

Lecture 22: Kernel and Null Space, Range and Column Space

1 Kernel

The **kernel** of a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the set of vectors \mathbf{v} such that $T(\mathbf{v}) = \mathbf{0}$. We write $\ker(T)$ for the kernel of T .

Example: Let $T(x, y, z) = (x + y, x + 2y - z)$. Then $\ker(T)$ is the set of vectors $\mathbf{v} = (x, y, z)$ such that $T(x, y, z) = (0, 0)$, or equivalently

which is a system of equations

which we solve by row reducing

so $\ker(T) =$

Recall that every linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be thought of as multiplication by a matrix $[T]$, that is

$$T(\mathbf{v}) = [T]\mathbf{v}$$

for every $\mathbf{v} \in \mathbb{R}^n$, where

$$[T] = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{pmatrix},$$

with $\mathbf{e}_1, \dots, \mathbf{e}_n$ the standard unit vectors.

For our transformation

$$T(x, y, z) = (x + y, x + 2y - z),$$

we make the matrix

So then the vectors \mathbf{v} that make $T(\mathbf{v}) = \mathbf{0}$ are the same vectors \mathbf{v} that make

$$[T]\mathbf{v} = \mathbf{0}$$

So you have a choice, either use the definition of \ker directly with the transformation T

Or, find the matrix $[T]$ associated with T and compute $\ker(T) =$

Notice that $\ker(T)$ always contains $\mathbf{0}_n$ since $T(\mathbf{0}_n) = [T]\mathbf{0}_n = \mathbf{0}_m$.

2 One-to-One Functions

Suppose we have a function $f: D \rightarrow E$.

We say that f is **one-to-one** if different elements in D get mapped to different elements in E .

In mathematical symbols: $f(a) \neq f(b)$ when $a \neq b$

Or equivalently, the only way to have $f(c) = f(d)$ is if $c = d$. (That is $f(c) = f(d)$ FORCES $c = d$.)

Example: $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = x^2$ is NOT one-to-one

Example: $g: \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = 5x + 3$ is one-to-one

3 One-to-One Linear Transformations

Let $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear transformation.

Let $[T]$ be the matrix

$$[T] = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{pmatrix},$$

associated with T .

Then the following are equivalent.

- (a). T is one-to-one
- (b). $\ker(T) = \{\mathbf{0}\}$
- (c). $\text{null}([T]) = \{\mathbf{0}\}$

We saw before that $\ker(T) = \text{null}([T])$, so the equivalence of the (b) and (c) is clear.

Since $\ker(T) = \text{null}([T])$ is the solution space for a homogeneous system, it is either $\{\mathbf{0}\}$ or contains infinitely many vectors.

Example I: Earlier, we found that the linear transformation

$$T(x, y, z) = (x + y, x + 2y - z),$$

has $\ker(T) = \text{span}\{(1, -1, -1)\}$. So this T is not one-to-one.

Example II: Suppose that

$$U(x, y) = (x + y, x + 2y).$$

Then $\ker(U) =$

4 Range of a Linear Transformation

We have already defined the range of a function in Lecture 20.

We give a definition for the specific case of a linear transformation.

The **range** of a linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the set of all vectors \mathbf{w} in \mathbb{R}^m such that $T(\mathbf{v}) = \mathbf{w}$ for at least one \mathbf{v} in \mathbb{R}^n .

That is, the range of T , written $\text{ran}(T)$, is the set of all actual outputs of T .

Example: Let $T(x, y, z) = (x - y, x - y + 2z, 4z)$. Is $(3, 5, 1)$ in $\text{ran}(T)$?

We want to see if there is any $\mathbf{v} = (x, y, z)$ such that $T(x, y, z) = (3, 5, 1)$, i.e.,

This is the same as the linear system

Note that asking whether $(3, 5, 1)$ is in $\text{ran}(T)$ is exactly the same as asking whether the system

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$$

is consistent.

So to check whether any (a, b, c) is in $\text{ran}(T)$, check the consistency of

Note that $\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$

So $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is in $\text{ran}(T)$ is exactly the same thing

as saying that $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is in

Principle: Let $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear transformation and let $[T]$ be the matrix

$$[T] = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{pmatrix},$$

associated with T .

Then $\text{ran}(T)$ is the column space of $[T]$, that is

$$\text{ran}(T) = \text{span}\{T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)\}.$$

We determined that for $T(x, y, z) = (x - y, x - y + 2z, 4z)$,

$$\text{ran}(T) = \text{span}\{(1, -1, 0), (1, -1, 2), (0, 0, 4)\}$$

The first two vectors span a plane, but the third vector is

$$(0, 0, 4) = -2(1, -1, 0) + 2(1, -1, 2)$$

so it is also in that plane. So the span is just a plane, even though there are three vectors.

