Solutions to Assignment 4

Applied Linear Algebra
Math 232 (Fall 2012)

Section 3.3

1. (a) This matrix is elementary; it is obtained from I_2 by adding -5 times the first row to the second row.
   (b) This matrix is not elementary since two row operations are needed to obtain it from I_2 (follow the one in part (a) by interchanging the rows).
   (c) This matrix is elementary; it is obtained from I_3 by interchanging the first and third rows.
   (d) This matrix is not invertible, and therefore not elementary, since it has a row of zeros.

3. (a) Add 3 times the first row to the second row.
   (b) Multiply the third row by $\frac{1}{3}$.
   (c) Interchange the first and fourth rows.
   (d) Add $\frac{1}{7}$ times the third row to the first row.

5. (a) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$
   (c) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$
   (d) $\begin{bmatrix} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

7. (a) B is obtained from A by interchanging the first and third rows; thus $EA = B$ where $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.
   (b) $EB = A$ where E is the same as in part (a).
   (c) C is obtained from A by adding -2 times the first row to the third row; thus $EA = C$ where $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$.
   (d) $EC = A$ where E is the inverse of the matrix in part (c), i.e. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

11. (a) Start with the partitioned matrix $[A \mid I]$.

$$\left[\begin{array}{rrr:rrr} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$

Interchange rows 1 and 2.

$$\left[\begin{array}{rrr:rrr} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$

Add -3 times row 1 to row 2. Add -2 times row 1 to row 3.

$$\left[\begin{array}{rrr:rrr} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$$

Add -1 times row 2 to row 3.

$$\left[\begin{array}{rrr:rrr} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \end{array}\right]$$

Add -4 times row 3 to row 2, then interchange rows 2 and 3.

$$\left[\begin{array}{rrr:rrr} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -10 & 5 & -7 & -4 \end{array}\right]$$

Multiply row 3 by $-\frac{1}{10}$, then add -3 times the new row 3 to row 1.

$$\left[\begin{array}{rrr:rrr} 1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right]$$

From this we conclude that A is invertible, and that $A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$.

(b) Start with the partitioned matrix $[A \mid I]$.

$$\left[\begin{array}{rrr:rrr} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right]$$

Multiply row 1 by -1. Add -2 times the new row 1 to row 2; add 4 times the new row 1 to row 3.

$$\left[\begin{array}{rrr:rrr} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right]$$

Add row 2 to row 3.

$$\left[\begin{array}{rrr:rrr} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right]$$

At this point, since we have obtained a row of zeros on the left side, we conclude that the matrix A is not invertible.

21. (a) The identity matrix can be obtained from A by first adding 5 times row 1 to row 3, and then multiplying row 3 by $\frac{1}{2}$. Thus if $E_1 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, then $E_2E_1A = I$.

(b) $A^{-1} = E_2E_1$ where E_1 and E_2 are as in part (a).

(c) $A = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

31. The augmented matrix $\left[\begin{array}{rrr|r} 1 & -2 & -1 & b_1 \\ -2 & 3 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{array}\right]$ can be row reduced to $\left[\begin{array}{rrr|c} 1 & -2 & -1 & b_1 \\ 0 & -1 & 0 & b_2 + 2b_1 \\ 0 & 0 & 0 & 2b_1 - b_2 + b_3 \end{array}\right]$. Thus the system is consistent if and only if $2b_1 - b_2 + b_3 = 0$.

D3. There is no nontrivial solution. From the last equation we see that $x_4 = 0$ and, from back substitution, it follows immediately that $x_3 = x_2 = x_1 = 0$ also. The coefficient matrix is invertible.

D5. (a) False. Only invertible matrices can be expressed as a product of elementary matrices.

(b) False. For example the product $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ cannot be obtained from the identity by a single elementary row operation.

(c) True. This row operation is equivalent to multiplying the given matrix by an elementary matrix and, since any elementary matrix is invertible, the product is still invertible.

(d) True. If A is invertible and $AB = 0$, then $B = IB = (A^{-1}A)B = A^{-1}(AB) = A^{-1}0 = 0$.

(e) True. If A is invertible then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution; otherwise (if A is singular) there are infinitely many solutions.

Section 3.4

1. (a) $(x_1, x_2) = t(1, -1)$; $x_1 = t$, $x_2 = -t$

(b) $(x_1, x_2, x_3) = t(2, 1, -4)$; $x_1 = 2t$, $x_2 = t$, $x_3 = -4t$

3. (a) $(x_1, x_2, x_3) = s(4, -4, 2) + t(-3, 5, 7)$; $x_1 = 4s - 3t$, $x_2 = -4s + 5t$, $x_3 = 2s + 7t$

5. (a) $\mathbf{u} = -2\mathbf{v}$; thus $\mathbf{u}$ is in the subspace $\text{span}\{\mathbf{v}\}$.

(b) $\mathbf{u} \neq k\mathbf{v}$ for any scalar k; thus $\mathbf{u}$ is not in the subspace $\text{span}\{\mathbf{v}\}$.

