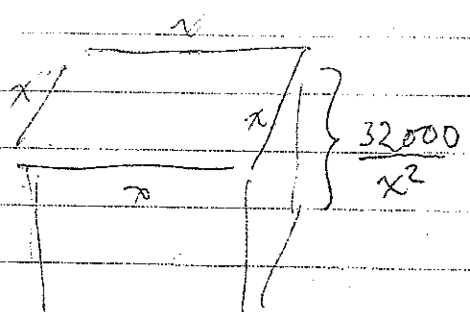
2

1

$\therefore$ $Q(x)$ has absolute minimum at $x = -50$

$\therefore$ numbers are $50, -50$

10. **5pt.**



Material used (Surface area) $= 4\left(\frac{32000}{x}\right) + x^2$

Minimize

$$Q(x) = \frac{4 \cdot 32000}{x} + x^2 \quad ; \quad x > 0.$$

2

$$Q'(x) = \frac{-4 \cdot 32000}{x^2} + 2x = \frac{1}{x^2}\left(2x^3 - 4 \cdot 32000\right) = 0$$

$$\Rightarrow x = \sqrt[3]{\frac{4 \cdot 32000}{2}} = \sqrt[3]{64000} = 40$$

	$Q'(x)$
$0 < x < 40$	$-$
$x > 40$	$+$

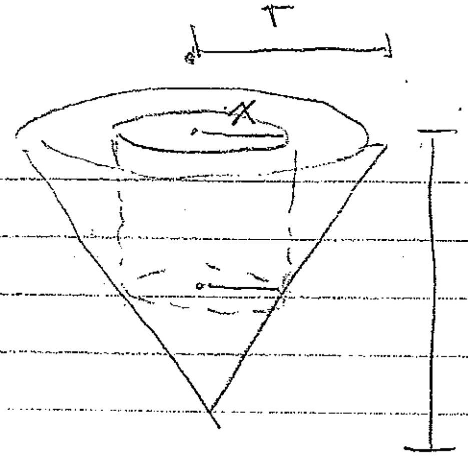
2

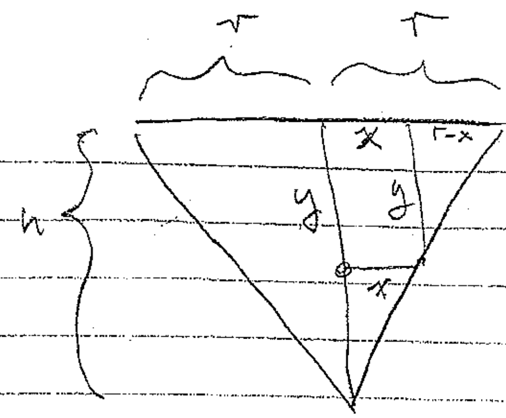
1

$\therefore$ $Q(x)$ has absolute min. at $x = 40$
and dimension of which box are $(40 \times 40) \times 40$
cube

26.

6pt.





then

$$\frac{y}{r-x} = \frac{h}{r}$$

$$\therefore y = \frac{h}{r}(r-x)$$

$V(x) =$ Volume of cylinder $= \pi x^2 y$

$$= \pi x^2 \frac{h}{r}(r-x) \quad ; \quad x \in [0, r]$$

$$= \frac{\pi h}{r}(r x^2 - x^3)$$

2

$$V'(x) = \frac{\pi h}{r}(2rx - 3x^2) = \frac{\pi h}{r} x (2r - 3x)$$

$$\Rightarrow x = 0, \ \frac{2r}{3}$$

2

$\frac{2r}{3}$ only critical pt. in $(0, r)$.

endpoints;

$$V(0) = 0$$

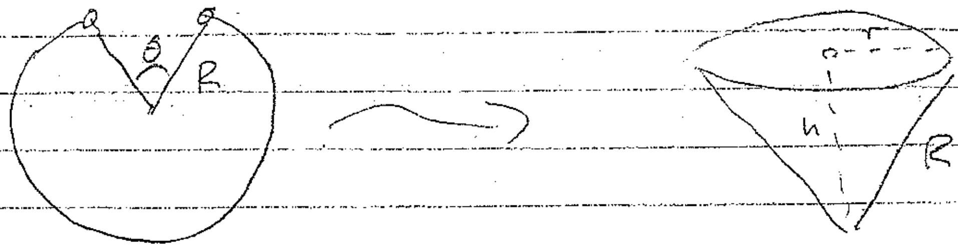
$$V(2/3 r) = \pi \left(\frac{2}{3}r\right)^2 \frac{h}{r}\left(\frac{1}{3}r\right) = \frac{4}{27}\pi h r^2$$

$$V(r) = 0$$

2

$\therefore$ Max Volume is $\boxed{\frac{4}{27}\pi h r^2}$

36.



length of edge after removing wedge $= (2\pi R - \theta R)$

$\therefore 2\pi r = 2\pi R - \theta R$

$\therefore r = \dfrac{R(2\pi - \theta)}{2\pi} = R\left(1 - \dfrac{\theta}{2\pi}\right)$

$h = \sqrt{R^2 - r^2}$

$= \sqrt{R^2 - R^2\left(1 - \frac{\theta}{2\pi}\right)^2}$

3

$= R\sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2}$

$\therefore$ Volume of cone

$= V(\theta) = \dfrac{\pi}{3} h r^2 = \dfrac{\pi}{3} R\sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2}\; R^2\left(1 - \frac{\theta}{2\pi}\right)^2$