That is, the three vectors are **not** linearly independent.

If we were asked to find $\text{ran}(T)$ as a span of a linearly independent set, what would be do?

One way is to analyze the vectors like we just did, and throw out redundant vectors. So we would eliminate $(0, 0, 4)$ from the set since it is already in the span of the other two.

$$\text{ran}(T) = \text{span}\{(1, -1, 0), (1, -1, 2)\}$$

and now the vectors are clearly independent since they are not parallel.

We noticed that our three vectors were not independent because they were simple.

In general, if we couldn't easily tell, we'd look for a dependence relation, i.e., we'd try to see if we can make

$$x(1, -1, 0) + y(1, -1, 2) + z(0, 0, 4) = (0, 0, 0)$$

in some way other than the obvious way with $x = y = z = 0$.

This is the same as solving the linear system

5 Onto Functions

A function $f: D \rightarrow E$ is said to be **onto** if its range is the entire codomain E .

That is, for every $y \in E$, there is some $x \in D$ such that $f(x) = y$.

Example I: $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^3$ is onto.

Example II: $g: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$ is not onto.

6 Onto Linear Transformations

Let $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear transformation.

Let $[T]$ be the matrix

$$[T] = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{pmatrix},$$

associated with T . Then the following are equivalent.

- (a). T is onto
- (b). $\text{ran}(T) = \mathbb{R}^n$
- (c). the column space of $[T]$ is \mathbb{R}^n
- (d). $\text{span}\{T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)\}$ is \mathbb{R}^n
- (e). $[T]\mathbf{v} = \mathbf{w}$ is consistent for any $\mathbf{w} \in \mathbb{R}^m$

The equivalence of (a) and (b) is just the definition of T being “onto”: the range of T is the entire codomain (which is \mathbb{R}^n).

From what we saw above, $\text{ran}(T)$ is the same as the column space of $[T]$, which is precisely $\text{span}\{T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)\}$, so (b), (c), and (d) are equivalent. We also saw that (e) is equivalent to (d).

Example I: For $T(x, y, z) = (x - y, x - y + 2z, 4z)$, we found that

$$\text{ran}(T) = \text{span}\{(1, -1, 0), (1, -1, 2)\}$$

which is a plane, not the whole codomain \mathbb{R}^3 .
So

Example II: For $T(x, y, z) = (x - y, x - y + 2z, 3z)$, we have

$$[T] =$$

so that $\text{ran}(T) =$

and these vectors are linearly independent (check at home!), so they span all of \mathbb{R}^3 . So

7 Relationship Between Being One-to-One and Onto

It is possible for a linear transformation to be one-to-one, but not onto.

Example: $T(x, y) = (x, y, 0)$.

It is also possible for a linear transformation to be onto but not one-to-one.

Example: $T(x, y, z) = (x, y)$. The outputs are clearly all of \mathbb{R}^2

A linear transformation can be neither one-to-one, nor onto

Example: $T(x, y) = (x, 0)$.

Or a linear transformation can be both one-to-one and onto

Example: $T(x, y) = (x, y)$

A SPECIAL CASE: When the domain and the codomain are the same, that is, when T is a linear operator, say from \mathbb{R}^n to \mathbb{R}^n .

Then T can be

- both one-to-one and onto
- neither one-to-one nor onto

but it CANNOT have only one of these two properties.

Example I: Recall that $T(x, y) = (x, 0)$ was neither one-to-one nor onto.

Example II: Recall that $T(x, y) = (x, y)$ was both one-to-one and onto.

8 Unifying Theorem

If A is an $n \times n$ matrix, and T_A is the linear transformation $T_A(\mathbf{v}) = A\mathbf{v}$, then the following statements are equivalent

- (a). The reduced row echelon form of A is I_n .
- (b). A is expressible as a product of elementary matrices.
- (c). A is invertible.
- (d). $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- (e). $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^n$.
- (f). $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b} \in \mathbb{R}^n$.
- (g). The columns of A are linearly independent.
- (h). The rows of A are linearly independent.
- (i). $\det(A) \neq 0$.
- (j). $\lambda = 0$ is not an eigenvalue of A .
- (k). T_A is one-to-one
- (l). T_A is onto.

We have proved the equivalence of (a)–(j) already.

We saw that part (k) is equivalent to $\ker(T_A) = \{\mathbf{0}\}$, which is equivalent to $\text{null}(A) = \{\mathbf{0}\}$, which is the same as (d)

We saw that part (l) is equivalent to (e) above. (See slide 20 and compare parts (a) and (e) there.)