Lecture 28: Orthogonal Projection

1 Orthogonal Projection onto the Axes

We have already seen orthogonal projection onto the x-axis and y-axis in $\mathbb{R}^2$.

For example, if $\mathbf{v} = (2,3)$, then the projection of $\mathbf{v}$ onto the y-axis is $(0,3)$:

More generally the projection onto the y-axis of (x, y) is

The matrix that accomplishes this projection is

because

2 Orthogonal Projection onto an Arbitrary Line

We want to make the concept of orthogonal projection more general.

Now we want to be able to project onto any line, not just the standard axes.

A line is just a span of a single nonzero vector, so imagine our line L as span$\{\mathbf{w}\}$

We want to project a vector $\mathbf{v}$ onto the line L

Idea of projection: express $\mathbf{v}$ as the sum of two vectors

with

and

So we need to find the scalar a such that

Now we can define **the orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$** (also known as the **orthogonal projection of $\mathbf{v}$ onto** $\text{span}\{\mathbf{w}\}$, or the **vector component of $\mathbf{v}$ along $\mathbf{w}$**)

$\text{proj}_{\mathbf{w}}(\mathbf{v}) =$

This is our vector $\mathbf{v}_1$.

The vector $\mathbf{v}_2$ is then

and is orthogonal to $\mathbf{w}$, and so is called **the vector component of $\mathbf{v}$ orthogonal to $\mathbf{w}$**.

Example: Find the orthogonal projection of $(4, 1)$ onto $(1, 3)$.

Example: Find the distance from the point $(4, 1)$ to the line $y = 3x$.

3 Orthogonal Projection onto an Arbitrary Subspace

Suppose that W is a subspace of $\mathbb{R}^n$, and that $\{\mathbf{w}_1, \ldots, \mathbf{w}_j\}$ is a basis of W.

Suppose that $\mathbf{v}$ is some vector in $\mathbb{R}^n$.

We would like to express $\mathbf{v}$ as the sum of two vectors

with

and

A vector in W is in the span of the basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_j\}$, so has the form

So $\mathbf{v}_1 = M\mathbf{a}$ for some $\mathbf{a} \in \mathbb{R}^j$.

$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$ should be orthogonal to the subspace W, that is, it should be in $W^\perp$.

Recall that $W^\perp$ is the same as

since $\{\mathbf{w}_1, \ldots, \mathbf{w}_j\}$ is a basis of W, so we need

This can be expressed in matrix form as

and since $\mathbf{v}_2 =$

we need

Now we can define **the orthogonal projection of $\mathbf{v}$ onto W**:
$\text{proj}_W(\mathbf{v}) =$

This is the vector $\mathbf{v}_1$ in our derivation

The vector $\mathbf{v}_2$ is then

and is orthogonal to W.

Example: Let P be the plane $x - y + 2z$ in $\mathbb{R}^3$. Find the projection of $(1, 3, -1)$ onto P.

Example: Let P be the plane $x - y + 2z$ in $\mathbb{R}^3$. Find the distance from P to the point $(1, 3, -1)$.

4 Projecting an Element Already in the Subspace

Suppose that W is a subspace of $\mathbb{R}^n$ and that $\mathbf{v}$ is in W.

Then $\text{proj}_W(\mathbf{v}) =$