Solutions to Assignment 7

Applied Linear Algebra
Math 232 (Fall 2012)

Section 4.4

1. (a) The matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ has nontrivial fixed points since $\det(I - A) = \begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix} = 0$. The fixed points are the solutions of the system $(I - A)\mathbf{x} = 0$, which can be expressed in vector form as $\mathbf{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $-\infty < t < \infty$.

(b) The matrix $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has nontrivial fixed points since $\det(I - B) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$. The fixed points are the solutions of the system $(I - B)\mathbf{x} = 0$, which can be expressed in vector form as $\mathbf{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $-\infty < t < \infty$.

3. We have $A\mathbf{x} = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 5\mathbf{x}$; thus $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 5$.

5. (a) The characteristic equation of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ is $\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} = (\lambda - 3)(\lambda + 1) = 0$. Thus $\lambda = 3$ and $\lambda = -1$ are eigenvalues of A; each has algebraic multiplicity 1.

(b) The characteristic equation is $\begin{vmatrix} \lambda - 10 & 9 \\ -4 & \lambda + 2 \end{vmatrix} = (\lambda - 10)(\lambda + 2) + 36 = (\lambda - 4)^2 = 0$. Thus $\lambda = 4$ is the only eigenvalue; it has algebraic multiplicity 2.

(c) The characteristic equation is $\begin{vmatrix} \lambda - 2 & 0 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 = 0$. Thus $\lambda = 2$ is the only eigenvalue; it has algebraic multiplicity 2.

7. (a) The characteristic equation is $\begin{vmatrix} \lambda - 4 & 0 & -1 \\ 2 & \lambda - 1 & 0 \\ 2 & 0 & \lambda - 1 \end{vmatrix} = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$. Thus $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$ are eigenvalues; each has algebraic multiplicity 1.

(b) The characteristic equation is $\begin{vmatrix} \lambda - 4 & 5 & 5 \\ -\frac{2}{5} & \lambda - 1 & 1 \\ -\frac{6}{5} & 3 & \lambda + 1 \end{vmatrix} = \lambda^3 - 4\lambda^2 + 4\lambda = \lambda(\lambda - 2)^2 = 0$. Thus $\lambda = 0$ and $\lambda = 2$ are eigenvalues; $\lambda = 0$ has algebraic multiplicity 1, and $\lambda = 2$ has multiplicity 2.

(c) The characteristic equation is $\begin{vmatrix} \lambda - 3 & -4 & 1 \\ 1 & \lambda + 2 & -1 \\ -3 & -9 & \lambda \end{vmatrix} = \lambda^3 - \lambda^2 - 8\lambda + 12 = (\lambda + 3)(\lambda - 2)^2 = 0$. Thus $\lambda = -3$ and $\lambda = 2$ are eigenvalues; $\lambda = -3$ has multiplicity 1, and $\lambda = 2$ has multiplicity 2.

9. (a) The eigenspace corresponding to $\lambda = 3$ is found by solving the system $\begin{bmatrix} 0 & 0 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This yields the general solution $x = t$, $y = 2t$; thus the eigenspace consists of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Geometrically, this is the line $y = 2x$ in the xy-plane.

The eigenspace corresponding to $\lambda = -1$ is found by solving the system $\begin{bmatrix} -4 & 0 \\ -8 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This yields the general solution $x = 0$, $y = t$; thus the eigenspace consists of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Geometrically, this is the line $x = 0$ (y-axis).

(b) The eigenspace corresponding to $\lambda = 4$ is found by solving the system $\begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This yields the general solution $x = 3t$, $y = 2t$; thus the eigenspace consist of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Geometrically, this is the line $y = \frac{2}{3}x$.

(c) The eigenspace corresponding to $\lambda = 2$ is found by solving the system $\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This yields the general solution $x = 0$, $y = t$; thus the eigenspace consists of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Geometrically, this is the line $x = 0$.

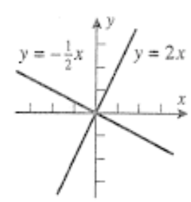
13. (a) The characteristic polynomial is $p(\lambda) = (\lambda + 1)(\lambda - 5)$. The eigenvalues are $\lambda = -1$ and $\lambda = 5$.

(b) The characteristic polynomial is $p(\lambda) = (\lambda - 3)(\lambda - 7)(\lambda - 1)$. The eigenvalues are $\lambda = 3$, $\lambda = 7$, and $\lambda = 1$.

(c) The characteristic polynomial is $p(\lambda) = (\lambda + \frac{1}{3})^2(\lambda - 1)(\lambda - \frac{1}{2})$. The eigenvalues are $\lambda = -\frac{1}{3}$ (with multiplicity 2), $\lambda = 1$, and $\lambda = \frac{1}{2}$.

14. Two examples are $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ -3 & 0 & -1 & 2 \end{bmatrix}$.

21. The eigenvalues are $\lambda = 0$ and $\lambda = 5$, with associated eigenvectors $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ respectively. Thus the eigenspaces correspond to the perpendicular lines $y = -\frac{1}{2}x$ and $y = 2x$.



