Lecture 24: Basis

1 Review of Span

The **span** of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ is the set of all linear combinations of the vectors

Here a_1, a_2, ..., a_k scalars; we allow each scalar to range over all of $\mathbb{R}$.

Example: $\operatorname{span}\{(1,2,0),(1,0,-1),(-1,2,2)\}$
is the set of all vectors of the form

2 Is a Vector in the Span?

Suppose we are asked whether $(3, 4, -1)$ is in $\operatorname{span}\{(1, 2, 0), (1, 0, -1), (-1, 2, 2)\}$.

Suppose we are asked whether $(2, 1, 0)$ is in $\operatorname{span}\{(1, 2, 0), (1, 0, -1), (-1, 2, 2)\}$.

We found that although we had a set of three vectors $\{(1, 2, 0), (1, 0, -1), (-1, 2, 2)\}$, we could not combine them to make the vector $(2, 1, 0)$.

It would be very different if our set of three vectors had been the standard unit vectors $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Then we could get any vector we want by combining them:

3 Review of Linear Independence

So we introduced another definition to distinguish these two scenarios.

We called $\{(1, 2, 0), (1, 0, -1), (-1, 2, 2)\}$ a **linearly dependent** set of vectors, and $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ a **linearly independent** set of vectors.

Let's recall the formal definition and see how it fits in with our calculations above.

A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of vectors is said to be **linearly dependent** if there are some scalars a_1, a_2, ..., a_k such that

and at least one of the scalars is nonzero.

If we find such a linear combination of the vectors that comes out to $\mathbf{0}$ (with at least one of the scalars nonzero), we call it a **dependence relation**.

Let's now check that $\{(1,0,0),(0,1,0),(1,0,0)\}$ are linearly independent.

Let's now check that $\{(1,2,0),(1,0,-1),(-1,2,2)\}$ is linearly dependent.

Now we see the reason why some of the vectors in $\mathbb{R}^3$ are not in $\operatorname{span}\{(1,2,0),(1,0,-1),(-1,2,2)\}$: the third vector is already a combination of the first two:

$$(-1,2,2) = (1,2,0) - 2(1,0,-1)$$

Thus, when we form the span of the three vectors, we are really only getting the span of two vectors.

The third vector adds nothing! So

We should then ask ourselves whether this smaller set is linearly independent or dependent.

A dependence relation would need to be of the form

Which would either mean

or

Neither of which happens.

This helps us recall that two vectors are linearly dependent if and only if one is a scalar multiple of the other (that is, if and only if they are **parallel**).

This brings up a good point: we can always cut down a linearly dependent set to get a linearly independent one. The method:

1. Find a dependence relation
2. Use this to express one of the vectors $\mathbf{v}$ as a linear combination of the others
3. Remove $\mathbf{v}$ from the set
4. Check whether the reduced set is independent: if it is, then you are done. If not, keep repeating steps 1 to 3 on smaller and smaller sets until you have an independent set.

4 Subspaces

Recall that a **subspace** of $\mathbb{R}^n$ is a subset S of $\mathbb{R}^n$ that is **closed** under addition and scalar multiplication:

- If you take any two vectors $\mathbf{u}$, $\mathbf{v}$ from S, then $\mathbf{u}+\mathbf{v}$ is also in S
- If you take any vector $\mathbf{u}$ from S, and any scalar a, then $a\mathbf{u}$ is also in S.

Geometrically, the subspaces of $\mathbb{R}^3$ are

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In $\mathbb{R}^2$, we have only $\{\mathbf{0}\}$, lines through the origin, and $\mathbb{R}^2$ itself.

In $\mathbb{R}^n$, we have subspaces ranging in dimension from the zero-dimensional $\{\mathbf{0}\}$ to the the n-dimensional $\mathbb{R}^n$ itself.

Once we know about span, we can give an alternative definition of subspace and dimension:

A **subspace** of $\mathbb{R}^n$ is either $\{\mathbf{0}\}$ or the span of a set of vectors.

For instance, any plane through the origin in $\mathbb{R}^3$ can be written as the span of two vectors in the plane (just make sure to pick non-parallel ones, otherwise the span would only be a line).

5 Basis

Let V be a subspace of $\mathbb{R}^n$ with $V \neq \{\mathbf{0}\}$.

A **basis** of V is a set $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_k$ of vectors such that

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The span part is important because we want to be able to write any vector $\mathbf{u}$ in V as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$

The independence part is important because we do not want any redundancy in our basis.

Example: Find a basis for the solution space of

$$\begin{aligned} w + 2x + y - z &= 0 \\ 2w + 3x + 2y &= 0 \end{aligned}$$

in $\mathbb{R}^4$.

You can get a different basis depending on precisely how you choose to solve the system (e.g., whether you go all the way to reduced row-echelon form or not).

For example, we found the basis

$$\{(-1, 0, 1, 0), (-3, 2, 0, 1)\}$$

Another set that works equally well as a basis is

$$\{(-1, 0, 1, 0), (0, 2, -3, 1)\}$$

- This new set is also linearly independent because its vectors are not parallel
- So this new set spans a plane
- Which must be the same plane as the original basis spans, because we can write both vectors of the original basis as linear combinations of vectors in the new basis

So a subspace can have **many different** bases! Here are some facts about bases

1. Every nonzero subspace V has infinitely many bases

2. If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_j\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ are both bases of V, then $j = k$. That is, any two bases of the same subspace have the same number of vectors.

Since all bases of a subspace have the same number of vectors, this number is important, and is called the **dimension** of the subspace.

For example, note that each line through the origin has a basis consisting of a single nonzero vector. For example, consider $y = x$.

By scaling, you can use **infinitely many different bases** for the same subspace: for example $\{(1,1,)\}$, $\{(-2,-2)\}$, and $\{(7,7)\}$ are all bases of our line.

But each basis has the same number of vectors: in this case 1 vector, so a line is 1-dimensional

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Similarly, note that there are many bases for any given plane through the origin, but each such basis has two vectors, since planes are two dimensional.

We saw this in $\mathbb{R}^4$ on slide 20: both

$$\{(-1,0,1,0),(-3,2,0,1)\}$$

and

$$\{(-1,0,1,0),(0,2,-3,1)\}$$

are bases of this plane: each has two vectors.