11. (a) A line (a 1-dimensional subspace) in R^4 that passes through the origin and is parallel to the vector $\mathbf{u} = (2, -3, 1, 4) = \frac{1}{2}(4, -6, 2, 8)$.

(b) A plane (a 2-dimensional subspace) in R^4 that passes through the origin and is parallel to the vectors $\mathbf{u} = (3, -2, 2, 5)$ and $\mathbf{v} = (6, -4, 4, 0)$.

13. The augmented matrix of the homogenous system is

$$\begin{bmatrix} 1 & 6 & 2 & -5 & 0 \\ -1 & -6 & -1 & -3 & 0 \\ 2 & 12 & 5 & -18 & 0 \end{bmatrix}$$

and the reduced row-echelon form of this matrix is

$$\begin{bmatrix} 1 & 6 & 0 & 11 & 0 \\ 0 & 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus a general solution of the system can be written in parametric form as

$$x_1 = -6s - 11t, \quad x_2 = s, \quad x_3 = 8t, \quad x_4 = t$$

or in vector form as

$$(x_1, x_2, x_3, x_4) = s(-6, 1, 0, 0) + t(-11, 0, 8, 1)$$

This shows that the solution space is span$\{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = (-6, 1, 0, 0)$ and $\mathbf{v}_2 = (-11, 0, 8, 1)$.

19. (a) These two vectors are linearly independent since neither is a scalar multiple of the other.

(b) $\mathbf{v}_1$ is a scalar multiple of $\mathbf{v}_2 (\mathbf{v}_1 = -3\mathbf{v}_2)$; thus these two vectors are linearly dependent.

(c) These three vectors are linearly independent since the system $\begin{bmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has only the trivial solution. The coefficient matrix, which is the matrix having the given vectors as its columns, is invertible.

(d) These four vectors are linearly dependent since any set of more than 3 vectors in R^3 is linearly dependent.

23. (a) This set of vectors is a subspace; it is closed under scalar multiplication and addition: $k(a, 0, 0) = (ka, 0, 0)$ and $(a_1, 0, 0) + (a_2, 0, 0) = (a_1 + a_2, 0, 0)$.

(b) This set of vectors is not a subspace; it is not closed under scalar multiplication.

(c) This set of vectors is a subspace. If $b = a + c$, then $kb = ka + kc$. If $b_1 = a_1 + c_1$, and $b_2 = a_2 + c_2$, then $(b_1 + b_2) = (a_1 + a_2) + (c_1 + c_2)$.

(d) This set of vectors is not a subspace; it is not closed under addition or scalar multiplication.

28. If $\lambda = \frac{1}{2}$, then the given vectors are clearly dependent. Otherwise, consider the matrix having these vectors as its columns. Are there any other values of λ for which it is singular? To determine this, it is convenient to start by multiplying all of the rows by 2.

$$\begin{bmatrix} 2\lambda & 1 & 1 \\ 1 & 2\lambda & 1 \\ 1 & 1 & 2\lambda \end{bmatrix}$$

D6. (a) False. For example, two of the vectors may lie on a line (so one is a scalar multiple of the other), but the third vector may not lie on this same line and therefore cannot be expressed as a linear combination of the other two.

(b) False. The set of all linear combinations of two vectors can be $\{\mathbf{0}\}$ (if both are $\mathbf{0}$), a line (if one is a scalar multiple of the other), or a plane (if they are linearly independent).

(c) False. For example, $\mathbf{v}$ and $\mathbf{w}$ might be linearly dependent (scalar multiples of each other). [But it is true that if $\{\mathbf{v}, \mathbf{w}\}$ is a linearly independent set, and if $\mathbf{u}$ cannot be expressed as a linear combination of $\mathbf{v}$ and $\mathbf{w}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set.]

(d) True. See Example 9.

(e) True. If $c_1(k\mathbf{v}_1) + c_2(k\mathbf{v}_2) + c_2(k\mathbf{v}_2) = \mathbf{0}$, then $k(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_2) = \mathbf{0}$. Thus, since $k \neq 0$, it follows that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_2 = \mathbf{0}$ and so $c_1 = c_2 = c_3 = 0$.

Section 3.5

1. (a) The reduced row echelon form of the augmented matrix of the homogeneous system is

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

thus a general solution is $x_1 = -\frac{2}{3}s + \frac{1}{3}t$, $x_2 = s$, $x_3 = t$; or (in column vector form)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

(b) $$\begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

(c) From (a) and (b), a general solution of the nonhomogeneous system is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

(d) The reduced row echelon form of the augmented matrix of the nonhomogeneous system is

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

thus a general solution is given by $x_1 = \frac{2}{3} - \frac{2}{3}s' + \frac{1}{3}t'$, $x_2 = s'$, $x_3 = t'$; or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 0 \\ 0 \end{bmatrix} + s'\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + t'\begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

This solution is related to the one in part (c) by the change of variable $s' = s$, $t' = t + 1$.

3. (a) The reduced row echelon form of the augmented matrix of the given system is

$$\begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

thus a general solution is given by $x_1 = \frac{1}{3} - \frac{4}{3}s - \frac{1}{3}t$, $x_2 = s$, $x_3 = t$, $x_4 = 1$; or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(b) A general solution of the associated homogeneous system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and a particular solution of the given nonhomogenous system is $x_1 = \frac{1}{3}$, $x_2 = 0$, $x_3 = 0$, $x_4 = 1$.

