Lecture 26: Row Space, Null Space, Column Space, and Rank

## 1 Row Space

The **row space** of an $m \times n$ matrix $A$ is the span of the rows of $A$.

We write row$(A)$ for the row space of $A$

Example: Let $A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 3 & 1 \\ 3 & 1 & -2 & 11 \end{pmatrix}$. Find row$(A)$.

We have already defined the **null space** of $A$, written null($A$) to be the set of vectors $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{0}$.

Example: Let $A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 3 & 1 \\ 3 & 1 & -2 & 11 \end{pmatrix}$. Find null($A$).

Now null($A$) is the set of all $\mathbf{v} = (w, x, y, z)$ such that

In other words

So null($A$) is also the orthogonal complement of span$\{(1, 0, -1, 2), (-2, 1, 3, 1), (3, 1, -2, 11)\}$, that is

So for our $m \times n$ matrix $A$, slide 10 of Lecture 26 tells us that:

(a). The only vector in both row($A$) and null($A$) is $\mathbf{0}$.

(b). If $\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$ is a basis of row($A$) and $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a basis of null($A$), then the combined set $\{\mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a basis of $\mathbb{R}^n$.

(c). $\dim(\text{row}(A)) + \dim(\text{null}(A)) = n$.

We call $\dim(\text{row}(A))$ the **rank** of $A$ (written $\text{rank}(A)$) and $\dim(\text{null}(A))$ the **nullity** of $A$ (written $\text{nullity}(A)$). So (c) can be restated as

To find bases of $\text{row}(A)$ and $\text{null}(A)$, we use the following fact:

If $B$ is obtained by applying a row operation to $A$, then

1. $\text{row}(B) = \text{row}(A)$
2. $\text{null}(B) = \text{null}(A)$

That is, row operations do not change the row space or null space.

Row operations don't change null space because $\text{null}(A)$ is the solution of

and row reducing $A$ doesn't change the solution of the system: **that's why row reduction is used to solve the system!**

Likewise, it is not hard to see that none of the three row operations change the span of the rows.

For example: if $A = \begin{pmatrix} \text{——————}\mathbf{a}_1\text{——————} \\ \text{——————}\mathbf{a}_2\text{——————} \\ \text{——————}\mathbf{a}_3\text{——————} \end{pmatrix}$,

then $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is no different than

- $\operatorname{span}\{\mathbf{a}_3, \mathbf{a}_2, \mathbf{a}_1\}$ (switch rows)
- $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, -5\mathbf{a}_3\}$ (scale one row)
- $\operatorname{span}\{\mathbf{a}_1 + 2\mathbf{a}_2, \mathbf{a}_2, \mathbf{a}_3\}$ (add a scalar multiple of one row to another)

Example: Let $A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 3 & 1 \\ 3 & 1 & -2 & 11 \end{pmatrix}$. Find bases for $\text{row}(A)$ and $\text{null}(A)$.

First we row reduce $A$

Now we know that $R = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ has the same row space and null space as $A$.

The nice thing about a row echelon form is that the nonzero rows are always linearly independent.

So $\text{row}(A) = \text{row}(R) =$

And $\text{null}(A) = \text{null}(R)$, which is the solution set for

Another nice thing about a row echelon form is that if you set the free variables equal to parameters and solve for the pivot variables, your general solution will always come out as the span of linearly independent vectors.

This brings up an important point: the rank of $A$ (the dimension of $\operatorname{row}(A) = \operatorname{row}(R)$) is the number of nonzero rows in $R$, which equals the number of pivot variables (one pivot per nonzero row).

And the nullity of $A$ (the dimension of $\operatorname{null}(A) = \operatorname{null}(R)$) is equal to the number of free variables in $R$.

So we have learned that for an $m \times n$ matrix $A$

- The rank of $A$ is equal to the number of pivot variables in a row echelon form of $A$.
- The nullity of $A$ is equal to the number of free variables in a row echelon form of $A$.

Recap: For our matrix $A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 3 & 1 \\ 3 & 1 & -2 & 11 \end{pmatrix}$,

the a row echelon form is $R = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

So the rank and nullity are

## 2 Column Space

Recall that the **column space** of a matrix $A$ is the span of the columns.

We write $\mathrm{col}(A)$ for the column space of $A$.

Note that $\mathrm{col}(A)$ is just $\mathrm{row}(A^T)$ because

Let $A$ be an $m \times n$ matrix. We just apply all our knowledge of row spaces to $A^T$ to obtain:

(a). The only vector in both $\mathrm{col}(A)$ and $\mathrm{null}(A^T)$ is $\mathbf{0}$.

(b). If $\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$ is a basis of $\mathrm{col}(A)$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a basis of $\mathrm{null}(A^T)$, then the combined set $\{\mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a basis of $\mathbb{R}^m$.

(c). $\dim(\mathrm{col}(A)) + \dim(\mathrm{null}(A^T)) = m$.

(d). column operations don't change the column space nor the nullspace of the transpose

The dimension of col($A$) is the same as the dimension of row($A^T$), so

The spaces row($A$), null($A$), col($A$) = row($A^T$), and null($A^T$) are called the **four fundamental subspaces of the matrix** $A$.

## 3 Transposition Does Not Change Rank

Suppose $A$ is an $m \times n$ matrix with row echelon form $R$.

We have already seen that row operations don't change the row space.

So row($A$) = row($R$), and so rank($A$) = rank($R$).

Row operations CAN change the column space, but they DO NOT change its dimension.

So it may be that col($A$) $\neq$ col($R$), but

Let's now look at $A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 3 & 1 \\ 3 & 1 & -2 & 11 \end{pmatrix}$, and its reduced row echelon form $R = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

As already noted, the rank equals the number of nonzero rows, which is equal to the number of pivot variables.

Note that the pivot columns are linearly independent and that all the other columns are linear combinations of these.

So the dimension of the column space is equal to the number of pivot variables also

But then

Example: We know that the row space of $A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 3 & 1 \\ 3 & 1 & -2 & 11 \end{pmatrix}$ has dimension 2 because its reduced row echelon form is $R = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

So $\text{rank}(A) = 2$, and this is also the dimension of the column space of $A$.

Indeed, the column space is

$\text{span}\{(1,-2,3),(0,1,1),(-1,3,-2),(2,1,11)\}$

It is clear that the first two vectors are not scalar multiples of each other, and

$$(-1,3,-2) = -(1,-2,3) + (0,1,1)$$
$$(2,1,11) = 2(1,-2,3) + 5(0,1,1)$$

so that the third and fourth vectors are already in the span of the first two.

Eliminating these, we see that

is a basis for col($A$): this gives direct confirmation that the column space has dimension 2.

## 4 Full Row Rank, Full Column Rank

Let $A$ be an $m \times n$ matrix

We say that $A$ has **full row rank** if $\text{rank}(A) = m$, i.e., equals the number of rows of $A$. This means that the $m$ rows of $A$

Likewise, we say that $A$ has **full column rank** if $\text{rank}(A) = n$, i.e., equals the number of columns of $A$. This means that the columns of $A$

Example: We found that the rank of

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 3 & 1 \\ 3 & 1 & -2 & 11 \end{pmatrix}$$

is 2. So