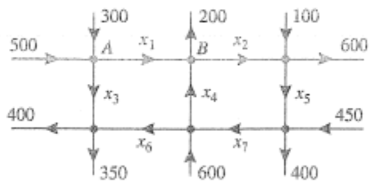
Solutions to Assignment 3

Applied Linear Algebra
Math 232 (Fall 2012)

Section 2.3

4. This problem is concerned with the one-way traffic network



in which the known traffic flow rates, and directions of the unknown rates, are as indicated.

(a) At each intersection the flow in must be equal to the flow out. Starting with intersection A and working clockwise, this leads to the following system of equations:

$$\begin{aligned}
x_1 \quad + x_3 \qquad\qquad\qquad\qquad &= 800\\
x_1 - x_2 \quad + x_4 \qquad\qquad\qquad &= 200\\
x_2 \qquad\quad - x_5 \qquad\qquad &= 500\\
x_5 \quad - x_7 &= -50\\
x_4 \quad + x_6 - x_7 &= 600\\
x_3 \qquad + x_6 \qquad &= 750
\end{aligned}$$

(b) The augmented matrix of the system is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 800\\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 200\\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 500\\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -50\\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 600\\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 750 \end{bmatrix}$$

and the reduced row echelon form is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 50\\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 450\\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 750\\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 600\\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -50\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we conclude that x_6 and x_7 are free variables, and the general solution of the system is

$$x_1 = 50 + s, \quad x_2 = 450 + t, \quad x_3 = 750 - s,$$
$$x_4 = 600 - s + t, \quad x_5 = -50 + t, \quad x_6 = s, \quad x_7 = t$$

(c) If the road from A to B is closed, then $x_1 = 50 + s = 0$. This requires that $x_6 = s = -50$, and since the streets are one-way, this is not permissible. Thus, without reversing the direction of traffic, it is not possible to close the road from A to B.

11. We must find positive integers x_1, x_2, x_3, and x_4 that balance the chemical equation

$$x_1(\mathrm{CH_3COF}) + x_2(\mathrm{H_2O}) \to x_3(\mathrm{CH_3COOH}) + x_4(\mathrm{HF})$$

For each of the elements in the equation (C, H, O, and F) the number of atoms on the left must equal the number of atoms on the right. This leads to the equations $x_1 = x_3$, $3x_1 + 2x_2 = 4x_3 + x_4$, $x_1 + x_2 = 2x_3$, and $x_1 = x_4$. This is a very simple system of equations having (by inspection) the general solution

$$x_1 = x_2 = x_3 = x_4 = t$$

Thus, taking $t = 1$, the balanced equation is

$$\mathrm{CH_3COF} + \mathrm{H_2O} \to \mathrm{CH_3COOH} + \mathrm{HF}$$

15. The graph of the polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ passes through the points $(-1,-1)$, $(0,1)$, $(1,3)$, and $(4,-1)$, if and only if the coefficients a_0, a_1, and a_2 satisfy the equations

$$\begin{aligned} a_0 - a_1 + a_2 - a_3 &= -1 \\ a_0 &= 1 \\ a_0 + a_1 + a_2 + a_3 &= 3 \\ a_0 + 4a_1 + 16a_2 + 64a_3 &= -1 \end{aligned}$$

The augmented matrix of this system is

$$\begin{bmatrix} 1 & -1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 3 \\ 1 & 4 & 16 & 64 & -1 \end{bmatrix}$$

Interchange rows 1 and 2. Add -1 times the new row 1 to each of the other rows.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 4 & 16 & 64 & -2 \end{bmatrix}$$

Multiply row 2 by -1. Add -1 times the new row 2 to row 3. Add -4 times the new row 2 to row 4.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 20 & 60 & -10 \end{bmatrix}$$

Multiply row 2 by $\frac{1}{2}$. Add -20 times the new row 3 to row 4.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 60 & -10 \end{bmatrix}$$

From the first row we see that $a_0 = 1$, and from last two rows we conclude that $a_2 = 0$ and $a_3 = -\frac{1}{6}$. Finally, from back substitution, it follows that $a_1 = 2 + a_2 - a_3 = \frac{13}{6}$. Thus the interpolating polynomial is $p(x) = 1 + \frac{13}{6}x - \frac{1}{6}x^3$.

