Solutions #6.

3.11

#31. $f(x) = x^5$; $f'(x) = 5x^4$

5pt.

$\therefore$ $f(x) \sim f'(2)(x-2) + f(2)$

$= 80(x-2) + 32$ 3

for x near 2

2

$$\boxed{\therefore f(2.001) \sim 80(.001) + 32 = 32.08}$$

#32. $f(x) = \sqrt{x}$; $f'(x) = \frac{1}{2\sqrt{x}}$

5pt.

$\therefore f(x) \sim f'(100)(x-100) + f(100)$

$= \frac{1}{20}(x-100) + 10$ 3

for x near 100

2

$$\boxed{\therefore f(99.8) \sim \frac{1}{20}(-.2) + 10 = 10 - .01 = 9.99}$$

#34. $f(x) = \frac{1}{x}$; $f'(x) = -\frac{1}{x^2}$

$\therefore f(x) \sim f'(1000)(x-1000) + f(1000)$

$= -10^{-6}(x-1000) + \frac{1}{1000}$

$$\boxed{\therefore f(1002) \sim -\frac{1}{1000^2}2 + \frac{1}{1000} = .000998}$$

Now using $T_2(x)$ for # 31, 32, 34

#31. $f''(x) = 20x^3$

$$\therefore f(x) \sim T_2(x) = \frac{f''(2)}{2}(x-2)^2 + f'(2)(x-2) + f(2)$$

$$= 80(x-2)^2 + 80(x-2) + 32 \quad 3$$

$$\boxed{\begin{aligned} \therefore f(2.001) &\sim 80(.001)^2 + 80(.001) + 32 \\ &= 32.08008 \end{aligned}} \quad 2$$

#32. $f''(x) = -\frac{1}{4x^{3/2}}$

$$\therefore f(x) \sim T_2(x) = \frac{f''(100)}{2}(x-100)^2 + f'(100)(x-100) + f(100)$$

$$= -.000125(x-100)^2 + \frac{1}{20}(x-100) + 10 \quad 3$$

$$\boxed{\begin{aligned} \therefore f(99.8) &\sim -.000125(-.2)^2 + \frac{1}{20}(-.2) + 10 \\ &= 9.989995 \end{aligned}} \quad 2$$

#34. $f''(x) = \frac{2}{x^3}$ $\therefore$ Similar to ~~pre~~ above,

$$f(x) \sim T_2(x) = 10^{-9}(x-1000)^2 - 10^{-6}(x-1000) + 10^{-3}$$

$$\boxed{\therefore f(1002) \sim .000998004}$$

4.4

2. a) yes   b) no (limit $= \infty$)

c) no (limit $= \infty$)

4. a) yes   b) no (limit $= 0$)

5pt.

c) yes   d) yes

e) no (limit $= \infty$)

18. type $\frac{\infty}{\infty}$

5pt.

$$\lim_{x\to\infty} \frac{[\log(\log x)]'}{x'} \quad 1$$

$$= \lim_{x\to\infty} \frac{\frac{1}{\log x}\cdot\frac{1}{x}}{1}$$

$$= \lim_{x\to\infty} \frac{1}{x\log x} \quad 2$$

$$= 0$$

$$\boxed{\therefore \lim \frac{\log(\log(x))}{x} = 0} \quad 2$$

by (HR)

20. type $\frac{0}{0}$

$$\lim_{x \to 1} \frac{(\log x)'}{(\sin \pi x)'} = \lim_{x \to 1} \frac{1/x}{\pi \cos \pi \cdot x} = \frac{1}{-\pi}$$

$$\boxed{\therefore \lim_{x \to 1} \frac{\log x}{\sin \pi x} = -\frac{1}{\pi}}$$ by (HR)

48. type $\infty - \infty$

5pt.