23. The invariant lines, if any, correspond to eigenspaces of the matrix.

(a) The eigenvalues are $\lambda = 2$ and $\lambda = 3$, with associated eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively. Thus the lines $y = 2x$ and $y = x$ are invariant under the given matrix.

(b) This matrix has no real eigenvalues, so there are no invariant lines.

(c) The only eigenvalue is $\lambda = 2$ (multiplicity 2), with associated eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus the line $y = 0$ is invariant under the given matrix.

25. The characteristic polynomial of A is $p(\lambda) = \lambda^2 - (b+3)\lambda + (3b - 2a)$, so A has the stated eigenvalues if and only if $p(2) = p(5) = 0$. This leads to the equations

$$\begin{aligned} -2a + b &= 2 \\ a + b &= 5 \end{aligned}$$

from which we conclude that $a = 1$ and $b = 4$.

27. If $A^2 = I$, then $A(\mathbf{x} + A\mathbf{x}) = A\mathbf{x} + A^2\mathbf{x} = A\mathbf{x} + \mathbf{x} = \mathbf{x} + A\mathbf{x}$; thus $\mathbf{y} = \mathbf{x} + A\mathbf{x}$ is an eigenvector of A corresponding to $\lambda = 1$. Similarly, $\mathbf{z} = \mathbf{x} - A\mathbf{x}$ is an eigenvector of A corresponding to $\lambda = -1$.

D1. (a) The characteristic polynomial $p(\lambda)$ has degree 6; thus A is a 6×6 matrix.
(b) Yes. From Theorem 4.4.12, we have $\det(A) = (1)(3)^2(4)^3 = 576 \neq 0$; thus A is invertible.

D3. Using Formula (22), the characteristic polynomial of A is $p(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$. Thus $\lambda = 2$ is the only eigenvalue of A (it has multiplicity 2).

D4. The eigenvalues of A (with multiplicity) are 3, 3, and $-2, -2, -2$. Thus, from Theorem 4.4.12, we have $\det(A) = (3)(3)(-2)(-2)(-2) = -72$ and $\operatorname{tr}(A) = 3 + 3 - 2 - 2 - 2 = 0$.

D6. The characteristic polynomial of A factors as $p(\lambda) = (\lambda - 1)(\lambda + 2)^3$; thus the eigenvalues of A are $\lambda = 1$ and $\lambda = -2$. It follows from Theorem 4.4.6 that the eigenvalues of A^2 are $\lambda = (1)^2 = 1$ and $\lambda = (-2)^2 = 4$.

Section 5.1

1. (a) and (c) are stochastic matrices. (b) and (d) are not stochastic.

5. (a) The entries of P are positive; thus P is a regular stochastic matrix.
(b) All positive powers of P have a zero in the upper right corner; thus P is not regular.
(c) The entries of $P^2 = \begin{bmatrix} \frac{21}{25} & \frac{1}{5} \\ \frac{4}{25} & \frac{4}{5} \end{bmatrix}$ are all positive; thus P is regular.

9. The system $(I - P)\mathbf{q} = \mathbf{0}$ can be written as $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{4} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ which has general solution $q_1 = \frac{4}{3}t$, $q_2 = \frac{4}{3}t$, $q_3 = t$. For $\mathbf{q}$ to be a probability vector, we must have $1 = q_1 + q_2 + q_3 = \frac{11}{3}t$, or $t = \frac{3}{11}$. Thus the steady-state vector is $\mathbf{q} = \begin{bmatrix} \frac{4}{11} \\ \frac{4}{11} \\ \frac{3}{11} \end{bmatrix}$.

11. (a) The probability of transition from state 1 to state 1.
(b) The probability of transition from state 2 to state 1.
(c) The probability of transition from state 1 to state 2 is $p_{21} = 0.8$.
(d) The initial state vector is $\mathbf{x}(0) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, and so $\mathbf{x}(1) = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.15 \\ 0.85 \end{bmatrix}$. Thus, after the first transition, the probability of the system being in state 2 at the next observation is 0.85.

15. This process can be described by a Markov chain with transition matrix $P = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$, and initial vector $\mathbf{x}(0) = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$ which represents the percentage of the total population of 125,000 that initially lives in the city (80% of total) and in the suburbs (20% of the total).

(a) The state vector $\mathbf{x}(1)$ is $\mathbf{x}(1) = P\mathbf{x}(0) = \begin{bmatrix} 0.766 \\ 0.234 \end{bmatrix}$ which, upon multiplying by 125,000, corresponds to populations of $\begin{bmatrix} 95750 \\ 29250 \end{bmatrix}$. Similarly, $\mathbf{x}(2) = P\mathbf{x}(1) = \begin{bmatrix} 0.73472 \\ 0.26528 \end{bmatrix}$ which corresponds to $\begin{bmatrix} 91840 \\ 33160 \end{bmatrix}$, etc. Proceeding in this fashion, one constructs the following table showing the populations of the city and its suburbs over a five-year period:

Year	1	2	3	4	5
City	95750	91840	88243	84933	81889
Suburbs	29250	33160	36757	40067	43111

(b) The system $(I - P)\mathbf{q} = \mathbf{0}$ can be written as $\begin{bmatrix} 0.05 & -0.03 \\ -0.05 & 0.03 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which has general solution $q_1 = \frac{3}{5}t$, $q_2 = t$. For $\mathbf{q}$ to be a probability vector, we must have $1 = q_1 + q_2 = \frac{8}{5}t$, or $t = \frac{5}{8}$. Thus the steady-state vector is $\mathbf{q} = \begin{bmatrix} \frac{3}{8} \\ \frac{5}{8} \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}$ which corresponds to populations of $\begin{bmatrix} 46875 \\ 78125 \end{bmatrix}$.