$= \dfrac{\pi}{3} R^3 \sqrt{1 - \left(1 - \theta/2\pi\right)^2}\; \left(1 - \theta/2\pi\right)^2$

3

$= \dfrac{\pi}{3} R^3 \sqrt{\dfrac{\theta}{\pi} - \dfrac{\theta^2}{4\pi^2}}\; \left(1 - \dfrac{\theta}{2\pi}\right)^2$

$\therefore V'(\theta) = \dfrac{\pi}{3} R^3 \left[\dfrac{\left(\frac{1}{\pi} - \frac{\theta}{2\pi^2}\right)}{2\sqrt{\theta/\pi - \theta^2/4\pi^2}} \left(1 - \dfrac{\theta}{2\pi}\right)^2 - \dfrac{1}{2\pi}\sqrt{\theta/\pi - \theta^2/4\pi^2}\; 2\left(1 - \theta/2\pi\right) \right]$

$= \dfrac{\pi}{3} R^3 \left[\dfrac{\left(1/\pi - \theta/2\pi^2\right)\left(1 - \theta/2\pi\right)^2 - \frac{2}{\pi}\left(\theta/\pi - \theta^2/4\pi^2\right)\left(1 - \theta/2\pi\right)}{2\sqrt{\theta/\pi - \theta^2/4\pi^2}} \right]$

$= 0 \implies$

set numerator to zero gives

$$\Rightarrow (1-\theta/2\pi)(1-\theta/2\pi) = 2/\pi\,(\theta-\theta^2/4\pi)$$

$$\Rightarrow 1-\theta/\pi+\theta^2/4\pi^2 = 2\theta/\pi - \theta^2/2\pi^2$$

$$\Rightarrow \frac{3}{4\pi^2}\theta^2 - \frac{3}{\pi}\theta + 1 = 0$$

$$\Rightarrow \theta = \frac{3/\pi \pm \sqrt{9/\pi^2 - 4(3/4\pi^2)}}{2(3/4\pi^2)}$$

$$= 2\pi\left(1 \pm \frac{\sqrt{6}}{3}\right)$$

$\Rightarrow 2\pi(1-\frac{\sqrt{6}}{3})$ is only critical pt. in $(0, 2\pi)$

Endpoints;

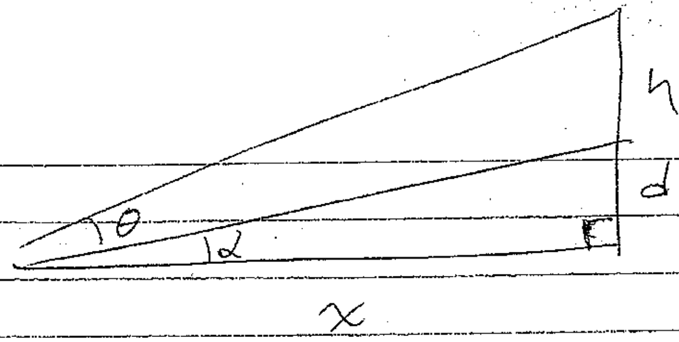
$V(0) = 0$

$V(2\pi(1-\sqrt{6}/3)) > 0$

$V(2\pi) = 0$

$$\boxed{\therefore\ V \text{ is maximal at } \theta = 2\pi(1-\sqrt{6}/3)}$$

58.



$$\tan\alpha = \frac{d}{x} \implies \alpha = \tan^{-1}\left(\frac{d}{x}\right)$$

$$\tan\theta = \frac{h+d}{x} \implies \theta = \tan^{-1}\left(\frac{h+d}{x}\right)$$

$$\boxed{\therefore\ \theta(x) = \tan^{-1}\left(\frac{h+d}{x}\right) - \tan^{-1}\left(\frac{d}{x}\right) \ ;}$$

now using $\left(\tan^{-1}(x)\right)' = \frac{1}{1+x^2}$ gives

$$\theta'(x) = \frac{1}{1+\left(\frac{h+d}{x}\right)^2}\left(-\frac{h+d}{x^2}\right) - \frac{1}{1+\left(\frac{d}{x}\right)^2}\left(-\frac{d}{x^2}\right)$$

$$= \frac{-(h+d)}{x^2+(h+d)^2} + \frac{d}{x^2+d^2}$$

$$= \frac{-(h+d)(x^2+d^2) + d\left(x^2+(h+d)^2\right)}{(x^2+d^2)\left(x^2+(h+d)^2\right)}$$

$$= \frac{-h(x^2+d^2) - d^3 + d(h+d)^2}{(\quad)(\quad)}$$

$$= 0$$

$$\implies h(x^2+d^2) = d(h+d)^2 - d^3$$

$$= \frac{-hx^2 + hd^2 + dh^2}{(\ldots)(\ldots)} = \frac{-h\left(x^2 - d(d+h)\right)}{(\ldots)(\ldots)} = 0$$

$$\Longrightarrow \quad x = \sqrt{d(d+h)}$$

	$\theta'(x)$
$0 < x < \sqrt{d(d+h)}$	$+$
$x > \sqrt{d(d+h)}$	$-$

$$\boxed{\therefore\ \theta(x) \text{ has absolute Max at } x = \sqrt{d(d+h)}}$$