Lecture 23: Composition of Linear Transformations and Invertible Linear Operators

1 Composition of Linear Transformations

In Lecture 21, beginning at slide 13, we studied what happens when we a succession of linear operators to a vector.

Now we expand the discussion to more general linear transformations

Remember that linear operators are the special class of linear transformations distinguished by having square matrices.

If $T_1 \colon \mathbb{R}^n \to \mathbb{R}^k$ and $T_2 \colon \mathbb{R}^k \to \mathbb{R}^m$ are linear transformations, then the **composition of T_2 with T_1**, written $T_2 \circ T_1$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ given by the rule

$$(T_2 \circ T_1)(\mathbf{v}) = T_2(T_1(\mathbf{v})).$$

First, let's figure out the domain and codomain of $T_2 \circ T_1$.

Example: Suppose $T_1 \colon \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$T_1(x, y) = (x, x - y, y)$$

and $T_2 \colon \mathbb{R}^3$ is given by

$$T_2(x, y, z) = (2x, y + z, y - z, x + y + 3z)$$

Then $(T_2 \circ T_1)$

We can also work in matrix form: then the rule for what order to multiply the matrices is the same one that we learned for linear operators.

Since $(T_2 \circ T_1)(\mathbf{v})$ is obtained by multiplying a matrix times $\mathbf{v}$, $T_2 \circ T_1$ is also linear! It gets its linearity from the linearity of T_1 and T_2.

More generally, you can compose more of linear transformations

$(T_3 \circ T_2 \circ T_1)(\mathbf{v}) =$

or in matrix form

2 Inverses of Linear Operators

If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one linear operator, then there is an operator called the **inverse** of T, written T^{-1}, such that

If T is not one-to-one, it has no inverse

If we write the matrix for T

$$[T] = \begin{pmatrix} T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ \cdots \ T(\mathbf{e}_n) \end{pmatrix}$$

so that $T(\mathbf{v}) = [T]\mathbf{v}$, and also write the matrix for T^{-1}

$$[T^{-1}] = \begin{pmatrix} T^{-1}(\mathbf{e}_1) \ T^{-1}(\mathbf{e}_2) \ \cdots \ T^{-1}(\mathbf{e}_n) \end{pmatrix}$$

so that $T^{-1}(\mathbf{v}) = [T^{-1}]\mathbf{v}$, then

$$(T^{-1} \circ T)(\mathbf{v}) = \mathbf{v}$$

Example: Let $T(x, y) = (x - y, x + 2y)$

Let's first check that T is one-to-one

The matrix $[T]$ for T is then

So the matrix $[T^{-1}]$ for T^{-1} is

So $T^{-1}(x, y) =$

3 How Linear Operators Modify Area

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator

Let P be any parallelogram (including the unit square, or other squares or rectangles) in the plane

Then $T(P)$ is a new parallelogram

Example: $T(x, y) = (x - y, x + 2y)$ and P is the parallelogram with two of its sides being $\mathbf{u} = (3, 0)$ and $\mathbf{v} = (1, 2)$ (with initial points at the origin)

Then $\begin{pmatrix} T(\mathbf{u}) & T(\mathbf{v}) \\ & \end{pmatrix} =$

So the relation between areas is