$$= \lim_{x \to 1} \frac{(x-1) - \log x}{(x-1)\log x}$$ (2) (type $\frac{0}{0}$)

$$\lim_{x \to 1} \frac{(x-1-\log x)'}{((x-1)\log x)'} = \lim_{x \to 1} \frac{1 - 1/x}{\log x + \frac{x-1}{x}}$$ (type $\frac{0}{0}$)

(1.5)

$$\lim_{x \to 1} \frac{(1-\frac{1}{x})'}{(\log + \frac{x-1}{x})'} = \lim_{x \to 1} \frac{\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}$$

(1.5)

$$\boxed{\therefore \lim_{x \to 1} \left(\frac{1}{\log x} - \frac{1}{x-1} \right) = \frac{1}{2}}$$

58. type ∞^0

$$\lim_{x\to\infty} \frac{1}{x}\log(e^x + x) \quad \left(\text{type } \frac{\infty}{\infty}\right)$$

$$\lim_{x\to\infty} \frac{[\log(e^x + x)]'}{x'} = \lim_{\infty} \frac{\frac{1}{e^x+x}\cdot(e^x+1)}{1}$$

$$\left(\text{type } \frac{\infty}{\infty}\right)$$

$$\lim_{x\to\infty} \frac{(e^x+1)'}{(e^x+x)'} = \lim_{x\to\infty} \frac{e^x}{e^x+1} = 1$$

$$\boxed{\therefore \lim_{x\to\infty} (e^x + x)^{1/x} = e^1} \quad \text{by (HR)}$$

70. 5pt. a) $\boxed{\lim_{t\to\infty} V = \frac{mg}{c}}$ 2

b) $$\lim_{m\to\infty} V = \lim_{m\to\infty} \frac{(1-e^{-ct/m})}{c/mg} \quad 1 \quad \left(\text{type } \frac{0}{0}\right)$$

$$\lim_{m\to\infty} \frac{(1-e^{-ct/m})'}{(c/mg)'} = \lim_{m\to\infty} \frac{-\frac{ct}{m^2}e^{-\frac{ct}{m}}}{-\frac{c}{m^2 g}} = gt \quad 2$$

$$\boxed{\therefore \lim_{m\to\infty} V = gt}$$

4.9

6. $f(x) = x^3 - x^2 - 1$; $f'(x) = 3x^2 - 2x$
5pt.

1

$x_1 = 1$

2 $x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{(-1)}{1} = 2$

2 $x_3 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{8} = \frac{13}{8}$

12. $f(x) = x^7 - 1000$; $f'(x) = 7x^6$

then root of f is $\sqrt[7]{1000}$.
Initial guess, --- let $x_0 = 2$.

then

$x_1 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-872}{448}$

$= 3.94643$

$x_2 = (3.9\ldots) - \frac{f(3.9\ldots)}{f'(3.9\ldots)} = 3.42047$

$x_3 = (3.42047) - \frac{f(3.42\ldots)}{f'(3.42\ldots)} = 3.02104$

Solutions #7

4.9

11. let $f(x) = x^3 - 30 \quad \therefore f'(x) = 3x^2$

Then $\sqrt[3]{30}$ is a root of $f(x)$.

let $x_0 = 3$,

$$\therefore \quad x_1 = 3 - \frac{f(3)}{f'(3)} = 3.11111111$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.107237339$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.107232506$$

