Lecture 25: Building Bases and Dimension

1 Cutting Down a Spanning Set to Get a Basis

Suppose that we know that $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and we want a basis of V. We can use the procedure from slide 14 of Lecture 24 to remove dependent vectors from our set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ without changing its span.

1. Find a dependence relation
2. Use this to express one of the vectors $\mathbf{v}$ as a linear combination of the others
3. Remove $\mathbf{v}$ from the set
4. Check whether the reduced set is independent: if it is, then you are done. If not, keep repeating steps 1 to 3 on smaller and smaller sets until you have an independent set.

For example, if

$$V = \text{span}\{(1,2,0),(1,0,-1),(-1,2,2)\},$$

then on slides 10–13 of Lecture 24, we show that

and that $\{(1,2,0),(1,0,-1)\}$ is linearly independent.

2 Building up an Independent Set to Get a Basis

Let $\mathbf{v}_1 = (1, 2, 0)$ and $\mathbf{v}_2 = (1, -1, 0)$. Form a basis of $\mathbb{R}^3$ that includes $\mathbf{v}_1$ and $\mathbf{v}_2$.

First of all, we notice that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent because

Next we note that they can not span all of $\mathbb{R}^3$ because

This brings up a good point: if we have a linearly independent set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$ from a subspace V, but if $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_j\} \neq V$, we can enlarge our set of vectors while retaining linear independence.

1. If $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_j\} \neq V$, then some $\mathbf{w} \in V$ is not in $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$.
2. Find such a $\mathbf{w}$.
3. Then the extended set $\{\mathbf{v}_1, \ldots, \mathbf{v}_j, \mathbf{w}\}$ is a larger linearly independent set of vectors in V.

(The reason step 3 is true is that $\mathbf{w}$ is not a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_j$, so there can be no dependence relation involving $\mathbf{w}$.)

If you want to extend an existing set of linearly independent vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$ to a basis of V, just keep repeating these three steps.

With each new vector, the span increases in size until it becomes all of V.

Example: Find a basis of $\mathbb{R}^3$ which includes the vectors $\mathbf{v}_1 = (1, 0, -1)$ and $\mathbf{v}_2 = (2, 0, 3)$.

First, note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent.

Now find a vector $\mathbf{w}$ that is in $\mathbb{R}^3$ but not in span$\{\mathbf{v}_1, \mathbf{v}_2\}$

3 Dimension and Basis

Let V be a k-dimensional subspace of $\mathbb{R}^n$

Then there are some quick ways to see if a set is NOT a basis of V.

- A set containing fewer than k vectors can not span V, so is not a basis of V
- A set containing more than k vectors of V must be linearly independent, so is not a basis of V

Example: Let H be the hyperplane given by

$$w + x + y + z = 0$$

in $\mathbb{R}^4$, and let $\mathbf{v}_1 = (1, -1, 0, 0)$ and $\mathbf{v}_2 = (1, 2, -4, 1)$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis of H?

Example: Let P be the plane given by

$$x + 2y + z = 0$$

in $\mathbb{R}^4$ and suppose that $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (0, 1, -2)$, $\mathbf{v}_3 = (2, 2, -6)$. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis of P?

4 A Two-Out-of-Three Criterion for a Basis

Let V be a subspace of $\mathbb{R}^n$. Then a set of k vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ that satisfies any two of the three conditions below will automatically satisfy the third, and will be a basis of V.

1. $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$
2. $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent
3. The number k of vectors in $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ equals the dimension of V

Example: Let H be the hyperplane given by

$$w + x + y + z = 0$$

in $\mathbb{R}^4$, and let $\mathbf{v}_1 = (1, -1, 0, 0)$, $\mathbf{v}_2 = (1, 2, -4, 1)$, $\mathbf{v}_3 = (1, 0, 0, -1)$. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis of V?