Solutions to Assignment 6

Applied Linear Algebra
Math 232 (Fall 2012)

Section 4.2

3. (a) $\begin{vmatrix} 3 & 1 & 3 & 33 \\ 0 & \frac{1}{3} & 9 & 22 \\ 0 & 0 & -2 & 12 \\ 0 & 0 & 0 & 2 \end{vmatrix} = (3)\left(\frac{1}{3}\right)(-2)(2) = -4$

(b) $\begin{vmatrix} 3 & 1 & 9 \\ -1 & 2 & -3 \\ 1 & 5 & 3 \end{vmatrix} = 0$ (first and third columns are proportional)

(c) $\begin{vmatrix} 3 & -17 & 4 \\ 0 & 5 & 1 \\ 0 & 0 & -2 \end{vmatrix} = (3)(5)(-2) = -30$

5. (a) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = (-1)\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = (-1)(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-1)(-1)(-6) = -6$

(b) $\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = (3)\begin{vmatrix} a & b & c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = (-3)\begin{vmatrix} a & b & c \\ d & e & f \\ 4g & 4h & 4i \end{vmatrix} = (-12)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-12)(-6) = 72$

(c) $\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$

(d) $\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix} = \begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g & h & i \end{vmatrix} = (-3)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-3)(-6) = 18$

9. If $x = 0$, the given matrix becomes $A = \begin{bmatrix} 0 & 0 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix}$ and, since the first and third rows are proportional, we have $\det(A) = 0$. If $x = 2$, the given matrix becomes $B = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix}$ and, since the first and second rows are proportional, we have $\det(B) = 0$.

15. We use the properties of determinants stated in Theorem 4.2.2. Corresponding row operations are as indicated.

$$\det(A) = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{vmatrix}$$

−5 times row 1 was added to row 2; row 1 was added to row 3; −2 times row 1 was added to row 4.

$$= \begin{vmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{vmatrix}$$

−12 times row 2 was added to row 4.

$$= \begin{vmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -13 \end{vmatrix}$$

36 times row 3 was added row 4.

$$= 39$$

21. $$\det(A) = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = (y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & z-y \end{vmatrix} = (y-x)(z-x)(z-y)$$

27. (a) $\det(3A) = 3^3\det(A) = (27)(7) = 189$ (b) $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{7}$

(c) $\det(2A^{-1}) = 2^3\det(A^{-1}) = (8)(\frac{1}{7}) = \frac{8}{7}$

(d) $\det((2A)^{-1}) = \dfrac{1}{\det(2A)} = \dfrac{1}{2^3\det(A)} = \dfrac{1}{(8)(7)} = \dfrac{1}{56}$

36. $\det(A^{-1}BA) = \det(A^{-1})\det(B)\det(A) = \dfrac{1}{\det(A)}\det(B)\det(A) = \det(B)$

D1. The matrices are singular if and only if the corresponding determinants are zero. This leads to the system of equations

$$\begin{vmatrix} 1 & 2 & s \\ 2 & 3 & t \\ 4 & 5 & 7 \end{vmatrix} = s\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} - t\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} + 7\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -2s + 3t - 7 = 0$$

$$\begin{vmatrix} 4 & 5 & 8 \\ s & 2 & 3 \\ t & 1 & 8 \end{vmatrix} = 4\begin{vmatrix} 2 & 3 \\ 1 & 8 \end{vmatrix} - s\begin{vmatrix} 5 & 8 \\ 1 & 8 \end{vmatrix} + t\begin{vmatrix} 5 & 8 \\ 2 & 3 \end{vmatrix} = 52 - 32s - t = 0$$

from which it follows that $s = \frac{149}{98}$ and $t = \frac{164}{49}$.

D2. Since $\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$, it is always true that $\det(AB) = \det(BA)$.

D7. (a) False. For example, if $A = I = I_2$, then $\det(I + A) = \det(2I) = 4$, whereas $1 + \det(A) = 2$.

