Lecture 26: Orthogonal Complements

1 Hyperplanes

Recall that a hyperplane through the origin is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to a single vector $\mathbf{a} = (a_1, \ldots, a_n)$.

That is, the hyperplane orthogonal to $\mathbf{a}$ is the set of all $\mathbf{x} = (x_1, \ldots, x_n)$ satisfying

We write $\mathbf{a}^\perp$ to denote this hyperplane.

Example: In $\mathbb{R}^4$, the hyperplane $(1, -2, 3, 5)^\perp$ is the set of (x_1, x_2, x_3, x_4) with

This is a ____-dimensional object.

2 Orthogonal Complement

More generally, if S is any set of vectors in $\mathbb{R}^n$, we write $S^\perp$ for the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that

The set S can be finite or infinite, but let's start with finite sets.

Example: Describe the set $\{(0,1,2),(-1,1,0)\}^{\perp}$ in $\mathbb{R}^3$

If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$ is our set of vectors, then this set

$$S^\perp = \{\mathbf{v}_1, \ldots, \mathbf{v}_j\}^\perp$$

is called

Example: from the previous two slides

3 Orthogonal Complement of a Subspace

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$ is a finite set of vectors, and

$$\mathbf{x} \in \{\mathbf{v}_1, \ldots, \mathbf{v}_j\}^{\perp}$$

then this means that

It turns out that $\mathbf{x}$ will also be orthogonal to any linear combination of $\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$

So if $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$, then

V is a subspace (spans of sets of vectors always are subspaces).

So to find the orthogonal complement of a subspace V, you should just find the orthogonal complement of a spanning set (such as a basis) of V.

Example: We already discovered that $\{(0,1,2), (-1,1,0)\}^{\perp} = \text{span}\{(2,2,-1)\}$.

Therefore, if $V = \text{span}\{(0,1,2), (-1,1,0)\}$, then $V^{\perp}$ is also $\text{span}\{(2,2,-1)\}$.

So $V = \text{span}\{(0,1,2),(-1,1,0)\}$ is a plane and $V^{\perp} = \text{span}\{(2,2,-1)\}$ is the line orthogonal to the plane.

Note that the line and plane intersect at

If you combine the bases of V and $V^{\perp}$ into one set: $\{(0,1,2),(-1,1,0),(2,2,-1)\}$, then this set is

And note that $\dim(V) + \dim(V^{\perp}) =$

Also note that the orthogonal complement of the line $V^{\perp}$ is just the plane V, that is

These observations are manifestations of some general facts:

If V is a subspace of $\mathbb{R}^n$, then

(a). The only vector in both V and $V^\perp$ is $\mathbf{0}$.

(b). If $\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$ is a basis of V and $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a basis of $V^\perp$, then the combined set $\{\mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a basis of $\mathbb{R}^n$.

(c). $\dim(V) + \dim(V^\perp) = n$.

(d). $(V^\perp)^\perp = V$