Lecture 29: Orthonormal Sets of Vectors

1 Recalling Definitions

Recall that a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors is said to be **orthogonal** if

And $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is said to be **orthonormal** if

Example: $\{(1,2,1,0),(1,-1,1,0),(0,1,1,0)\}$ is

Example: $\{(1,2,1,0),(1,-1,1,0),(1,0,-1,0)\}$ is

$\{(1,2,1,0),(1,-1,1,0),(1,0,-1,0)\}$ is not orthonormal, since

but

2 Orthogonal Sets are Always Linearly Independent

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set of **nonzero** vectors, then we claim that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent

Example: we showed that the set

$$\{(1,2,1,0),(1,-1,1,0),(1,0,-1,0)\}$$

is orthogonal, and since the vectors are nonzero, the set is linearly independent.

WARNING: this does not work the other way: plenty of linearly independent sets are NOT orthogonal, for example

3 Orthogonal and Orthonormal Bases

If V is a subspace of $\mathbb{R}^n$, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a set of vectors from V, then this set is called

- an **orthogonal basis** of V if it is both an orthogonal set of vectors and a basis of V
- an **orthonormal basis** of V if it is both an orthonormal set of vectors and a basis of V

Since orthogonal sets of nonzero vectors and all orthonormal sets are linearly independent, to check that such a set is a basis of V you only need to

4 Projection Using Orthogonal Bases

Suppose that W is a subspace of $\mathbb{R}^n$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an orthogonal basis of W.

If we have a vector $\mathbf{v}$ in $\mathbb{R}^n$, and want to compute $\mathrm{proj}_W(\mathbf{v})$, we use the formula from the previous lecture:

5 Projection Using Orthonormal Bases

An orthonormal basis is orthogonal, so if $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is orthonormal, we can use the same formula as we just obtained.

It becomes simpler because orthonormality means that

6 An Example

Example: Let P be the subspace of $\mathbb{R}^3$ whose basis is $\{(1, 3, 1), (1, -1, 2)\}$. Compute the orthogonal projection of $(3, 2, 2)$ onto W.

Note that our basis is

We could have normalized the basis to obtain

and then computed $\text{proj}_W(3, 2, 2) =$

7 Expressing a Vector as a Linear Combination of an Orthogonal or Orthonormal Basis

If W is a subspace of $\mathbb{R}^n$, and if $\mathbf{v}$ is in W, then recall that

So if $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an orthogonal basis of W, then

And in the very special case where $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an orthonormal basis

Example: Express the vector $(3, 4, 5)$ as a linear combination of the basis

$$\{(1, 3, 1), (1, -1, 2), (7, -1, -4)\}$$

of $\mathbb{R}^3$.

8 The Gram-Schmidt Orthogonalization Procedure

If you are given a non-orthogonal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of a subspace W of $\mathbb{R}^n$, but you want an orthogonal basis of W, use the following procedure. We will build an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ out of the non orthogonal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$.

- Set $\mathbf{v}_1 =$

- Set $\mathbf{v}_2 =$

  This guarantees that $\mathbf{v}_2 \cdot \mathbf{v}_1 = 0$ because it replaces $\mathbf{w}_2$ with its vector component orthogonal to $\mathbf{v}_1$.

- Set $\mathbf{v}_3 =$

  This guarantees that $\mathbf{v}_3 \cdot \mathbf{v}_1 = 0$ and $\mathbf{v}_3 \cdot \mathbf{v}_2 = 0$ because it replaces $\mathbf{w}_3$ with its vector component orthogonal to span$\{\mathbf{v}_1, \mathbf{v}_2\}$.

- Continue in this fashion until you have a new basis.

Example: Let H be the hyperplane in $\mathbb{R}^4$ with basis $\{(1,1,1,1),(0,2,1,1),(1,-1,1,0)\}$. Produce an orthogonal basis of H.

Example: Let H be the hyperplane in $\mathbb{R}^4$ with basis $\{(1,1,1,1),(0,2,1,1),(1,-1,1,0)\}$. Produce an orthonormal basis of H.

One consequence of the Gram-Schmidt procedure is that every nonzero subspace of $\mathbb{R}^n$ has an orthonormal basis, because every nonzero subspace has a basis, and then we can apply the G-S procedure and normalize to get an orthonormal basis.