Section 3.1

5. (a) $A+2B = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -6 & 2 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -7 & 4 \\ 9 & 1 \end{bmatrix}$

(b) $A - B^T$ is not defined

(c) $4D - 3C^T = \begin{bmatrix} 4 & 4 \\ -12 & 12 \end{bmatrix} - \begin{bmatrix} 3 & 9 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -12 & 15 \end{bmatrix}$

(d) $D - D^T = \begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$

(e) $G + (2F^T) = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 6 \\ 10 & 0 & 4 \\ 4 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 8 & -1 & 9 \\ 9 & 1 & 6 \\ 8 & 3 & 11 \end{bmatrix}$

(f) $(7A - B) + E$ is not defined

7. (a) $CD = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$

(b) $AE = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 12 & 6 \\ 5 & -2 & 8 \\ 4 & 5 & 7 \end{bmatrix}$

(c) $FG = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix}$

(d) $B^T F = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 18 & 17 \\ 0 & 5 & 3 \end{bmatrix}$

(e) $BB^T = \begin{bmatrix} 2 & 1 \\ -3 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 & 4 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -5 & 8 \\ -5 & 10 & -12 \\ 8 & -12 & 16 \end{bmatrix}$

(f) GE is not defined

11. (a) $\begin{bmatrix} 2 & -3 & 5 \\ 9 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$

13.
$$\begin{aligned} 5x_1 + 6x_2 - 7x_3 &= 2 \\ -x_1 - 2x_2 + 3x_3 &= 0 \\ 4x_2 - x_3 &= 3 \end{aligned}$$

18. (a) $\mathbf{r}_1(BA) = \mathbf{r}_1(B)A = \begin{bmatrix} 6 & -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 6 & -6 & 70 \end{bmatrix}$

(b) $\mathbf{r}_3(BA) = \mathbf{r}_3(B)A = \begin{bmatrix} 7 & 7 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 63 & 41 & 122 \end{bmatrix}$

(c) $\mathbf{c}_2(BA) = B\mathbf{c}_2(A) = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 17 \\ 41 \end{bmatrix}$

23. $\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} k+1 \\ k+2 \\ -1 \end{bmatrix} = k(k+1) + (k+2) - 1 = k^2 + 2k + 1 =$ $(k+1)^2 = 0$ if and only if $k = -1$.

D2. If $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then $AA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

D4. The matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the only 3×3 with this property. Here is a proof:

Since $x\mathbf{c}_1(A) + y\mathbf{c}_2(A) + z\mathbf{c}_3(A) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we must have $x\mathbf{c}_1(A) + y\mathbf{c}_2(A) + z\mathbf{c}_3(A) = \begin{bmatrix} x+y \\ x-y \\ 0 \end{bmatrix}$ for all x, y, and z. Taking $x = 1$, $y = 0$, $z = 0$, it follows that $\mathbf{c}_1(A) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Similarly, $\mathbf{c}_2(A) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{c}_3(A) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

D9. (a) False. For example, if A is 2×3 and B is 3×2, then AB and BA are both defined.

(b) True. If AB and BA are both defined and A is $m \times n$, then B must be $n \times m$; thus AB is $m \times m$ and BA is $n \times n$. If, in addition, $AB + BA$ is defined then AB and BA must have the same size, i.e. $m = n$.

(c) True. From the column rule, $\mathbf{c}_j(AB) = A\mathbf{c}_j(B)$. Thus if B has a column of zeros, then AB will have a column of zeros.

(d) False. For example, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(e) True. If A is $n \times m$, then A^T is $m \times n$, AA^T is $n \times n$, and A^TA is $m \times m$. Thus A^TA and AA^T both are square matrices, and so $\operatorname{tr}(A^TA)$ and $\operatorname{tr}(AA^T)$ are both defined.

(f) False. If $\mathbf{u}$ and $\mathbf{v}$ are $1 \times n$ row vectors, then $\mathbf{u}^T\mathbf{v}$ is an $n \times n$ matrix.

Section 3.2

7. (a) A matrix X satisfies the equation $\text{tr}(B)A + 3X = BC$ if and only if $3X = BC - \text{tr}(B)A$, i.e.

$$3X = \begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix} \begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix} - 15 \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -18 & -62 & -33 \\ 7 & 17 & 22 \\ 11 & -42 & 38 \end{bmatrix} - \begin{bmatrix} 30 & -15 & 45 \\ 0 & 60 & 75 \\ -30 & 15 & 60 \end{bmatrix} = \begin{bmatrix} -48 & -47 & -78 \\ 7 & -47 & -53 \\ 41 & -42 & -22 \end{bmatrix}$$

in which case we have $X = \frac{1}{3}\begin{bmatrix} -48 & -47 & -78 \\ 7 & -47 & -53 \\ 41 & -42 & -22 \end{bmatrix}$