$$\boxed{x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 3.107232506}$$

33. Newton's Method gives

6pt

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad 2$$

$$= x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}}$$

$$= x_n - 3x_n$$

$$= -2x_n \quad 2$$

if they just draw appropriate graphs, but can't do the calculation give 2 pts

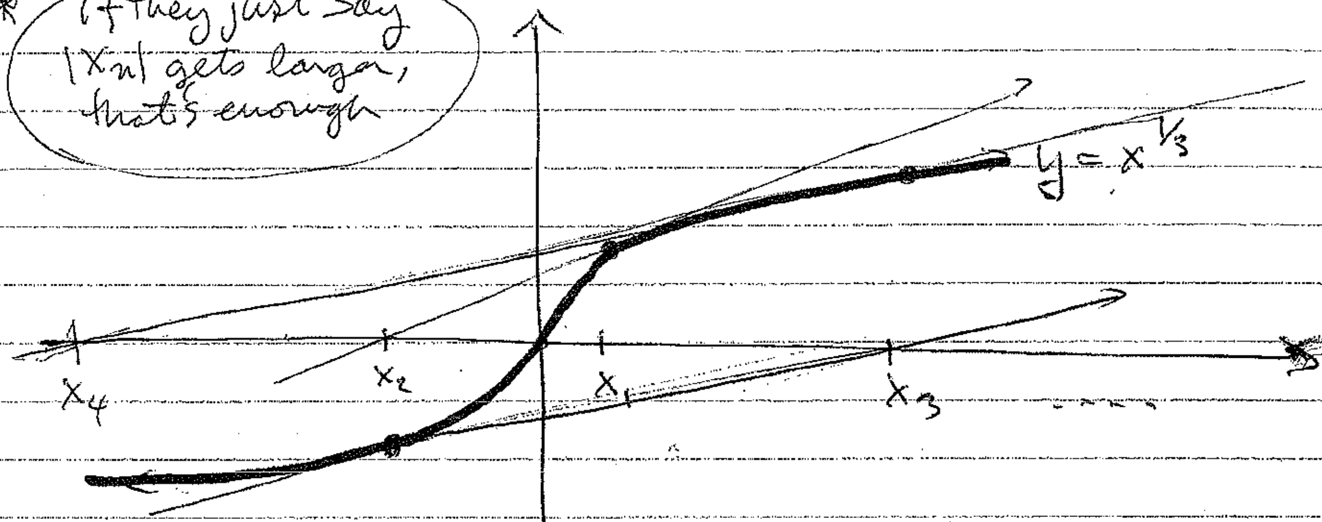
**

$\therefore \quad |X_{n+1}| = 2|X_n|$

$\therefore \quad \lim_{n\to\infty} |X_n| = \infty$ provided $X_1 \neq 0$.

2

** if they just say $|X_n|$ gets larger, that's enough



$|X_n|$ gets larger as $n \to \infty$.

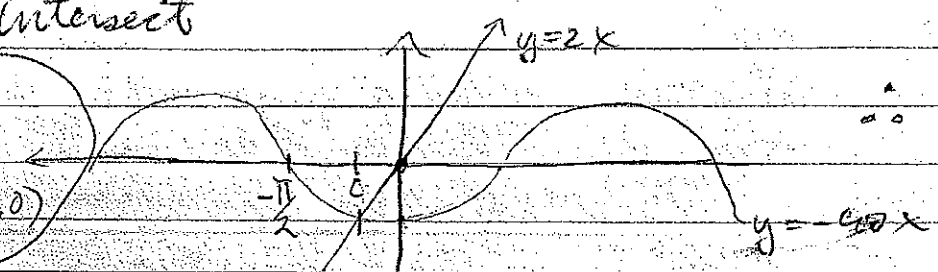
36. Assume $x^2 + \sin x$ does have an absolute minimum at $x = c$.
Then we must have

$$f'(c) = 0$$

$\therefore$ 2 $\quad 2c + \cos c = 0 \quad (2c = -\cos c)$

$\therefore$ c is where graphs $y = 2x$ and $y = -\cos x$ intersect

** give 2 pts as long as ~~xxx~~ $c \in (-\pi/2, 0)$



$\therefore c \sim -\pi/4$ 2

now use Newtons Method to approx the root of $f'(x)$ using $x_0 = -\pi/4$.