(b) True. From Theorem 4.2.5 it follows that $\det(A^n) = (\det(A))^n$ for every $n = 1, 2, 3, \ldots$.

(c) False. From Theorem 4.2.3(c), we have $\det(3A) = 3^n\det(A)$ where n is the size of A. Thus the statement is false except when $n = 1$ or $\det(A) = 0$.

(d) True. If $\det(A) = 0$, the matrix is singular and so the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

D8. (a) True. If A is invertible, then $\det(A) \neq 0$. Since $\det(ABA) = \det(A)\det(B)\det(A)$ it follows that if A is invertible and $\det(ABA) = 0$, then $\det(B) = 0$.

(b) True. If $A = A^{-1}$, then since $\det(A^{-1}) = \frac{1}{\det(A)}$, it follows that $(\det(A))^2 = 1$ and so $\det(A) = \pm 1$.

(c) True. If the reduced row echelon form of A has a row of zeros, then A is not invertible.

(d) True. Since $\det(A^T) = \det(A)$, it follows that $\det(AA^T) = \det(A)\det(A^T) = (\det(A))^2 \geq 0$.

(e) True. If $\det(A) \neq 0$ then A is invertible, and an invertible matrix can always be written as a product of elementary matrices.

Section 4.3

27. $V = |\det(A)|$ where $A = \begin{bmatrix} 2 & 0 & 2 \\ -6 & 4 & 2 \\ 2 & -2 & -4 \end{bmatrix}$; thus $V = |-16| = 16$.

29. The vectors lie in the same plane if and only if the parallelepiped that they determine is degenerate in the sense that its "volume" is zero. In this example, we have

$$V = \begin{vmatrix} -1 & 3 & 5 \\ -2 & 0 & -4 \\ 1 & -2 & 0 \end{vmatrix} = 16$$

and so the vectors do not lie in the same plane.

33. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -6 \\ 2 & 3 & 6 \end{vmatrix} = 36\mathbf{i} - 24\mathbf{j}$ $\qquad \sin\theta = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\sqrt{1296 + 576}}{\sqrt{49}\sqrt{49}} = \frac{\sqrt{1872}}{49} = \frac{12\sqrt{13}}{49}$

35. (a) $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -3 \\ 2 & 6 & 7 \end{vmatrix} = (14 + 18)\mathbf{i} - (0 + 6)\mathbf{j} + (0 - 4)\mathbf{k} = 32\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$

(b) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 32 & -6 & -4 \end{vmatrix} = (-8 - 6)\mathbf{i} - (-12 + 32)\mathbf{j} + (-18 - 64)\mathbf{k} = -14\mathbf{i} - 20\mathbf{j} - 82\mathbf{k}$

(c) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 0 & 2 & -3 \end{vmatrix} = (-6 + 2)\mathbf{i} - (-9 + 0)\mathbf{j} + (6 - 0)\mathbf{k} = -4\mathbf{i} + 9\mathbf{j} + 6\mathbf{k}$

$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 9 & 6 \\ 2 & 6 & 7 \end{vmatrix} = (63 - 36)\mathbf{i} - (-28 - 12)\mathbf{j} + (-24 - 18)\mathbf{k} = 27\mathbf{i} + 40\mathbf{j} - 42\mathbf{k}$

37. (a) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 4 & 2 \\ 3 & 1 & 5 \end{vmatrix} = 18\mathbf{i} + 36\mathbf{j} - 18\mathbf{k} = (18, 36, -18)$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

43. (a) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = -7\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ $\qquad A = \|\mathbf{u} \times \mathbf{v}\| = \sqrt{49 + 1 + 9} = \sqrt{59}$

53. From Theorem 4.3.9, we know that $\mathbf{v} \times \mathbf{w}$ is orthogonal to the plane determined by $\mathbf{v}$ and $\mathbf{w}$. Thus a vector lies in the plane determined by $\mathbf{v}$ and $\mathbf{w}$ if and only if it is orthogonal to $\mathbf{v} \times \mathbf{w}$. Therefore, since $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is orthogonal to $\mathbf{v} \times \mathbf{w}$, it follows that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane determined by $\mathbf{v}$ and $\mathbf{w}$.