(b) A matrix X satisfies the equation $B + (A + X)^T = C$ if and only if

$$(A + X)^T = C - B$$
$$A + X = ((A + X)^T)^T = (C - B)^T$$
$$X = (C - B)^T - A = C^T - B^T - A$$

Thus $X = \begin{bmatrix} 0 & 1 & 3 \\ -2 & 7 & 5 \\ 3 & 4 & 9 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 4 \\ -3 & 1 & -7 \\ -5 & 2 & 6 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -10 & 2 & -4 \\ 1 & 2 & 7 \\ 10 & 1 & -1 \end{bmatrix}$.

14. $X = (C - B)^{-1}AB = \frac{1}{22}\begin{bmatrix} -5 & -7 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} 10 & -5 \\ 18 & -7 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -88 & 37 \\ 66 & -29 \end{bmatrix}$

17. $(AB)^{-1}(AC^{-1})(D^{-1}C)^{-1}D^{-1} = B^{-1}A^{-1}AC^{-1}C^{-1}DD^{-1} = B^{-1} = \frac{1}{20}\begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix}$

20. (a) Given that $(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$ it follows that $5A^T = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}^{-1} = (-1)\begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix}$,

and so $A = \frac{1}{5}\begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix}^T = \frac{1}{5}\begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix}$.

(b) Given $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$, we have $I + 2A = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}^{-1} = \frac{1}{13}\begin{bmatrix} -5 & 2 \\ 4 & 1 \end{bmatrix}$ and so it follows that

$A = \frac{1}{2}\left(\frac{1}{13}\begin{bmatrix} -5 & 2 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{13}\begin{bmatrix} -9 & 1 \\ 2 & -6 \end{bmatrix}$.

21. The matrix $A = \begin{bmatrix} c & 1 \\ c & c \end{bmatrix}$ is invertible if and only if $\det(A) = c^2 - c \neq 0$, i.e. if and only if $c \neq 0, 1$.

24. One such example is $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 2 \\ 3 & -2 & 0 \end{bmatrix}$. In general, any matrix of the form $\begin{bmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{bmatrix}$.

32. (a) If $A^2 = I$, then $(I - A)^2 = I^2 - 2A + A^2 = I - 2A + A = I - A$; thus $I - A$ is idempotent.

(b) If $A^2 = I$, then $(2A - I)(2A - I) = 4A^2 - 4A + I^2 = 4A - 4A + I = I$; thus $2A - I$ is invertible and $(2A - I)^{-1} = 2A - I$.

35. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $p(x) = x^2 - (a+b)x + (ad - bc)$, then

$$p(A) = A^2 - (a+b)A + (ad-bc)I = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 - (a+b)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} - \begin{bmatrix} a^2+da & ab+db \\ ac+dc & ad+d^2 \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} bc-da & 0 \\ 0 & cb-ad \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

D1. **(a)** Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, and $A^2 - B^2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. On the other hand, $(A+B)(A-B) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$.

(b) $(A+B)(A-B) = A^2 - AB + BA - B^2$

(c) $(A+B)(A-B) = A^2 - B^2$ if and only if $AB = BA$.

D4. No. First note that if A^3 is defined then A must be square. Thus if $A^3 = AA^2 = I$, it follows that A is invertible with $A^{-1} = A^2$.

D5. (a) False. $(AB)^2 = (AB)(AB) = A(BA)B$. If A and B commute then $(AB)^2 = A^2B^2$, but if $BA \neq AB$ then this will not in general be true.

(b) True. Expanding both expressions, we have $(A-B)^2 = A^2 - AB - BA + B^2$ and $(B-A)^2 = B^2 - BA - AB + A^2$; thus $(A-B)^2 = (B-A)^2$.

(c) True. The basic fact (from Theorem 3.2.11) is that $(A^T)^{-1} = (A^{-1})^T$, and from this it follows that $(A^{-n})^T = ((A^n)^{-1})^T = ((A^n)^T)^{-1} = ((A^T)^n)^{-1} = (A^T)^{-n}$.

(d) False. For example, if $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then $\mathrm{tr}(AB) = \mathrm{tr}\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right) = 3$, whereas $\mathrm{tr}(A)\mathrm{tr}(B) = (2)(2) = 4$.

(e) False. For example, if $B = -A$ then $A + B = 0$ is not invertible (whether A is invertible or not).