This gives

$$x_0 = -\pi/4 = -.7853982$$

4

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = -.4663529$$

$$x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} = -.4502314$$

$$x_3 = x_2 - \frac{f'(x_2)}{f''(x_2)} = -.4501836$$

$$\boxed{x_4 = x_3 - \frac{f'(x_3)}{f''(x_3)} = -.4501836}$$

(plug x_4 into your calculator and see how close $f'(x_4)$ is to zero!)

4.1 50. critical points in $(-1,4)$

5pt

$$f'(x) = 3x^2 - 12x + 9$$

$$= 3(x+1)(x+3) = 0$$

2 $\Rightarrow x = -1, -3$ $\therefore$ no critical points in $(-1,4)$

endpoints

1

2 $f(-1) = -1 - 6 - 9 + 2 = -14$ ← absolute Min

$f(4) = 64 - 96 + 36 + 2 = 6$ ← absolute Max

$\therefore$ absolute Max $= 6$
" Min $= -14$

52. 1. critical points in $(-1, 2)$
5pt

$$f'(x) = 3(x^2-1)^2(2x)$$
$$= 0$$
$$\Rightarrow x = \pm 1, 0$$

2

2. Endpoints

$$f(-1) = 0$$
$$f(0) = -1$$
$$f(1) = 0$$
$$f(2) = 27$$

2

$\therefore$ absolute Max is 27
" Min is 0

1

70. 1. critical points in $(0, \pi/2)$
8pt.

$$\frac{dF}{d\theta} = \frac{-\mu W}{(\mu \sin\theta + \cos\theta)^2}(\mu\cos\theta - \sin\theta) = 0$$

$$\Rightarrow (\mu\cos\theta - \sin\theta) = 0$$

2

$$\Rightarrow \mu = \frac{\sin\theta}{\cos\theta} = \tan\theta, \quad \theta \in (0, \pi/2)$$

2. Endpoints.

let critical point be θ_1. Then

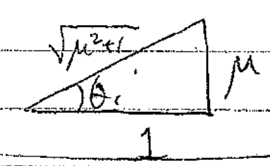
$$F(\theta_1) = \frac{\mu W}{\mu \sin\theta_1 + \cos\theta_1}$$

$$= \frac{\left(\frac{\sin\theta_1}{\cos\theta_1}\right) W}{\left(\frac{\sin\theta_1}{\cos\theta_1}\right)\sin\theta_1 + \cos\theta_1}$$

$$= \frac{\sin\theta_1 W}{\sin^2\theta_1 + \cos^2\theta_1} = \sin\theta_1 W$$ 2

but $\tan\theta_1 = \mu \Rightarrow \sin\theta_1 = \frac{\mu}{\sqrt{\mu^2+1}} < \mu$

since



2

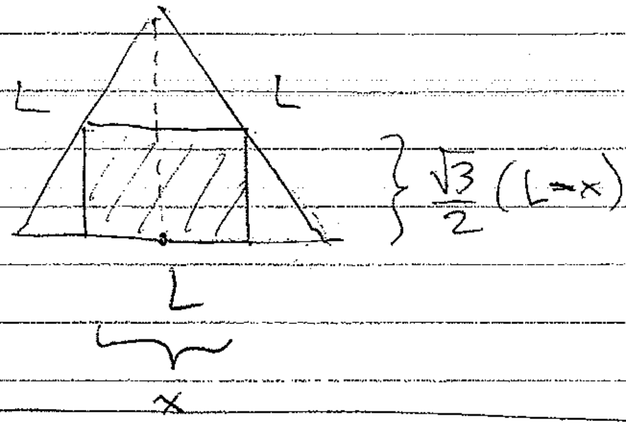
$\therefore \quad F(\theta_1) < \mu W = F(0)$ 2