7. The vector $\mathbf{w}$ is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ if and only if the system

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -3 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$$

is consistent. The row reduced echelon form of the augmented matrix of this system is

$$\begin{bmatrix} 1 & 0 & 2 & -1 & -2 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we conclude that the system has infinitely many solutions; thus $\mathbf{w}$ is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

9. **(a)** The hyperplane $\mathbf{a}^\perp$ consists of all vectors $\mathbf{x} = (x, y)$ such that $\mathbf{a} \cdot \mathbf{x} = 0$, i.e. $-2x + 3y = 0$. This corresponds to the line through the origin with parametric equations $x = \frac{3}{2}t$, $y = t$.

(b) The hyperplane $\mathbf{a}^\perp$ consists of all vectors $\mathbf{x} = (x, y, z)$ such that $\mathbf{a} \cdot \mathbf{x} = 0$, i.e. $4x - 5z = 0$. This corresponds to the plane through the origin with parametric equations $x = \frac{5}{4}t$, $y = s$, $z = t$.

(c) $\mathbf{a}^\perp$ consists of all vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ such that $\mathbf{a} \cdot \mathbf{x} = 0$, i.e. $x_1 + 2x_2 - 3x_3 + 7x_4 = 0$. This is a hyperplane in R^4 with parametric equations $x_1 = -2r + 3s - 7t$, $x_2 = r$, $x_3 = s$, $x_4 = t$.

19. **(a)** A vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$ is orthogonal to $\mathbf{v}_1 = (1, 1, 2, 2)$ and $\mathbf{v}_2 = (5, 4, 3, 4)$ if and only if

$$\begin{aligned} x_1 + x_2 + 2x_3 + 2x_4 &= 0 \\ 5x_1 + 4x_2 + 3x_3 + 4x_4 &= 0 \end{aligned}$$

(b) The solution space is the plane (2 dimensional subspace) in R^4 that passes through the origin and is perpendicular to the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

(c) The reduced row echelon form of the augmented matrix of the system is

$$\begin{bmatrix} 1 & 0 & -5 & -4 & 0 \\ 0 & 1 & 7 & 6 & 0 \end{bmatrix}$$

and so a general solution of the system is given by $(x_1, x_2, x_3, x_4) = s(5, -7, 1, 0) + t(4, -6, 0, 1)$. Note that the vectors $(5, -7, 1, 0)$ and $(4, -6, 0, 1)$ are orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$.

D2. If $\mathbf{v}$ is orthogonal to every row of A, then $A\mathbf{v} = \mathbf{0}$, and so (since A is invertible) $\mathbf{v} = \mathbf{0}$.

D4. (a) True. The solution set of $A\mathbf{x} = \mathbf{b}$ is of the form $\mathbf{x}_0 + W$ where W is the solution space of $A\mathbf{x} = \mathbf{0}$.

(b) False. For example, the system $\begin{matrix} x - y = 0 \\ x - y = 1 \end{matrix}$ is inconsistent, but the associated homogeneous system has infinitely many solutions.

(c) True. Each hyperplane corresponds to a single homogeneous linear equation in four variables, and there must be a least four equations in order to have a unique solution.

(d) True. Every plane in R^3 corresponds to a equation of the form $ax + by + cz = d$.

(e) False. A vector $\mathbf{x}$ is orthogonal to $\text{row}(A)$ if and only if $\mathbf{x}$ is a solution of the homogenous system $A\mathbf{x} = \mathbf{0}$.

Homework 4
Solutions to Instructor's Questions

A1. If A is invertible, its rows must be linearly independent. This implies that A cannot have any zero rows, and so there must be at least one nonzero entry in every row. Similarly, if A is invertible, its columns must be linearly independent, and so there must be at least one nonzero entry in every column. Therefore A must have at least n nonzero entries (placed such that there is a nonzero entry in every row and column), and so the maximum number of zeros cannot exceed $n^2 - n$.

To show that the maximum number of zeros is not less than $n^2 - n$, we simply give an example of an invertible $n \times n$ matrix with exactly $n^2 - n$ zeros, such as the $n \times n$ identity matrix.

A2. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ denote the columns of A. Then

$$\begin{aligned}(A^T A)_{ij} &= r_i(A^T) \cdot c_j(A) \\ &= c_i(A) \cdot c_j(A) \\ &= \mathbf{v}_i \cdot \mathbf{v}_j.\end{aligned}$$

Now if $i \neq j$ we have $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ since $\mathbf{v}_i$ and $\mathbf{v}_j$ are orthogonal.

On the other hand, $i = j$, we have $\mathbf{v}_i \cdot \mathbf{v}_j = \mathbf{v}_j \cdot \mathbf{v}_j = ||\mathbf{v}_j||^2 = 1$ since $\mathbf{v}_j$ is a unit vector.

Thus,

$$(A^T A)_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

and so $A^T A = I$.