D3. $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$ does not make sense since the first factor is a scalar rather than a vector.

D4. If either $\mathbf{u}$ or $\mathbf{v}$ is the zero vector, then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero then, from Theorem 4.3.10, we have $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Thus if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, with $\mathbf{u}$ and $\mathbf{v}$ not zero, then $\sin\theta = 0$ and so $\mathbf{u}$ and $\mathbf{v}$ are parallel.

Homework 6
Solutions to Instructor's Questions

A1 (a): $\mathrm{Re}(3) = 3$, $\mathrm{Re}(0 - i) = 0$, $\mathrm{Re}(2 + 3i) = 2$, $\mathrm{Re}(1 + i) = 1$
A1 (b): $\mathrm{Im}(3 + 0i) = 0$, $\mathrm{Im}(0 - 1i) = -1$, $\mathrm{Im}(2 + 3i) = 3$, $\mathrm{Im}(1 + 1i) = 1$
A1 (c): $\overline{3 + 0i} = 3 - 0i = 3$, $\overline{0 - i} = 0 + i = i$, $\overline{2 + 3i} = 2 - 3i$, $\overline{1 + i} = 1 - i$
A1 (d): $|3| = 3$. $|-i| = \sqrt{0^2 + (-1)^2} = 1$, $|2 + 3i| = \sqrt{2^2 + 3^2} = \sqrt{13}$, $|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$
A1 (e): $-(3) = -3$, $-(-i) = i$, $-(2 + 3i) = -2 - 3i$, $-(1 + i) = -1 - i$
A1 (f): $\frac{1}{3}$, $\frac{1}{-i} = \frac{i}{(-i)i} = \frac{i}{1} = i$, $\frac{1}{2+3i} = \frac{2-3i}{(2+3i)(2-3i)} = \frac{2-3i}{13} = \frac{2}{13} - \frac{3}{13}i$, $\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i$

A2 (a): $(2 + 3i) + (1 + i) = (2 + 1) + (3 + 1)i = 3 + 4i$
A2 (b): $(2 + 3i) - (1 + i) = (2 - 1) + (3 - 1)i = 1 + 2i$
A2 (c): $(2+3i)(1+i) = (2)(1)+(2)(i)+(3i)(1)+(3i)(i) = 2+2i+3i-3 = -1 + 5i$.
A2 (d): $\frac{2+3i}{1+i} = \frac{(2+3i)(1-i)}{(1+i)(1-i)} = \frac{(2)(1)+(3i)(1)+2(-i)+(3i)(-i)}{2} = \frac{2+3i-2i+3}{2} = \frac{5+i}{2}$
A2 (f): We have

$$\begin{aligned}
\frac{\overline{z_4} - z_2}{z_2 z_3 + z_1} &= \frac{\overline{1+i} - (-i)}{(-i)(2+3i)+3} \\
&= \frac{\overline{1+i} + i}{(-i)(2+3i)+3} \\
&= \frac{1 - i + i}{(-i)(2+3i)+3} \\
&= \frac{1}{(-i)(2+3i)+3} \\
&= \frac{1}{-2i+3+3} \\
&= \frac{1}{6-2i} \\
&= \frac{6+2i}{(6-2i)(6+2i)} \\
&= \frac{6+2i}{40} \\
&= \frac{3}{20} + \frac{1}{20}i
\end{aligned}$$