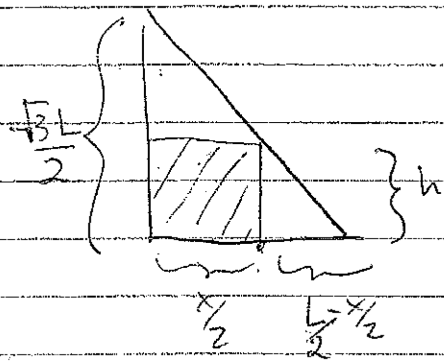
also $F(\theta_1) < W = F(\pi/2)$

$\therefore$ $F(\theta_1) = \sin\theta_1 W$ is absolute Min of F.
and $\tan\theta_1 = \mu$

(P337)

21.





$$\therefore \quad \frac{\frac{\sqrt{3}\,\cancel{L}}{\cancel{2}}}{\frac{\cancel{L}}{\cancel{2}}} = \frac{h}{\frac{L}{2}-\frac{x}{2}}$$

$$\therefore \quad h = \frac{\sqrt{3}}{2}(L-x)$$

$\therefore$ $A(x)$ = area of rectangle

$$= x\left(\frac{\sqrt{3}}{2}(L-x)\right)$$

$$= \frac{\sqrt{3}}{2}(Lx - x^2) \qquad ; \quad x \in [0, L]$$

1. critical point

$$A'(x) = \frac{\sqrt{3}}{2}(L-2x) = 0$$

$$\Longrightarrow x = L/2$$

2. endpoints

$$A(0) = 0$$

$$A(L/2) = L/2 \, \sqrt{3}/2 \, (L/2) = \frac{\sqrt{3}L^2}{8}$$

$$A(L) = 0$$

$$\boxed{\therefore \text{ Max area } = A(L/2) = \frac{\sqrt{3}}{8}L^2}$$

4.2

6. $f'(x) = -2(x-1)^{-3} = \dfrac{-2}{(x-1)^3}$

5pt.

$$\begin{cases} \therefore f'(x) > 0 & \text{when } x < 1 \\ f'(x) < 0 & \text{when } x \geq 1 \\ f'(x) \text{ DNE} & \text{at } x = 1 \end{cases}$$ 2

$\therefore f'(x)$ is never zero.

f(x) is not differentiable at $x=1$, $\therefore$ can't apply MVT on (0,2) 3

7. want $f'(c) = \frac{f(8)-f(0)}{8-0} = \frac{6-4}{8} = \frac{1}{4}$

$c \sim .9,\ 3.1,\ 4.2,\ 6.1$

("roughly")

17. 1. $f(x)$ has at least one root since;

6pt.

*** Dont deduct marks if they dont say (IVT), just check they get a root in some interval.

$f(x)$ is continuous

2

$f(-1) = -6 < 0 < f(1) = 8$.

$\boxed{\therefore \text{ by (IVT) } f \text{ has root in } (-1, 8)}$

2. $f(x)$ has at most one root since;

** dont deduct for not saying (MVT)

if $f(x)$ had 2 roots.

2 $\Longrightarrow f'(c) = 0$ for some c by (MVT)

(or Rolles Theorem)

2 $\Longrightarrow f'(c) = 2 + 3x^2 + 20x^4 = 0$ for some c.

But $f'(c) > 2$ for all c. $\therefore$ we get contradiction $\therefore f$ has at most one root.

(assume $a<b$)

** don't deduct for not saying MVT

29. 5pt

for any a, b, by (MVT) we know there is $c \in (a,b)$ such that

(2)

$$f'(c) = \frac{f(b)-f(a)}{b-a}. \quad (\text{where } f = \sin x)$$

$$\Rightarrow \cos(c) = \frac{\sin b - \sin a}{b-a}$$

(1) $$\Rightarrow |\cos c| = \frac{|\sin b - \sin a|}{|b-a|}$$

(2) $$\boxed{\Rightarrow \quad 1 \geq \frac{|\sin b - \sin a|}{|b-a|}}$$

$$\left(\text{i.e. } \quad 1 \geq \frac{|\sin a - \sin b|}{|a-b|}\right)$$