D2. $MP = M = [1 \ 1 \ \cdots \ 1]$

Section 6.1

1. (a) $T_A : R^2 \to R^3$; domain $= R^2$, codomain $= R^3$

(b) $T_A : R^3 \to R^2$; domain $= R^3$, codomain $= R^2$

(c) $T_A : R^3 \to R^3$; domain $= R^3$, codomain $= R^3$

3. The domain of T is R^2, the codomain of T is R^3, and $T(1, -2) = (-1, 2, 3)$.

5. (a) $T(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(b) $T(\mathbf{x}) = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \end{bmatrix}$

7. **(a)** We have $T_A(\mathbf{x}) = \mathbf{b}$ if and only if $\mathbf{x}$ is a solution of the linear system

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 2 & 5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

The reduced row echelon form of the augmented matrix of the above system is

$$\begin{bmatrix} 1 & 0 & 6 & -1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and it follows that the system has the general solution $x_1 = -1 - 6t$, $x_2 = 1 + 3t$, $x_3 = t$. Thus any vector of the form $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}$ will have the property that $T_A(\mathbf{x}) = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

9. (a), (c), and (d) are linear transformations. (b) is not linear; neither homogeneous nor additive.

17. $[T] = [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2)] = \begin{bmatrix} 3 & -1 \\ 2 & 3 \\ 4 & 0 \end{bmatrix}$

21. **(a)** The standard matrix of the transformation is $[T] = \begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}$.

(b) If $\mathbf{x} = (-1, 2, 4)$ then, using the equations, we have

$$T(\mathbf{x}) = (3(-1) + 5(2) - (4), 4(-1) - (2) + (4), 3(-1) + 2(2) - (4)) = (3, -2, -3)$$

On the other hand, using the matrix, we have

$$T(\mathbf{x}) = T\left(\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

23. (a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

25. (a) $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{7}{\sqrt{2}} \end{bmatrix} \approx \begin{bmatrix} -0.707 \\ 4.950 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

35. (a) $T_A(\mathbf{e}_1) = \mathbf{c}_1(A) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$ $\quad T_A(\mathbf{e}_2) = \mathbf{c}_2(A) = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$ $\quad T_A(\mathbf{e}_3) = \mathbf{c}_3(A) = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$

(b) $T_A(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = T_A(\mathbf{e}_1) + T_A(\mathbf{e}_2) + T_A(\mathbf{e}_3) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$

(c) $T_A(7\mathbf{e}_3) = 7T_A(\mathbf{e}_3) = 7\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ -21 \end{bmatrix}$

39. $T(x, y) = (-x, 0)$; thus $[T] = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$.

D7. Since T is linear, we have $T(\mathbf{x}_0 + t\mathbf{v}) = T(\mathbf{x}_0) + tT(\mathbf{v})$. Thus, if $T(\mathbf{v}) \neq \mathbf{0}$, the image of the line $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ is the line $\mathbf{y} = \mathbf{y}_0 + t\mathbf{w}$ where $\mathbf{y}_0 = T(\mathbf{x}_0)$ and $\mathbf{w} = T(\mathbf{v})$. If $T(\mathbf{v}) = \mathbf{0}$, then the image of $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ is the point $\mathbf{y}_0 = T(\mathbf{x}_0)$.

Homework 7
Solutions to Instructor's Questions

A1. Let $\mathbf{v} = (1, 1, \ldots, 1)$ in $\mathbb{R}^n$. Then

$$A\mathbf{v} = c_1(A) + c_2(A) + \cdots + c_n(A) = (s, s, \ldots, s) = s\mathbf{v}$$

since the sum of every row is s. Therefore, s is an eigenvalue of A.

A2.

$$\begin{aligned}
\lambda \text{ is an eigenvalue of } A &\Leftrightarrow \det(\lambda I - A) = 0 \\
&\Leftrightarrow \det[(\lambda I - A)^T] = 0 \\
&\Leftrightarrow \det[(\lambda I)^T - A^T] = 0 \\
&\Leftrightarrow \det(\lambda I - A^T) = 0 \\
&\Leftrightarrow \lambda \text{ is an eigenvalue of } A^T
\end{aligned}$$

A3. Suppose that all of the column sums of A equal s. This implies that all of the row sums of A^T equal s. By the result from Exercise A1, it follows that s is an eigenvalue of A^T. By the result from Exercise A2, s is also an eigenvalue of A.