A3 (a): $|z_1| = 3$, and it is on the negative x-axis, so it has principal argu-

ment π.
A3 (b): $|z_2| = \sqrt{0^2 + (-1)^2} = 1$, and it is on the negative y-axis, so it has principal argument $-\pi/2$.
A3 (c): $|-1+i| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$. It is in the second quadrant, with principal argument $3\pi/4$.
A3 (d): Conjugation reflects across the real axis, so $|\overline{z_3}|$ is the same as $|z_3| = \sqrt{2}$. This reflection flips the sign of the principal argument (unless it is equal to $-\pi$) so the principal argument of $\overline{z_3}$ equals $-3\pi/4$.
A3 (e): $|1/z_3| = 1/|z_3| = 1/\sqrt{2}$. Inversion flips the sign of the principal argument (unless it is equal to $-\pi$) so the principal argument of $1/z_3$ equals $-3\pi/4$.
A3 (f): $|z_2z_3| = |z_2||z_3| = \sqrt{2}$. Add the arguments of z_2 and z_3 to get $\pi/4$ for the argument of z_2z_3. This is in the right range, so it is the principal argument.
A3 (g): $|z_2/z_3| = |z_2|/|z_3| = 1/\sqrt{2}$. Subtract the argument of z_3 from that of z_2 to get $-5\pi/4$ for the argument of z_2/z_3. This is not in the right range to be a principal argument, so add 2π to get $3\pi/4$ for the principal argument of z_2/z_3.
A3 (h): $|z_3/z_2| = |z_3|/|z_2| = \sqrt{2}$. Subtract the argument of z_2 from that of z_3 to get $5\pi/4$ for the argument of z_2/z_3. This is not in the right range to be a principal argument, so subtract 2π to get $-3\pi/4$ for the principal argument of z_3/z_2.

A4 (a): z_1 has absolute value $\sqrt{1^2+3} = 2$ and argument $-\pi/3$:

$$z_1 = 2\Big(\cos(-\pi/3) + i\sin(-\pi/3)\Big),$$

so then

$$\begin{aligned} z_1^6 &= 2^6\Big(\cos(6(-\pi/3)) + i\sin(6(-\pi/3))\Big) \\ &= 64(1+0i) \\ &= 64. \end{aligned}$$

A4 (b): $z_1^7 = z_1^6 z_1 = 64z_1 = 64(1-\sqrt{3}i) = 64 - 64\sqrt{3}i$
A4 (c): z_2 has absolute value $\sqrt{0^2+1^2} = 1$ and argument $\pi/2$:

$$z_2 = \cos(\pi/2) + i\sin(\pi/2),$$

so then

$$\begin{aligned} z_2^{103} &= \cos(103(\pi/2)) + i\sin(103(\pi/2)) \\ &= \cos(50\pi + \frac{3\pi}{2}) + i\sin(50\pi + \frac{3\pi}{2}) \\ &= \cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2}) \\ &= 0 + i(-1) \\ &= -i. \end{aligned}$$

Or recognize that $i^2 = -1$, so $i^4 = 1$. Thus $i^{100} = (i^4)^{25} = 1$, so then $i^{103} = i^{100}i^3 = (1)(i^3) = i^3 = (i^2)i = -1(i) = -i$.
A4 (d): z_3 has absolute value $\sqrt{\frac{1}{2} + \frac{1}{2}} = 1$ and argument $-\pi/4$:

$$z_3 = \cos(-\pi/4) + i\sin(-\pi/4),$$

so then

$$\begin{aligned} z_3^{29} &= \cos(29(-\pi/4)) + i\sin(29(-\pi/4)) \\ &= \cos(-6\pi - \frac{5\pi}{4}) + i\sin(-6\pi - \frac{5\pi}{4}) \\ &= \cos(-\frac{5\pi}{4}) + i\sin(-\frac{5\pi}{4}) \\ &= \cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4}) \\ &= -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}. \end{aligned}$$

A5 (a): $e^{2-\frac{\pi}{2}i} = e^2 e^{-\frac{\pi}{2}i} = e^2(\cos(-\pi/2) + i\sin(-\pi/2)) = e^2 i(-1) = -ie^2$.
A5 (b): $e^{-5+7\pi i} = e^{-5}e^{7\pi i} = e^{-5}(\cos(7\pi) + i\sin(7\pi) = e^{-5}(-1) = -e^{-5}$.
A5 (c): $e^{z_1+z_2} = e^{z_1}e^{z_2} = (-ie^2)(-e^{-5}) = ie^2e^{-5} = ie^{2-5} = ie^{-3}$.