Homework 5
Solutions to Instructor's Questions

**A1.** Let $A = (a)_{ij}$. Then expanding the right-hand-side gives

$$2x^2+6y^2+3xy-5xz+4yz = a_{11}x^2+a_{22}y^2+a_{33}z^2+(a_{12}+a_{21})xy+(a_{13}+a_{31})xz+(a_{23}+a_{32})yz.$$

Since $A$ is symmetric, equating coefficients yields

$$\begin{aligned} a_{11} &= 2 \\ a_{22} &= 6 \\ a_{33} &= 0 \\ a_{12} = a_{21} &= 3/2 \\ a_{13} = a_{31} &= -5/2 \\ a_{23} = a_{32} &= 2 \end{aligned}$$

and so the matrix is

$$A = \begin{bmatrix} 2 & 3/2 & -5/2 \\ 3/2 & 6 & 2 \\ -5/2 & 2 & 0 \end{bmatrix}.$$

**A2.** We have

$$\begin{aligned} (A - A^T)^T &= A^T - (A^T)^T \\ &= A^T - A \\ &= -(A - A^T). \end{aligned}$$

Hence, $A - A^T$ is skew-symmetric.

# Solutions to Assignment 5

## Applied Linear Algebra
## Math 232 (Fall 2012)

## Section 3.6

**1.** (a) This matrix has a row of zeros; it is not invertible.

(b) A diagonal matrix with nonzero entries on the diagonal; $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

**5.** (a) $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 16 \end{bmatrix}$ (b) $A^{-2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{16} \end{bmatrix}$ (c) $A^{-k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-\frac{1}{2})^k & 0 \\ 0 & 0 & (\frac{1}{4}^k) \end{bmatrix}$

**7.** $A$ is invertible if and only if $x \neq 1, -2, 4$ (the diagonal entries must be nonzero).

**13.** The matrix $A$ is symmetric if and only if $a$, $b$, and $c$ satisfy the following equations:

$$\begin{aligned} a - 2b + 2c &= 3 \\ 2a + b + c &= 0 \\ a \quad\quad + c &= -2 \end{aligned}$$

The augmented matrix of this system is

$$\begin{bmatrix} 1 & -2 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & -2 \end{bmatrix}$$

and the reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -13 \end{bmatrix}$$

Thus, in order for the matrix $A$ to be symmetric we must have $a = 11$, $b = -9$, and $c = -13$.

**17.** $AB = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 2 & -8 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 12 & 17 \\ 0 & 2 & 10 \\ 0 & 0 & -12 \end{bmatrix}$

**19.** If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, then $A^{-5} = \begin{bmatrix} (1)^{-5} & 0 & 0 \\ 0 & (-1)^{-5} & 0 \\ 0 & 0 & (-1)^{-5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

21. The lower triangular matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ is invertible because of the nonzero diagonal entries.

Thus, from Theorem 3.6.5, $AA^T$ and $A^TA$ are also invertible; furthermore, since these matrices are symmetric, their inverses are also symmetric by Theorem 3.6.4. Following are the matrices $AA^T$, $A^TA$, and their inverses.

$$A^TA = \begin{bmatrix} 11 & 6 & 1 \\ 6 & 10 & 3 \\ 1 & 3 & 1 \end{bmatrix} \qquad (A^TA)^{-1} = \begin{bmatrix} 1 & -3 & 8 \\ -3 & 10 & -27 \\ 8 & -27 & 74 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 10 & 6 \\ 1 & 6 & 11 \end{bmatrix} \qquad (AA^T)^{-1} = \begin{bmatrix} 74 & -27 & 8 \\ -27 & 10 & -3 \\ 8 & -3 & 1 \end{bmatrix}$$

D10. (a) False. If $A$ is not square then $A$ is not invertible; it doesnt matter whether $AA^T$ (which is always square) is invertible or not. [But if $A$ is square and $AA^T$ is invertible, then $A$ is invertible by Theorem 3.6.5.]

(b) False. For example if $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $A + B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is symmetric.

(c) True. If $A$ is both symmetric and triangular, then $A$ must be a diagonal matrix. Thus $A = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$, and so $p(A) = \begin{bmatrix} p(d_1) & 0 & \cdots & 0 \\ 0 & p(d_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(d_n) \end{bmatrix}$ is also a diagonal matrix (both symmetric and triangular).

(d) True. For example, in the $3 \times 3$ case, we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

(e) True. If $A\mathbf{x} = 0$ has only the trivial solution, then $A$ is invertible. But if $A$ is invertible then so is $A^T$ (Theorem 3.2.11); thus $A^T\mathbf{x} = 0$ has only the trivial solution.

# Section 4.1

3. $\begin{vmatrix} -5 & 7 \\ -7 & -2 \end{vmatrix} = (-5)(-2) - (7)(-7) = 10 + 49 = 59$

7. $\begin{vmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{vmatrix} = (-20 - 7 + 72) - (20 + 84 + 6) = 45 - 110 = -65$

17. We have $\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = x(1-x) + 3 = -x^2 + x + 3$, and

$$\begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix} = ((x(x-5) + 0 - 18) - (-3x - 18 + 0) = x^2 - 2x$$

Thus the given equation is valid if and only if $-x^2 + x + 3 = x^2 - 2x$, i.e. if $2x^2 - 3x - 3 = 0$. The roots of this quadratic equation are $x = \frac{3 \pm \sqrt{33}}{4}$.

**19.** (a) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1)(-1)(1) = -1$ (b) $\begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 1 & 2 & 3 & 8 \end{vmatrix} = 0$

(c) $\begin{vmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{vmatrix} = (1)(1)(2)(3) = 6$

**23.** (a) $M_{13} = \begin{vmatrix} 0 & 0 & 3 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = (0+0+12) - (12+0+0) = 0 \qquad C_{13} = 0$

(b) $M_{23} = \begin{vmatrix} 4 & -1 & 6 \\ 4 & 1 & 14 \\ 4 & 1 & 2 \end{vmatrix} = (8 - 56 + 24) - (24 + 56 - 8) = -96 \qquad C_{23} = 96$

(c) $M_{22} = \begin{vmatrix} 4 & 1 & 6 \\ 4 & 0 & 14 \\ 4 & 3 & 2 \end{vmatrix} = (0 + 56 + 72) - (0 + 8 + 168) = -48 \qquad C_{22} = -48$

(d) $M_{21} = \begin{vmatrix} -1 & 1 & 6 \\ 1 & 0 & 14 \\ 1 & 3 & 2 \end{vmatrix} = (0 + 14 + 18) - (0 + 2 - 42) = 72 \qquad C_{21} = -72$

**31.** Using column 3: $\det(A) = (-3)\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 2 & 10 & 2 \end{vmatrix} - (3)\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & -2 \\ 4 & 1 & 0 \end{vmatrix} = (-3)(128) - (3)(-48) = -240$

**D3.** A $3 \times 3$ matrix $A$ can have as many as six zeros without having $\det(A) = 0$. For example, let $A$ be a diagonal matrix with nonzero diagonal entries.